EWCE 101

Topic: Centroid

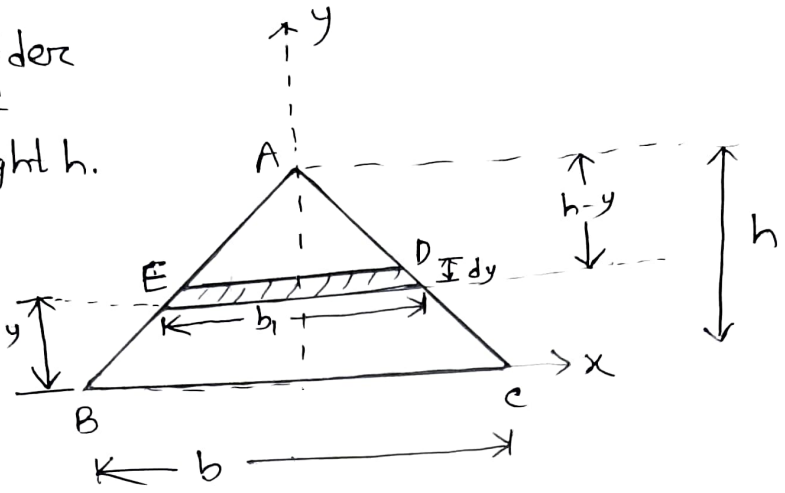
Reference Book: Analytical Mechanics (Virgil Moring Faires)

Department of Environmental, Water Resources and Coastal Engineering (EWCE)

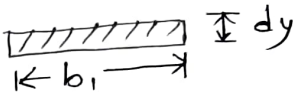
Military Institute of Science and Technology

Centroid of a Triangle

Let us consider a triangle of base b & height h.



Area of element, $dA = b_1 dy$



from similar triangle: ($\triangle AED$ & $\triangle ABC$)

$$\frac{b_1}{h-y} = \frac{b}{h}$$

$$\Rightarrow b_1 = b\left(1-\frac{y}{h}\right)$$

[In this figure, y is considered as axis of symmetry]

$$\therefore \int dA = A = \int_0^h b\left(1-\frac{y}{h}\right)dy$$

$$= b\left[y - \frac{y^2}{2h}\right]_0^h$$

$$= b\left[h - \frac{h^2}{2h}\right] = \frac{1}{2}bh$$

$$\int y\,dA = \int_0^h y\, b\left(1-\frac{y}{h}\right)dy$$

$$= b\int_0^h \left(y - \frac{y^2}{h}\right)dy$$

$$= b\left[\frac{y^2}{2} - \frac{y^3}{3h}\right]_0^h$$

$$= b\left[\frac{h^2}{2} - \frac{h^3 h^2}{3h}\right]$$

$$= b\left[\frac{3h^2 - 2h^2}{6}\right]$$

$$= \frac{bh^2}{6}$$

Now,

$$\bar{y} = \frac{\int y\,dA}{A}$$

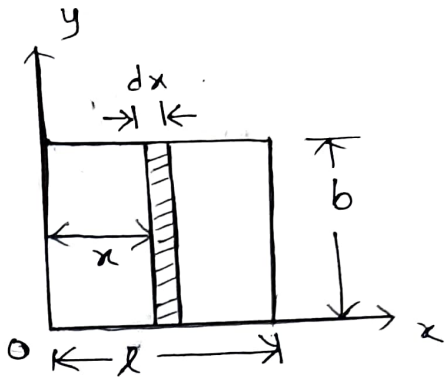
$$= \frac{\frac{bh^2}{6}}{\frac{1}{2}bh}$$

$$= \left(\frac{bh^2}{6} \times \frac{2}{bh}\right)$$

$$= \frac{h}{3}$$ (From the base)

Centroid of a Rectangle:

Let us consider a rectangle of length l and depth b.



Area of the strip $dA = b \cdot dx$

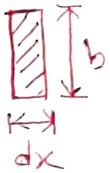
Area of Rectangle,

$$\int dA = A = \int b\,dx$$

$$= b\int_0^l dx$$

$$= b[x]_0^l$$

$$= bl$$



$$\therefore \int x\,dA = \int x \cdot b\,dx = b\int_0^l x\,dx = b\left[\frac{x^2}{2}\right]_0^l$$

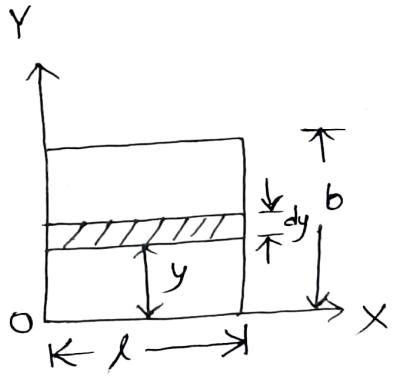
$$= \frac{bl^2}{2}$$

$$\therefore \bar{x} = \frac{\int x\,dA}{A} = \frac{\frac{bl^2}{2}}{bl}$$

$$= \left(\frac{bl^2}{2} \times \frac{1}{bl}\right) = \frac{l}{2}$$

Now, for $\bar{y}$

Area of the strip $dA = l\,dy$



Area of Rectangle,

$$\int dA = \int_0^b l\,dy$$

$$\Rightarrow A = l[y]_0^b$$

$$= lb$$



$$\therefore \int y\,dA = \int_0^b y \cdot l\,dy$$

$$= l\left[\frac{y^2}{2}\right]_0^b$$

$$= \frac{lb^2}{2}$$

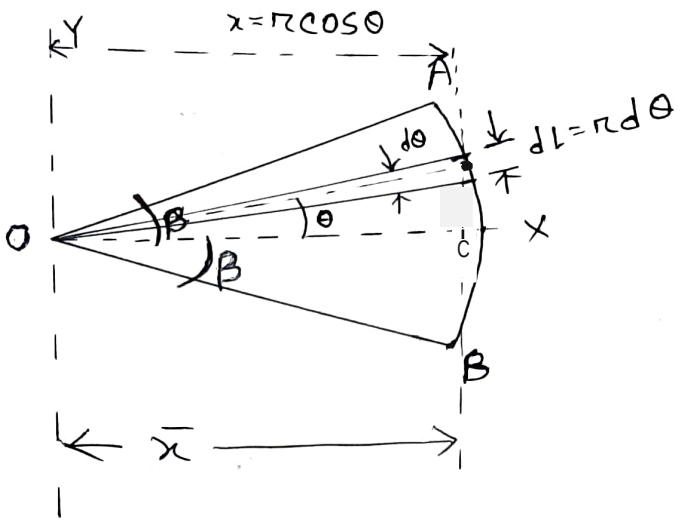
$$\therefore \bar{y} = \frac{\int y\,dA}{A}$$

$$= \frac{\frac{lb^2}{2}}{lb}$$

$$= \left(\frac{lb^2}{2} \times \frac{1}{lb}\right)$$

$$= \frac{b}{2}$$

Centroid of Arc of a Circle:



In the figure, we have chosen the x axis as the bisector of angle 2β, because this line is a line of symmetry and therefore contains the centroid of the arc AB. Since the centroid lies on line OX, it remains only to find the distance $\bar{x}$.

Here, $\angle AOC = \angle BOC = \beta$

$\therefore \angle AOB = \beta + \beta = 2\beta$

$$\therefore \int x\,dL = \int_{-\beta}^{+\beta} (r\cos\theta)\,(r\,d\theta)$$

$$= r^2 \int_{-\beta}^{+\beta} \cos\theta\, d\theta$$

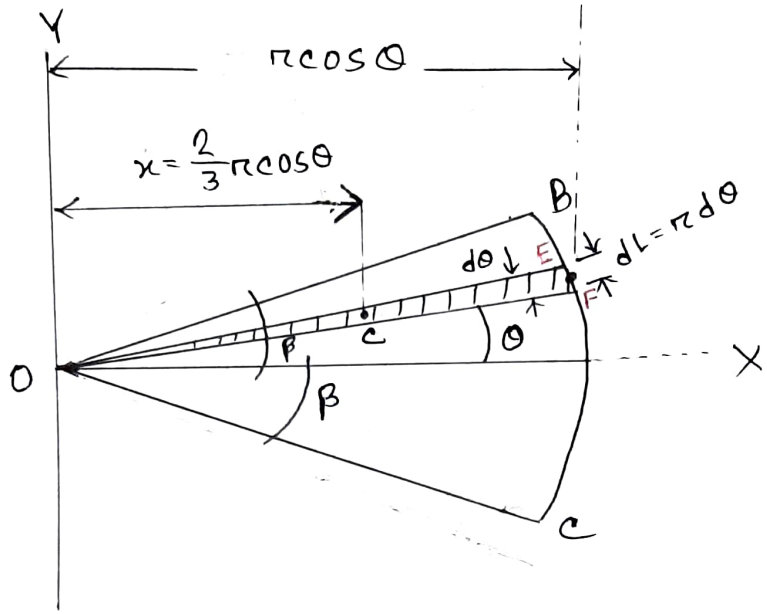
$$= r^2 [\sin\theta]_{-\beta}^{+\beta}$$

$$= 2r^2 \sin\beta$$

$$\therefore L = \int_{-\beta}^{+\beta} r\,d\theta = r[\theta]_{-\beta}^{+\beta} = 2r\beta$$

$$\therefore \bar{x} = \frac{\int x\,dL}{L} = \frac{2r^2\sin\beta}{2r\beta} = \frac{r\sin\beta}{\beta}$$

Centroid of Sector of a Circle:



If we choose the axis of symmetry as the x axis; only one integration is necessary because $\bar{y} = 0$, Here, the elementary area is OEF.

The area of elementary area $OEF = \frac{1}{2} \cdot r \cdot r\,d\theta$

$$= \frac{r^2 d\theta}{2}$$

$$\therefore \bar{x} = \frac{\int x\,dA}{\int dA} = \frac{\int_{-\beta}^{+\beta} \left(\frac{2}{3} r\cos\theta\right)\left(\frac{r^2 d\theta}{2}\right)}{\int_{-\beta}^{+\beta} \frac{r^2 d\theta}{2}}$$

$$= \frac{\frac{1}{3} r^3 \int_{-\beta}^{+\beta} \cos\theta\, d\theta}{\frac{1}{2} r^2 \int_{-\beta}^{+\beta} d\theta}$$

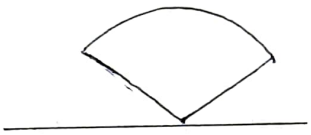
$$= \frac{2}{3} r \frac{[\sin\theta]_{-\beta}^{+\beta}}{[\theta]_{-\beta}^{+\beta}}$$

$$= \frac{2}{3} r \times \frac{2\sin\beta}{2\beta}$$

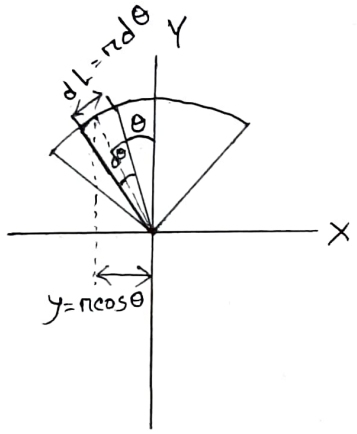
$$= \frac{2r\sin\beta}{3\beta}$$

Lines by Integration Sample Problems

(1) Find by integration, the centroid of a circular arc which subtends an angle of 150°, using the y axis as an axis of symmetry.



Ans:



since, y is the axis of symmetry so,

$\bar{x} = 0$

Now, we know,

$$\bar{y} = \frac{\int y\,dL}{L}$$

Here, $\frac{150^\circ}{2} = 75^\circ = \frac{5}{12}\pi$

$$\int y\,dL = \int_{-\frac{5}{12}\pi}^{\frac{5}{12}\pi} r\cos\theta\, r\,d\theta = r^2 \int_{-\frac{5}{12}\pi}^{\frac{5}{12}\pi} \cos\theta\,d\theta$$

$$= 2r^2 \sin\frac{5\pi}{12}$$

$$\therefore L = \int dL = \int_{-\frac{5}{12}\pi}^{\frac{5}{12}\pi} r\,d\theta = 2r \times \frac{5\pi}{12} = \frac{5\pi}{6} r$$

$$\therefore \bar{y} = \frac{\int y\,dL}{L} = \frac{2r^2 \sin\frac{5\pi}{12}}{\frac{5\pi}{6} r} = 0.737r$$ [Calc in rad mood]

(Ans.)

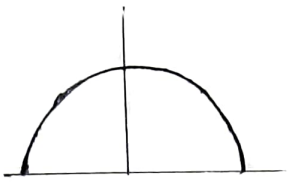


$\text{angle} = \frac{\text{arc}}{\text{radius}}$

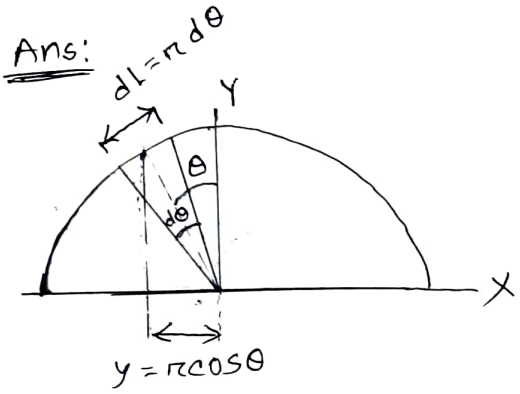
$\therefore \text{arc} = \text{angle} \times \text{radius}$

$\therefore dL = d\theta \times r$

2:Ques: Find by integration, the centroid of a circular arc which subtends an angle of 180°, using y axis as an axis of symmetry.



Ans:



since, y is the axis of symmetry,

so, $\bar{x} = 0$

$$\bar{y} = \frac{\int y\,dL}{L}$$

$$\therefore \int y\,dL = \int_{-\frac{\pi}{2}}^{\pi/2} r\cos\theta\, r\,d\theta$$

$$= r^2 \int_{-\pi/2}^{\pi/2} \cos\theta\, d\theta$$

$$= r^2 \left[\sin\theta\right]_{-\pi/2}^{\pi/2} = 2r^2 \sin\frac{\pi}{2}$$

$$= 2r^2$$

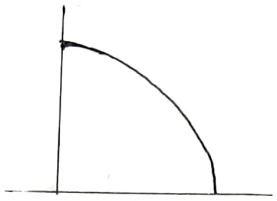
$$\frac{180^\circ}{2} = 90^\circ = \frac{\pi}{2}$$

$$\therefore L = \int dL = \int_{-\pi/2}^{\pi/2} r\,d\theta = r\left[\theta\right]_{-\pi/2}^{\pi/2} = \pi r$$

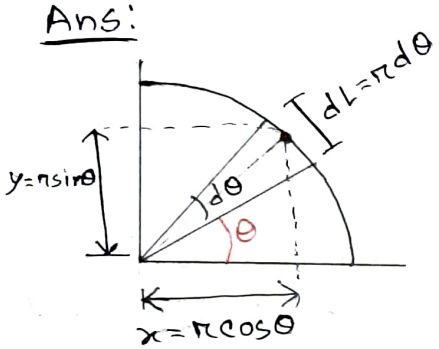
$$\therefore \bar{y} = \frac{\int y\,dL}{L} = \frac{2r^2}{\pi r} = \frac{2}{\pi} r = 0.637\,r$$

(Ans.)

(3) Locate the centroid of a quadrant of a circular arc by integration when the x and y axes mark the boundaries of the quadrant.



Ans:



radius = r

$$\bar{x} = \frac{\int x\,dL}{L} \; ; \quad \bar{y} = \frac{\int y\,dL}{L}$$

$$\int x\,dL = \int_0^{\pi/2} r\cos\theta \cdot r\,d\theta = r^2 \int_0^{\pi/2} \cos\theta\,d\theta$$

$$= r^2 \sin\frac{\pi}{2}$$

$$= r^2$$

$$\int y\,dL = \int_0^{\pi/2} r\sin\theta\, r\,d\theta = r^2 \int_0^{\pi/2} \sin\theta\,d\theta$$

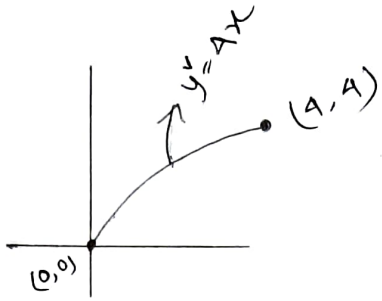
$$= r^2 \cos 0 = r^2$$

$$\therefore L = \int dL = \int_0^{\pi/2} r\,d\theta = r[\theta]_0^{\pi/2} = \frac{\pi}{2} \cdot r$$

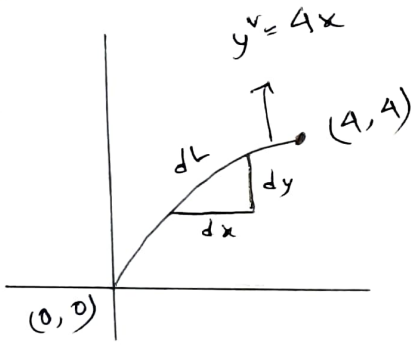
$$\therefore \bar{x} = \frac{\int x\,dL}{L} = \frac{r^2}{\pi/2 r} = 0.637r \; ; \quad \bar{y} = \frac{\int y\,dL}{L}$$

$$= \frac{r^2}{\pi/2 r} = 0.637r$$

4:Ques: Locate by integration, the centre of gravity of a slender wire bent into a parabolic curve whose equation is $y^2 = 4x$ and whose ends are defined by points (0, 0) and (4, 4) inch.



⇒ Soln:



Here, $y^2 = 4x$

$\Rightarrow 2y\,dy = 4\,dx$

$\Rightarrow \frac{dy}{dx} = \frac{2}{y}$

Here, $dL^2 = dx^2 + dy^2$

$\Rightarrow \left(\frac{dL}{dx}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2$

$\Rightarrow dL = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$

$\Rightarrow dL = \sqrt{1 + \left(\frac{2}{y}\right)^2} \cdot \frac{y}{2}\,dy$

$= \sqrt{1 + \frac{4}{y^2}} \cdot \frac{y}{2} \cdot dy$

$= \frac{\sqrt{y^2+4}}{y} \cdot \frac{y}{2}\,dy$

$= \frac{1}{2}\sqrt{y^2+4}\,dy$

$\therefore L = \int_0^4 \frac{1}{2}\sqrt{y^2+4}\,dy = \frac{1}{2}\int_0^4 \sqrt{y^2+4}\,dy = 5.915$ in [using calculator]

$\therefore \int x\,dL = \int x \frac{1}{2}\sqrt{y^2+4}\,dy$

$= \int_0^4 \left(\frac{y^2}{4}\right)\left(\frac{1}{2}\sqrt{y^2+4}\right)dy$

$= \frac{1}{8}\int_0^4 y^2\sqrt{y^2+4}\,dy$

$= 9.701$ in^2 [using calculator]

$\int y\,dL$

$= \int_0^4 y \frac{1}{2}\sqrt{y^2+4}\,dy$

$= \frac{1}{2}\int_0^4 y\sqrt{y^2+4}\,dy$

$= 13.57$ in^2 [using calculator]

$\therefore \bar{x} = \frac{\int x\,dL}{L} = \frac{9.701}{5.915} = 1.64$ inch ; $\bar{y} = \frac{\int y\,dL}{L} = \frac{13.57}{5.915}$

$= 2.30$ inch