EWCE 101

ANALYTICAL MECHANICS

3 Credit Hour

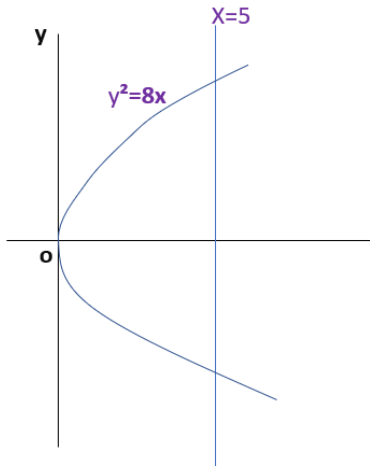
Lecturer
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Military Institute of Science and Technology

Areas By Integration

Ques 1: An area is bounded by the parabola $y^2 = 8x$ and the straight line x=5 in. Determine by integration the centroid of the area.



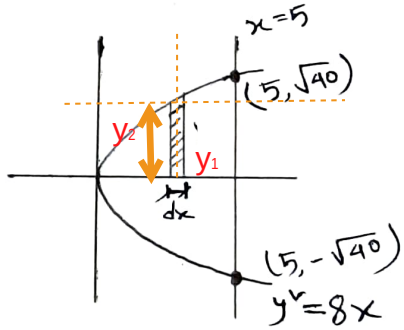
Soln: Here, x is the axis of symmetry therefore $\bar{y} = 0$.

Now, Given, $y^2 = 8x$ & $x = 5$

$\Rightarrow y^2 = (8 \times 5)$

$\Rightarrow y^2 = 40$

$\therefore y = \sqrt{40}$



$$\bar{x} = \frac{\int x\,dA}{A}$$

$$\int x\,dA = 2\int x \cdot (y_2 - y_1)\,dx = 2\int_0^5 x \cdot (\sqrt{8x} - 0)\,dx$$

$$= 2\sqrt{8}\int_0^5 x^{\frac{3}{2}}\,dx$$

$$= 126.49 \text{ in}^3$$

$$A = \int dA = 2\int_0^5 (y_2 - y_1)\,dx$$

$$= 2\int_0^5 (\sqrt{8x} - 0)\,dx$$

$$= 2\sqrt{8}\int_0^5 x^{\frac{1}{2}}\,dx$$

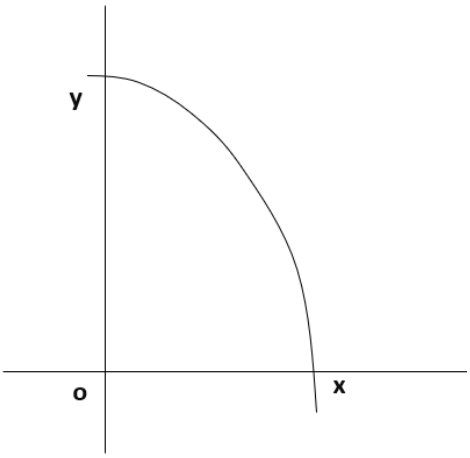
$$= 42.16 \text{ in}^2$$

$$\bar{x} = \frac{\int x\,dA}{A}$$

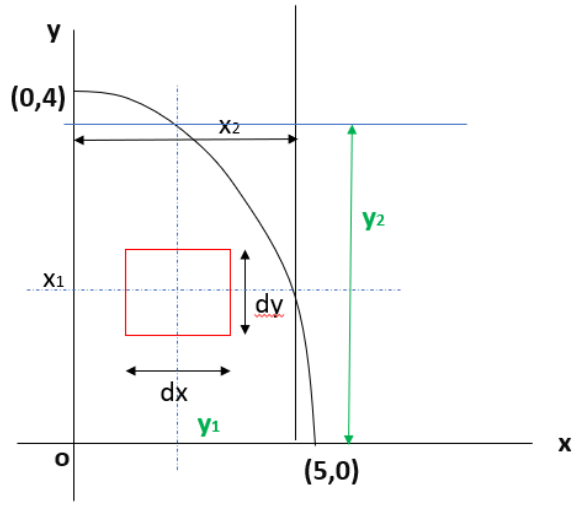
$$= \frac{126.49}{42.16}$$

$$= 3 \text{ in}$$

Ques 2: The equation of an ellipse is $x^2 / 25 + y^2 / 16 = 1$. Locate by integration the centroid of that part of the area of this ellipse which falls in the first quadrant. The linear unit is in foot.



$\Rightarrow$



$\Rightarrow \frac{x^2}{25} + \frac{y^2}{16} = 1$

$\Rightarrow \frac{x^2}{25} = 1 - \frac{y^2}{16}$

$\Rightarrow x^2 = 25\left(1 - \frac{y^2}{16}\right)$

$\therefore x = 5\sqrt{1 - \frac{y^2}{16}}$

$\frac{x^2}{25} + \frac{y^2}{16} = 1$

$\Rightarrow \frac{y^2}{16} = 1 - \frac{x^2}{25}$

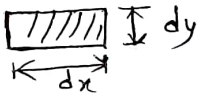
$\Rightarrow y^2 = 16\left(1 - \frac{x^2}{25}\right)$

$\therefore y = 4\sqrt{1 - \frac{x^2}{25}}$

$\frac{x^2}{5^2} + \frac{y^2}{4^2} = 1$

(5,0) & (0,4)

Here, $dA = dx\,dy$



$$\int x\,dA = \int_0^4 \int_{x_1}^{x_2} x\,dx\,dy$$

$$= \int_0^4 \int_0^{5\sqrt{1-\frac{y^2}{16}}} x\,dx\,dy$$

$$= \left[\frac{x^2}{2}\right]_0^{5\sqrt{1-\frac{y^2}{16}}} \cdot \int_0^4 dy$$

$$= \left[\frac{25\left(1-\frac{y^2}{16}\right)}{2} - 0\right] \cdot \int_0^4 dy$$

$$= \frac{1}{2} \times 25\left(1-\frac{y^2}{16}\right) \cdot \int_0^4 dy$$

$$= \frac{1}{2}\int_0^4 25\left(1-\frac{y^2}{16}\right) dy$$

$$= \frac{1}{2}\int_0^4 25\,dy - \frac{1}{2}\int_0^4 \frac{25y^2}{16}\,dy$$

$$= \frac{1}{2} \times 25\left[y\right]_0^4 - \frac{25}{32}\left[\frac{y^3}{3}\right]_0^4$$

$$= \frac{25}{2}\left[4\right] - \frac{25}{32}\left[\frac{4^3}{3}\right]$$

$$= 33.33\ ft^3$$

$$\int y\,dA = \int_0^5 \int_{y_1}^{y_2} y\,dy\,dx$$

$$= \int_0^5 \int_{y_1}^{y_2} y\,dy\,dx$$

$$= \int_0^5 \left[\frac{y^2}{2}\right]_0^{4\sqrt{1-\frac{x^2}{25}}} dx$$

$$= \int_0^5 \left[\frac{16\left(1-\frac{x^2}{25}\right)}{2} - 0\right] dx$$

$$= \frac{1}{2}\left[\int_0^5 16\,dx - \int_0^5 \frac{16}{25}x^2\,dx\right]$$

$$= \frac{1}{2}\times 16\left[x\right]_0^5 - \frac{1\times 16}{2\times 25}\left[\frac{x^3}{3}\right]_0^5$$

$$= 26.67\ ft^3$$

$$A = \int dA$$

$$= \int_0^4 \int_{x_1}^{x_2} dx\,dy$$

$$= \int_0^4 \int_0^{5\sqrt{1-\frac{y^2}{16}}} dx\,dy$$

$$= \int_0^4 \left[x\right]_0^{5\sqrt{1-\frac{y^2}{16}}} dy$$

$$= \int_0^4 \left[5\sqrt{1-\frac{y^2}{16}} - 0\right] dy$$

$$= 15.7079\ ft^2$$

$$\therefore \bar{x} = \frac{\int x\,dA}{A}$$

$$= \frac{33.33}{15.7079} = 2.12\ ft$$

Or,

$$A = \int dA$$

$$= \int_0^5 \int_{y_1}^{y_2} dy\,.\,dx$$

$$= \int_0^5 \int_0^{4\sqrt{1-\frac{x^2}{25}}} dy\,.\,dx$$

$$= \int_0^5 \left[y\right]_0^{4\sqrt{1-\frac{x^2}{25}}} dx$$

$$= \int_0^5 4\sqrt{1-\frac{x^2}{25}}\,dx$$

$$= 15.7079\ ft^2$$

$$\bar{y} = \frac{\int y\,dA}{A}$$

$$= \frac{26.67}{15.7079} = 1.7\ ft$$

Ques 3: Find the co-ordinates of the centroid of the area bounded by the curves $y^2=16x$ and $y=2x$.

Here,

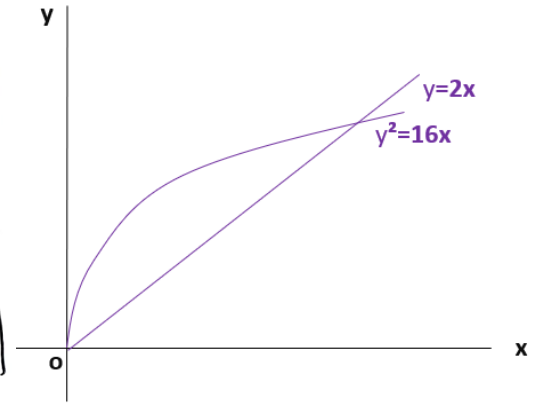
$y = 2x$

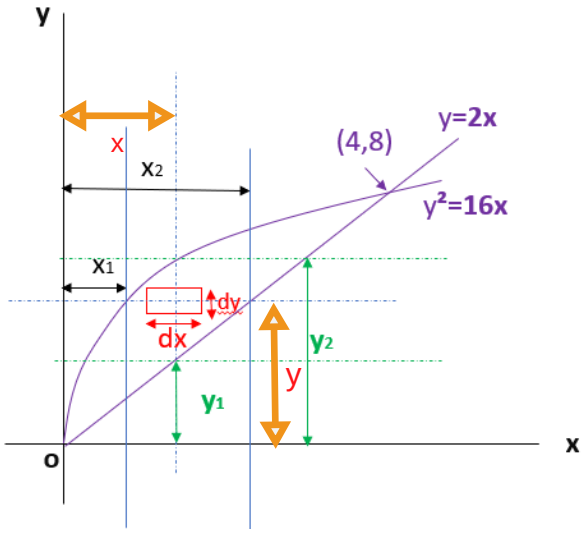
$y^2 = 16x$

$\Rightarrow 4x^2 = 16x$

$\therefore x = 4$

$\therefore y = (2\times 4) = 8$





$dA = dx.dy$; $y = 2x$; $y^2 = 16x$

$$\int x\,dA = \int_0^8 \int_{x_1}^{x_2} x\,dx\,dy$$

$$= \int_0^8 \int_{\frac{y^2}{16}}^{\frac{y}{2}} x\,dx\,dy$$

$$= \int_0^8 \left[\frac{x^2}{2}\right]_{\frac{y^2}{16}}^{\frac{y}{2}} dy$$

$$= \int_0^8 \left[\frac{y^2}{4\times 2} - \frac{y^4}{2\times(16)^2}\right] dy$$

$$= \int_0^8 \frac{y^2}{4\times 2}\,dy - \int_0^8 \frac{y^4}{2\times(16)^2}\,dy$$

$$= \left[\frac{y^3}{3\times 4\times 2}\right]_0^8 - \left[\frac{y^5}{5\times 2\times(16)^2}\right]_0^8$$

$$= \frac{64}{3} - \frac{64}{5}$$

$$= 8.533 \text{ in}^3$$

$$\int y\,dA = \int_0^4 \int_{y_1}^{y_2} y\,dy\,dx$$

$$= \int_0^4 \int_{2x}^{4x^{\frac{1}{2}}} y\,dy\,dx$$

$$= \int_0^4 \left[\frac{y^2}{2}\right]_{2x}^{4x^{\frac{1}{2}}} dx$$

$$= \frac{1}{2}\int_0^4 \left[16x - 4x^2\right] dx$$

$$= \frac{1}{2}\left[16.\frac{x^2}{2} - 4.\frac{x^3}{3}\right]_0^4$$

$$= \frac{1}{2}\left[16.\frac{4^2}{2} - 4.\frac{4^3}{3}\right]$$

$$= 21.33 \text{ in}^3$$

In these maths x and y are centroidal distance of the strip from x and y axis. shown in orange color

$$A = \int dA$$

$$= \int_0^8 \int_{x_1}^{x_2} dx\,dy$$

$$= \int_0^8 \int_{\frac{y^2}{16}}^{\frac{y}{2}} dx\,dy$$

$$= \int_0^8 \left[x\right]_{\frac{y^2}{16}}^{\frac{y}{2}} dy$$

$$= \int_0^8 \left[\frac{y}{2} - \frac{y^2}{16}\right] dy$$

$$= \int_0^8 \frac{y}{2}\,dy - \int_0^8 \frac{y^2}{16}\,dy$$

$$= \left[\frac{y^2}{2\times 2}\right]_0^8 - \left[\frac{y^3}{3\times 16}\right]_0^8$$

$$= \left[\frac{8^2}{4} - 0\right] - \left[\frac{8^3}{48} - 0\right]$$

$$= 5.33 \text{ in}^2$$

OR,

$$A = \int dA$$

$$= \int_0^4 \int_{y_1}^{y_2} dy\,dx$$

$$= \int_0^4 \int_{2x}^{4x^{\frac{1}{2}}} dy\,dx$$

$$= \int_0^4 \left[y\right]_{2x}^{4x^{\frac{1}{2}}} dx$$

$$= \int_0^4 \left[4x^{\frac{1}{2}} - 2x\right] dx$$

$$= \int_0^4 4x^{\frac{1}{2}}\,dx - \int_0^4 2x\,dx$$

$$= 5.33 \text{ in}^2$$

$$\therefore \bar{x} = \frac{\int x\,dA}{A} = \frac{8.533}{5.33} = 1.6 \text{ in}$$

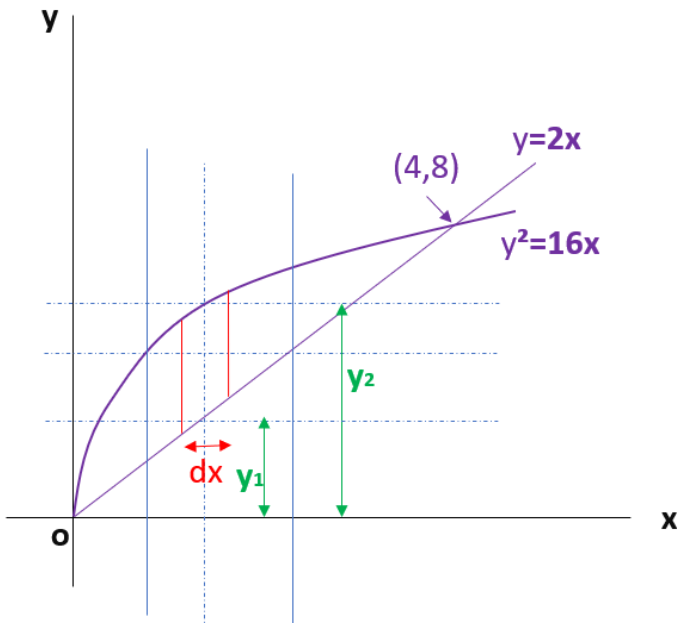
$$\bar{y} = \frac{\int y\,dA}{A} = \frac{21.33}{5.33} = 4 \text{ in}$$

Alternative Procedure

$$\int x\,dA = \int x\cdot(y_2 - y_1)\,dx$$

$$= \int_0^4 x\left(4\sqrt{x} - 2x\right)dx$$

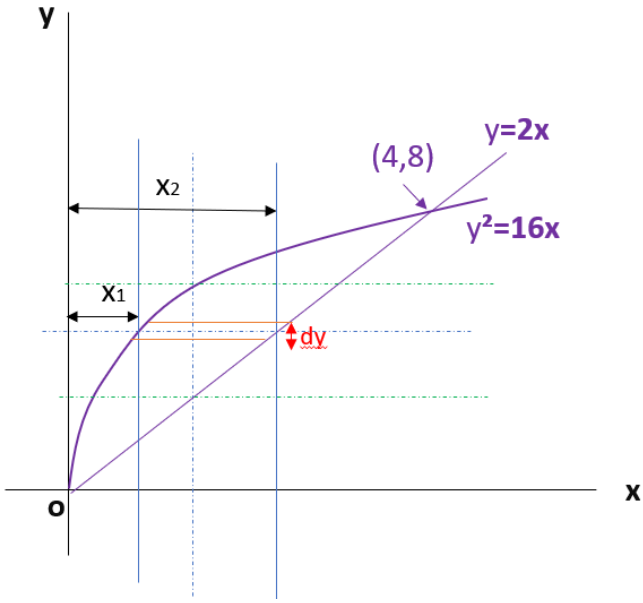
$$= 8.533\ in^3$$



$$\int y\,dA = \int_0^8 y\cdot(x_2 - x_1)\,dy$$

$$= \int_0^8 y\cdot\left(\frac{y}{2} - \frac{y^2}{16}\right)dy$$

$$= 21.33\ in^3$$



$$\int dA = \int_0^4 (y_2 - y_1)\,dx$$

$$= \int_0^4 \left(4\sqrt{x} - 2x\right)dx$$

$$= 5.33\ in^2$$

or,

$$\int dA = \int_0^8 (x_2 - x_1)\,dy$$

$$= \int_0^8 \left(\frac{y}{2} - \frac{y^2}{16}\right)dy$$

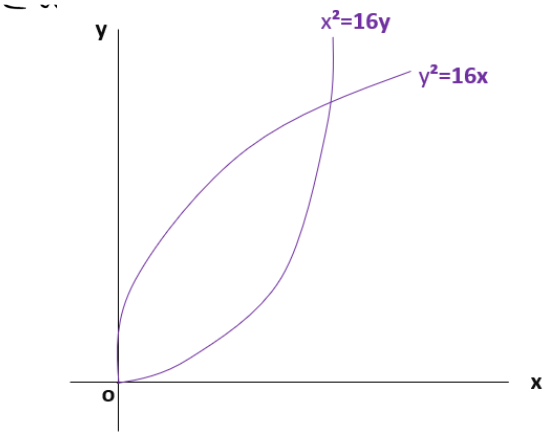
$$= 5.33\ in^2$$

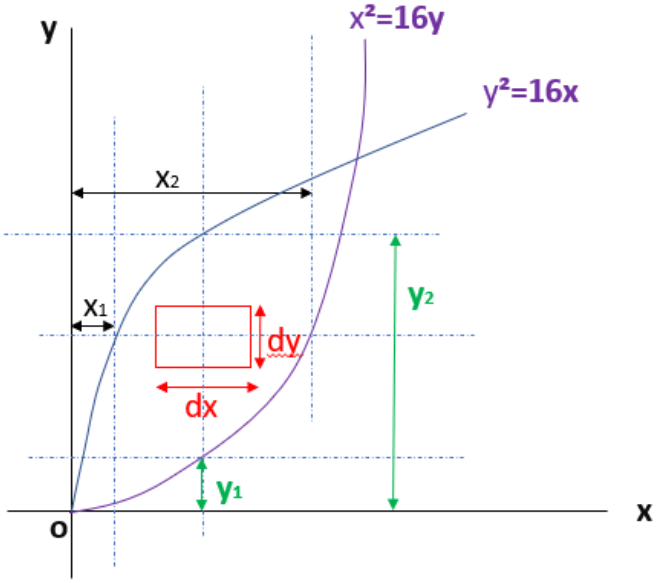
(This overall solution is different from the previous procedure. In previous procedure we used the concept of dA=dx.dy). But at last answer is same in both the procedures

$$\therefore \bar{x} = \frac{\int x\,dA}{A} = \frac{8.533}{5.33} = 1.6\ in$$

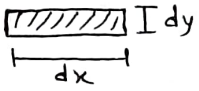
$$\bar{y} = \frac{\int y\,dA}{A} = \frac{21.33}{5.33} = 4\ in$$

Ques 4: Find the co-ordinates of the centroid of the area bounded by the curves $y^2=16x$ and $x^2=16y$





Soln: $dA = dx \cdot dy$



$x^2 = 16y$ & $y^2 = 16x$

$\Rightarrow \left(\frac{y^2}{16}\right)^2 = 16y$ $\quad \Rightarrow x = \frac{y^2}{16}$

$\Rightarrow \frac{y^4}{(16)^2} = 16y$ $\quad \Rightarrow x = \frac{(16)^2}{16}$

$\Rightarrow y^3 = 4096$ $\quad \therefore x = 16$

$\therefore y = 16$

$$\int x\,dA = \int_0^{16}\int_{x_1}^{x_2} x\,dx\,dy$$

$$= \int_0^{16}\int_{\frac{y^2}{16}}^{4y^{\frac{1}{2}}} x\,dx\,dy$$

$$= \int_0^{16}\left[\frac{x^2}{2}\right]_{\frac{y^2}{16}}^{4y^{\frac{1}{2}}} dy$$

$$= \int_0^{16}\frac{1}{2}\left[x^2\right]_{\frac{y^2}{16}}^{4y^{\frac{1}{2}}} dy$$

$$= \int_0^{16}\cdot\frac{1}{2}\cdot\left[16y - \frac{y^4}{(16)^2}\right] dy$$

$$= \frac{1}{2}\int_0^{16}\left[16y - \frac{y^4}{(16)^2}\right] dy$$

$$= \frac{1}{2}\left[16\cdot\frac{y^2}{2} - \frac{y^5}{5\times(16)^2}\right]_0^{16}$$

$$= \frac{1}{2}\left[16\cdot\frac{(16)^2}{2} - \frac{(16)^5}{5\times(16)^2}\right]$$

$$= 614.4 \text{ in}^3$$

$$\int y\,dA = \int_0^{16}\int_{y_1}^{y_2} y\,dy\,dx$$

$$= \int_0^{16}\int_{\frac{x^2}{16}}^{4x^{\frac{1}{2}}} y\,dy\,dx$$

$$= \int_0^{16}\left[\frac{y^2}{2}\right]_{\frac{x^2}{16}}^{4x^{\frac{1}{2}}} dx$$

$$= \int_0^{16}\frac{1}{2}\cdot\left[16x - \frac{x^4}{(16)^2}\right] dx$$

$$= \frac{1}{2}\int_0^{16}\left[16x - \frac{x^4}{(16)^2}\right] dx$$

$$= \frac{1}{2}\left[16\cdot\frac{x^2}{2} - \frac{x^5}{5\cdot(16)^2}\right]_0^{16}$$

$$= \frac{1}{2}\cdot\left[16\cdot\frac{(16)^2}{2} - \frac{(16)^5}{5\cdot(16)^2}\right]$$

$$= 614.4\ in^3$$

$$A = \int dA$$

$$= \int_0^{16}\int_{x_1}^{x_2} dx\,dy$$

$$= \int_0^{16}\int_{\frac{y^2}{16}}^{4y^{\frac{1}{2}}} dx\,dy$$

$$= \int_0^{16}\left[x\right]_{\frac{y^2}{16}}^{4y^{\frac{1}{2}}} dy$$

$$= \int_0^{16}\left[4y^{\frac{1}{2}} - \frac{y^2}{16}\right] dy$$

$$= \left[4\cdot\frac{y^{3/2}}{3/2} - \frac{y^3}{3\times16}\right]_0^{16}$$

$$= 85.33\ in^2$$

or,

$$A = \int dA$$

$$= \int_0^{16}\int_{y_1}^{y_2} dy\,dx$$

$$= \int_0^{16}\int_{\frac{x^2}{16}}^{4x^{\frac{1}{2}}} dy\,dx$$

$$= \int_0^{16}\left[y\right]_{\frac{x^2}{16}}^{4x^{\frac{1}{2}}} dx$$

$$= \int_0^{16}\left[4x^{\frac{1}{2}} - \frac{x^2}{16}\right] dx$$

$$= \left[4\cdot\frac{x^{3/2}}{3/2} - \frac{x^3}{3\times16}\right]_0^{16}$$

$$= 85.33\ in^2$$

$$\therefore \bar{x} = \frac{\int x\,dA}{A} = \frac{614.4}{85.33} = 7.2 \text{ in}$$

$$\bar{y} = \frac{\int y\,dA}{A} = \frac{614.4}{85.33} = 7.2 \text{ in}$$