Alexander-Sadiku Fundamentals of Electric Circuits

Chapter 17

The Fourier Series

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The Fourier Series - Chapter 17

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17.1 Trigometric Fourier Series (1)

- The Fourier series of a periodic function *f(t)* is a representation that resolves *f(t)* into a dc component and an ac component comprising an infinite series of harmonic sinusoids.

- Given a periodic function *f(t)=f(t+nT)* where n is an integer and T is the period of the function.

$$f(t) = \underbrace{a_0}_{dc} + \underbrace{\sum_{n=1}^{\infty} (a_0 \cos n\omega_0 t + b_n \sin n\omega_0 t)}_{ac}$$

where $\omega_0 = 2\Pi/T$ is called the fundamental frequency in radians per second.

17.1 Trigometric Fourier Series (1)

- and a_n and b_n are as follow

$$a_n = \frac{2}{T}\int_0^T f(t)\cos(n\omega_o t)dt \qquad b_n = \frac{2}{T}\int_0^T f(t)\sin(n\omega_o t)dt$$

- in alternative form of *f(t)*

$$f(t) = \underbrace{a_0}_{dc} + \underbrace{\sum_{n=1}^{\infty}(A_n\cos(n\omega_0 t + \phi_n)}_{ac}$$

where $A_n = \sqrt{a_n^2 + b_n^2}$, $\phi_n = -\tan^{-1}(\frac{b_n}{a_n})$

17.1 Trigometric Fourier Series (2)

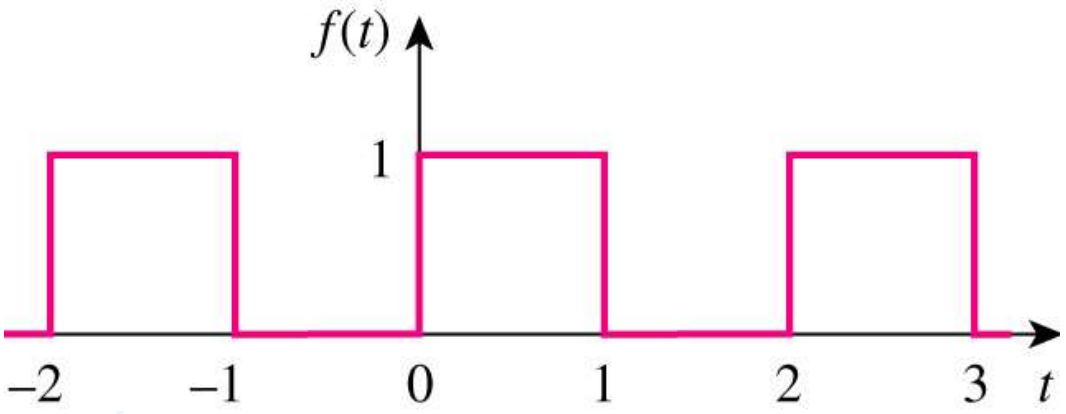
Conditions (Dirichlet conditions) on *f(t)* to yield a convergent Fourier series:

1. *f(t)* is single-valued everywhere.
2. *f(t)* has a finite number of finite discontinuities in any one period.
3. *f(t)* has a finite number of maxima and minima in any one period.
4. The integral $\int_{t_0}^{t_0+T} |f(t)|dt < \infty$ for any t_0.

17.1 Trigometric Fourier Series (3)

Example 1

Determine the Fourier series of the waveform shown below. Obtain the amplitude and phase spectra



17.1 Trigometric Fourier Series (4)

Solution:

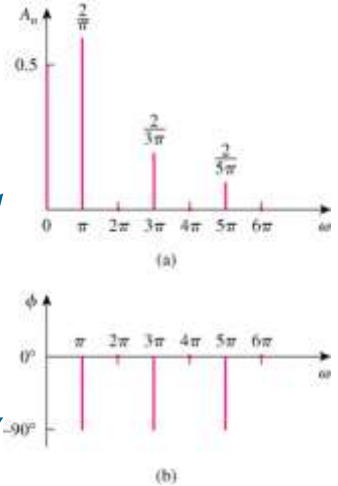
$$f(t)=\begin{cases}1, & 0<t<1\\ 0, & 1<t<2\end{cases} \text{ and } f(t)=f(t+2)$$

$$a_n=\frac{2}{T}\int_0^T f(t)\cos(n\omega_0 t)dt=0 \text{ and}$$

$$b_n=\frac{2}{T}\int_0^T f(t)\sin(n\omega_0 t)dt=\begin{cases}2/n\pi, & \text{n = odd}\\ 0, & \text{n = even}\end{cases}$$

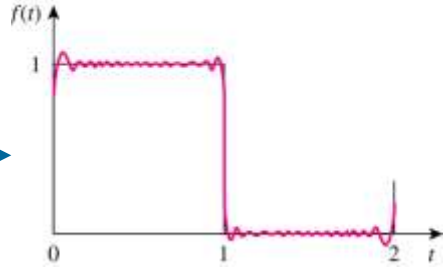
$$A_n=\begin{cases}2/n\pi, & \text{n = odd}\\ 0, & \text{n = even}\end{cases}$$

$$\phi_n=\begin{cases}-90^\circ, & \text{n = odd}\\ 0, & \text{n = even}\end{cases}$$



a) Amplitude and
b) Phase spectrum

$$f(t)=\frac{1}{2}+\frac{2}{\pi}\sum_{k=1}^{\infty}\frac{1}{n}\sin(n\pi t),\quad n=2k-1$$



Truncating the series at N=11

17.2 Symmetry Considerations (1)

Three types of symmetry

1. Even Symmetry : a function *f(t)* if its plot is symmetrical about the vertical axis.

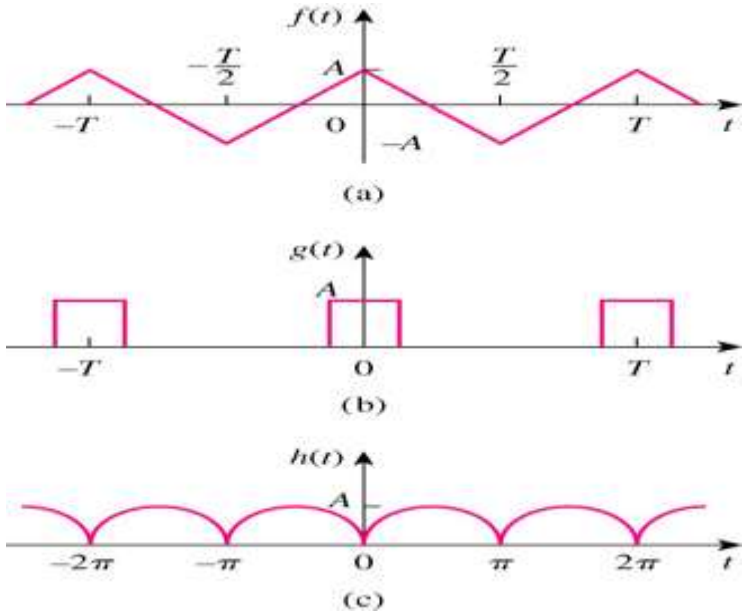
$$f(t) = f(-t)$$

In this case,

$$a_0 = \frac{2}{T}\int_0^{T/2} f(t)dt$$

$$a_n = \frac{4}{T}\int_0^{T/2} f(t)\cos(n\omega_0 t)dt$$

$$b_n = 0$$



Typical examples of even periodic function

17.2 Symmetry Considerations (2)

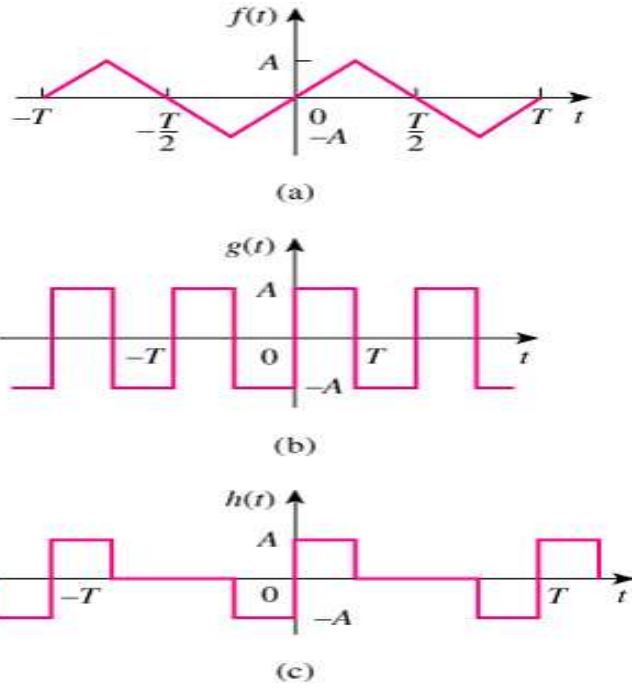
2. Odd Symmetry : a function *f(t)* if its plot is anti-symmetrical about the vertical axis.

$$f(-t) = -f(t)$$

In this case,

$$a_0 = 0$$

$$b_n = \frac{4}{T}\int_0^{T/2} f(t)\sin(n\omega_0 t)dt$$



Typical examples of odd periodic function

17.2 Symmetry Considerations (3)

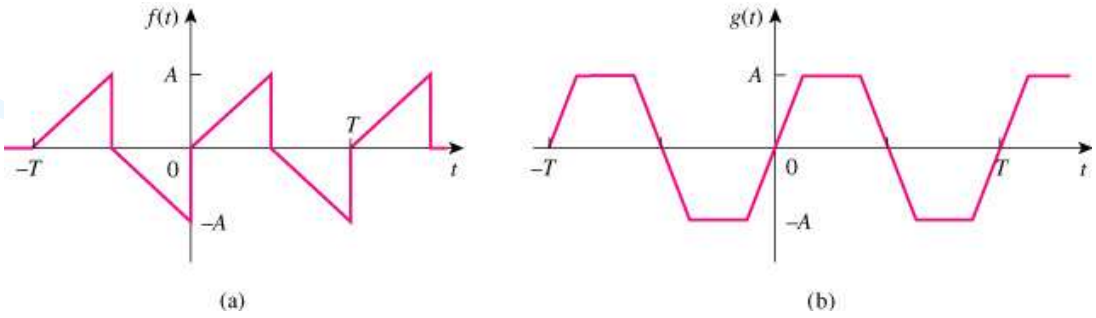
3. Half-wave Symmetry : a function *f(t)* if

$$f(t-\frac{T}{2}) = -f(t)$$

$$a_0 = 0$$

$$a_n = \begin{cases} \frac{4}{T}\int_0^{T/2} f(t)\cos(n\omega_0 t)dt & \text{, for n odd} \\ 0 & \text{, for an even} \end{cases}$$

$$b_n = \begin{cases} \frac{4}{T}\int_0^{T/2} f(t)\sin(n\omega_0 t)dt & \text{, for n odd} \\ 0 & \text{, for an even} \end{cases}$$

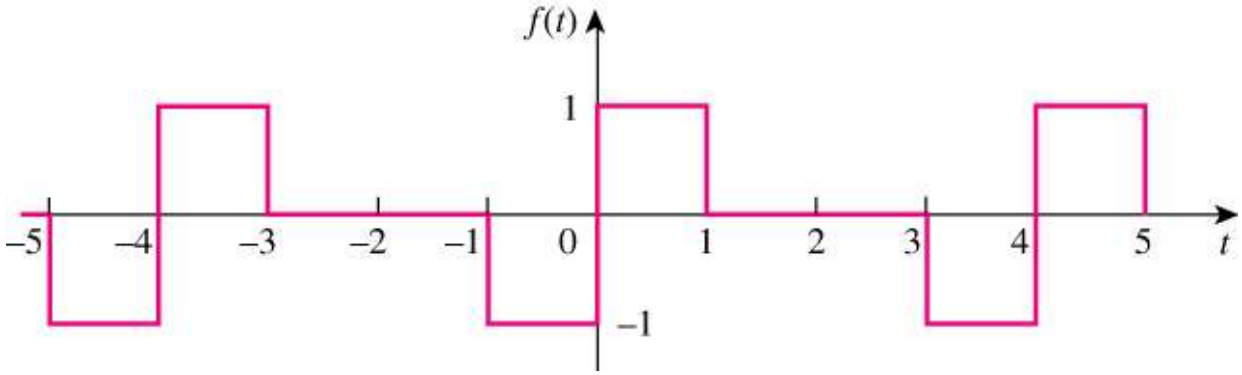


Typical examples of half-wave odd periodic functions

17.2 Symmetry Considerations (4)

Example 2

Find the Fourier series expansion of f(t) given below.



Ans:

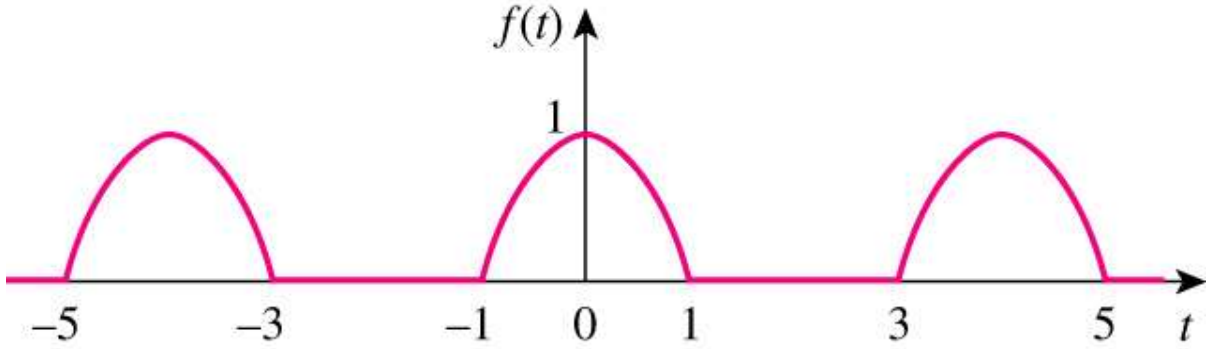
$$f(t) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left(1-\cos\frac{n\pi}{2}\right)\sin\left(\frac{n\pi}{2}t\right)$$

*Refer to in-class illustration, textbook

17.2 Symmetry Considerations (5)

Example 3

Determine the Fourier series for the half-wave cosine function as shown below.



Ans: $$f(t) = \frac{1}{2} - \frac{4}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{n^2}\cos nt,\ n = 2k-1$$

*Refer to in-class illustration, textbook

17.3 Circuit Applications (1)

Steps for Applying Fourier Series

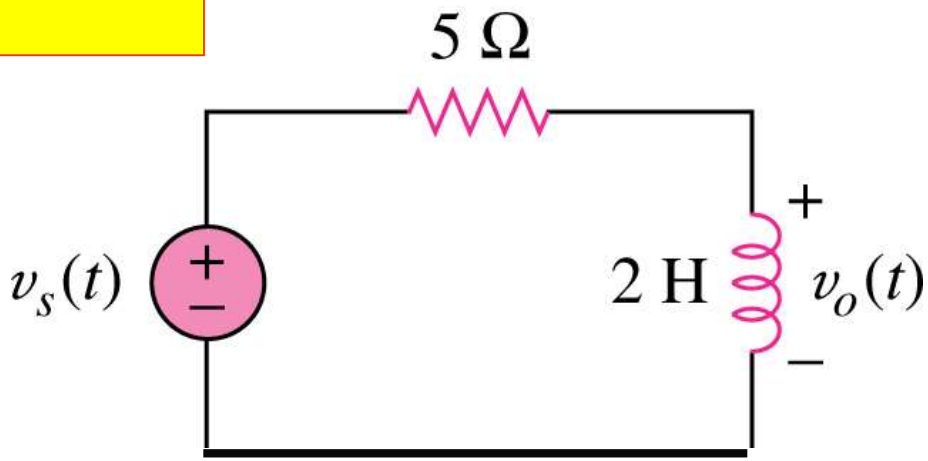
1. Express the excitation as a Fourier series.
2. Transform the circuit from the time domain to the frequency domain.
3. Find the response of the dc and ac components in the Fourier series.
4. Add the individual dc and ac response using the superposition principle.

17.3 Circuit Applications (2)

Example 4

Find the response $v_0(t)$ of the circuit below when the voltage source $v_s(t)$ is given by

$$v_s(t) = \frac{1}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin(n\pi\omega t),\ n = 2k-1$$



17.3 Circuit Applications (3)

Solution

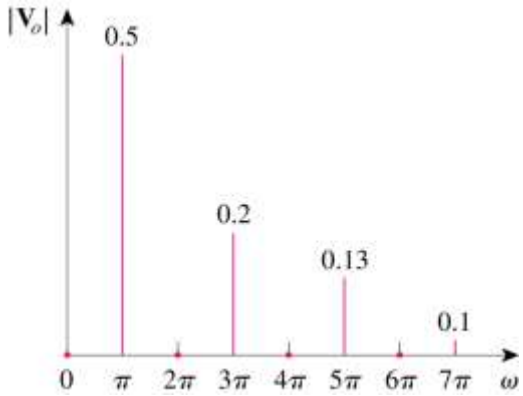
Phasor of the circuit $\mathbf{V}_0 = \dfrac{j2n\pi}{5 + j2n\pi}\mathbf{V}_s$

For dc component, (ω_n=0 or n=0), V_s = ½ => V_o = 0

For nth harmonic, $\mathbf{V}_S = \dfrac{2}{n\pi}\angle -90°, \quad \mathbf{V}_0 = \dfrac{4\angle -\tan^{-1} 2n\pi/5}{\sqrt{25+4n^2\pi^2}}\mathbf{V}_s$

In time domain,

$$v_0(t) = \sum_{k=1}^{\infty} \frac{4}{\sqrt{25+4n^2\pi^2}} \cos(n\pi t - \tan^{-1}\frac{2n\pi}{5})$$



Amplitude spectrum of the output voltage

17.4 Average Power and RMS Values (1)

Given:

$$v(t) = \mathrm{V_{dc}} + \sum_{n=1}^{\infty} \mathrm{V_n} \cos(n\omega_0 t - \theta_n) \text{ and } i(t) = \mathrm{I_{dc}} + \sum_{n=1}^{\infty} \mathrm{I_m} \cos(m\omega_0 t - \phi_m)$$

The average power is

$$\mathrm{P} = \mathrm{V_{dc}}\mathrm{I_{dc}} + \frac{1}{2}\sum_{n=1}^{\infty} \mathrm{V_n}\mathrm{I_n} \cos(\theta_n - \phi_n)$$

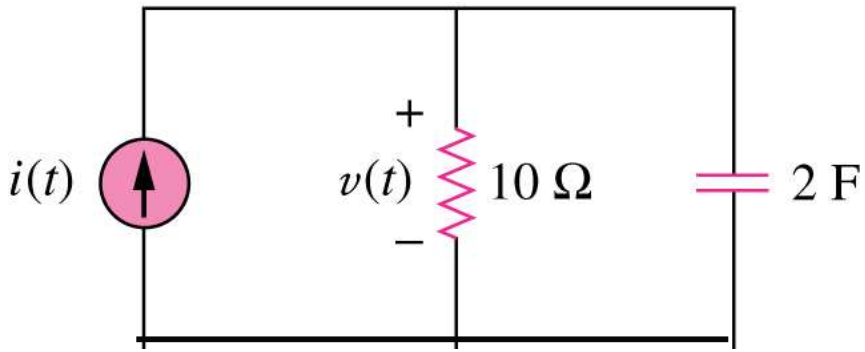
The rms value is

$$F_{rms} = \sqrt{a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)}$$

17.4 Average Power and RMS Values (2)

Example 5:

Determine the average power supplied to the circuit shown below if $i(t)=2+10\cos(t+10°)+6\cos(3t+35°)$ A



Ans: 41.5W

*Refer to in-class illustration, textbook

17.5 Exponential Fourier Series (1)

- The exponential Fourier series of a periodic function *f(t)* describes the spectrum of *f(t)* in terms of the amplitude and phase angle of ac components at positive and negative harmonic.

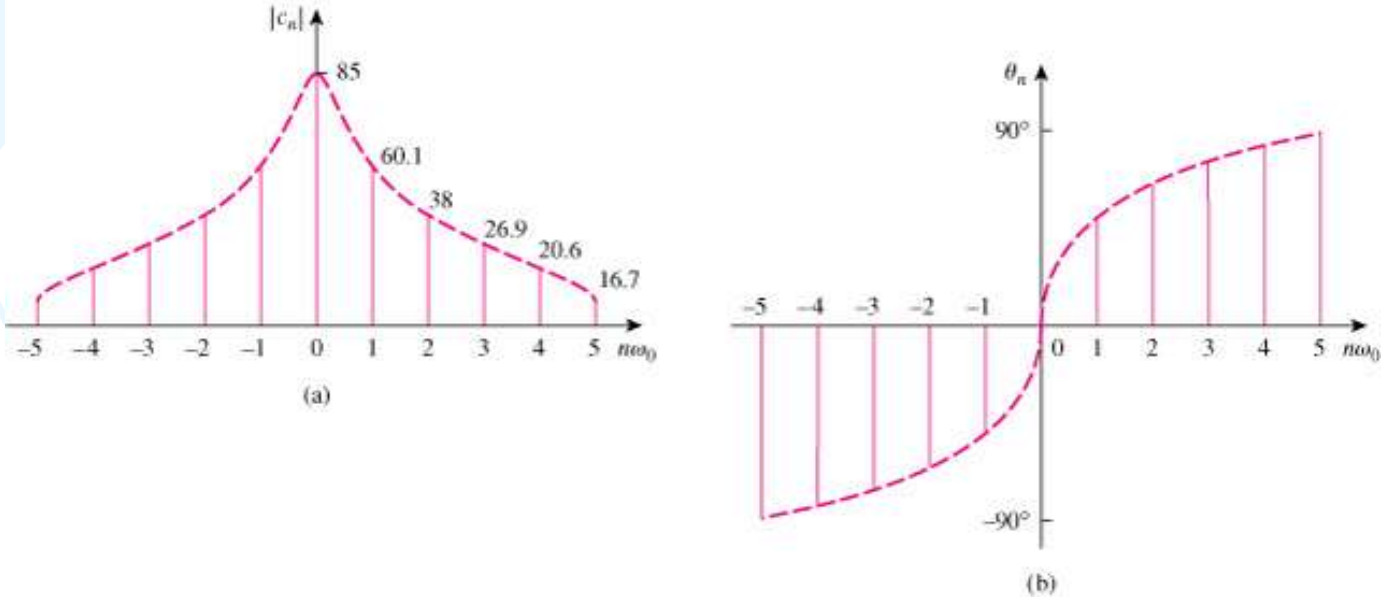
$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_o t}$$

$$c_n = \frac{1}{T}\int_0^T f(t)e^{-jn\omega_0 t}dt, \text{ where } \omega_0 = 2\pi/T$$

- The plots of magnitude and phase of c_n versus $n\omega_0$ are called the complex amplitude spectrum and complex phase spectrum of *f(t)* respectively.

17.5 Exponential Fourier Series (1)

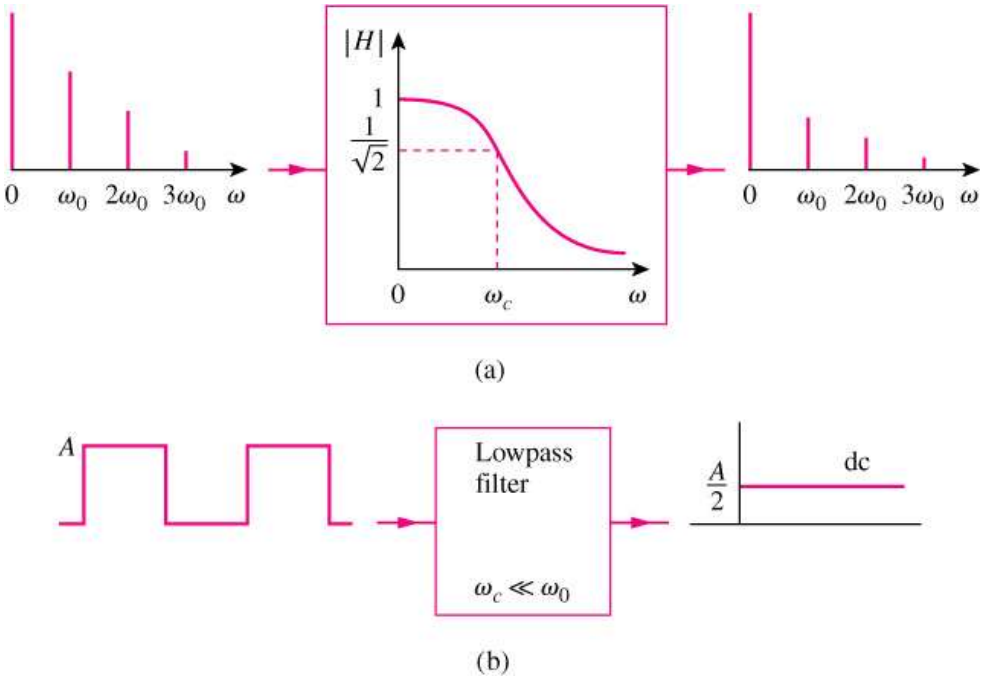
- The complex frequency spectrum of the function $f(t)=e^t$, $0<t<2\pi$ with $f(t+2\pi)=f(t)$



(a) Amplitude spectrum; (b) phase spectrum

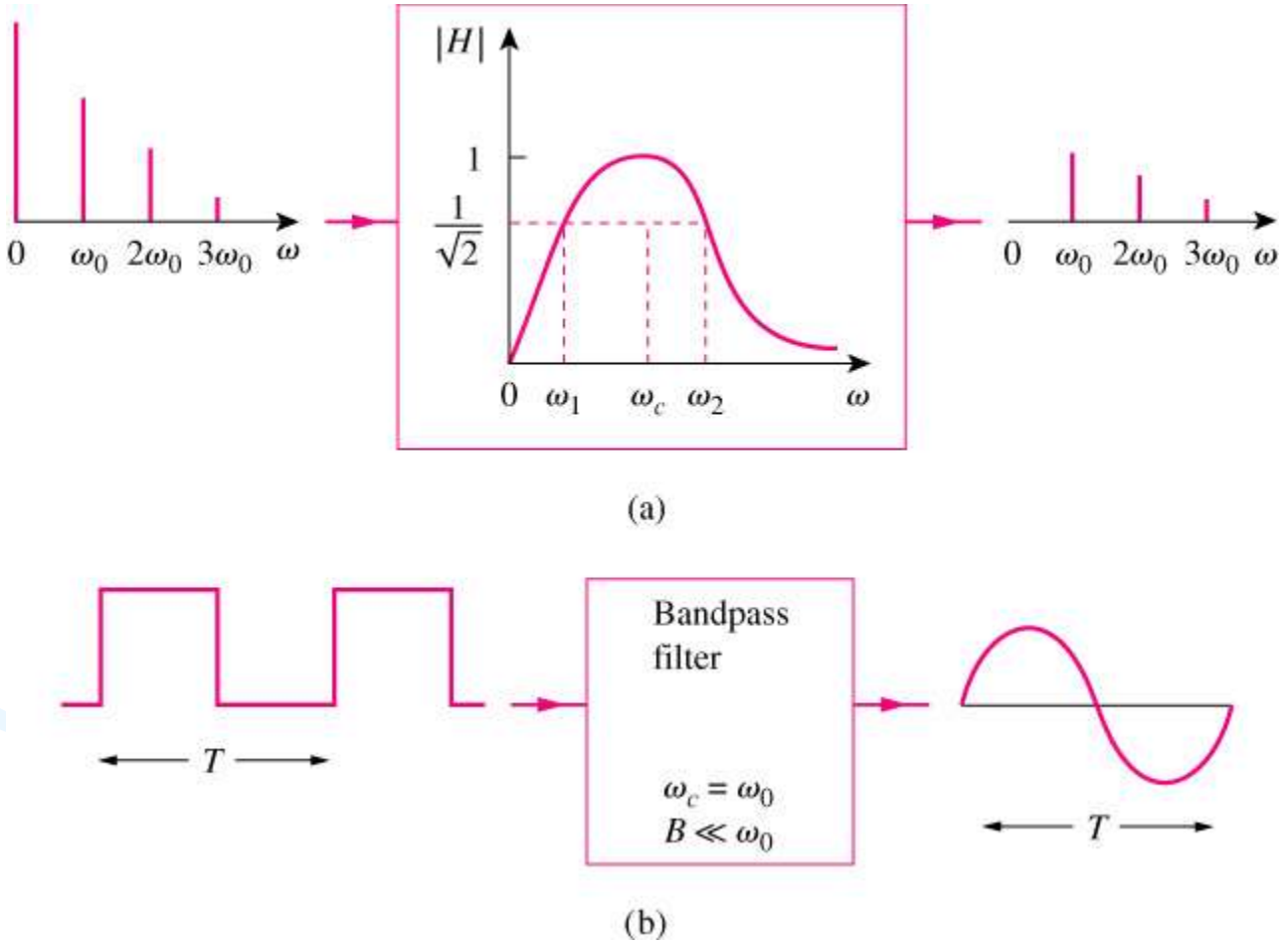
17.6 Application – filter (1)

- Filter are an important component of electronics and communications system.
- This filtering process cannot be accomplished without the Fourier series expansion of the input signal.
- For example,



(a) Input and output spectra of a lowpass filter, (b) the lowpass filter passes only the dc component when $\omega_c << \omega_0$

17.6 Application – filter (2)



(a) Input and output spectra of a bandpass filter, (b) the bandpass filter passes only the dc component when B << ω_0