EWCE 101

ANALYTICAL MECHANICS

3 Credit Hour

Department of Environmental, Water Resources, and Coastal Engineering (EWCE)

Military Institute of Science and Technology

Chapter
Moment of Inertia of Areas

Reference Book: Analytical Mechanics by Virgil Moring Faires

131. Moment of Inertia. The definition of moment of inertia is mathematical:

(28) $$\text{Moment of inertia of an area} = \int a^2\, dA,$$

where a is the distance of an element dA of an area from an axis about which the moment of inertia is desired. Generally,

The axis is in the plane of the area, in which case it is called a *rectangular moment of inertia* I, or

The axis is perpendicular to the area, in which case it is called a *polar moment of inertia* J.

The limits of the integral in (28) are of course such as to include the entire area whose moment of inertia is needed.

If we remember that in finding centroids we used the form $\int a\,dA$, which is sometimes termed the *first* moment of the area, we may call $\int a^2\,dA$ the *second* moment of the area. Observation of the similarities of the procedures of this chapter with those of the preceding chapter will be helpful. The first important *difference* that might be noted is that moment of inertia is *always* positive, involving as it does a^2, which is positive whether a is positive or negative. Observe too that if the linear unit used is inches, the moment

132. Rectangular Moment of Inertia.

Consider the area A, Fig. 384, lying in the xy plane. The differential element dA is at a distance y from the x axis. Hence, the moment of inertia of the area about the x axis is

(a) $$I_x = \int y^2 \, dA;$$

and, similarly, that about the y axis is

(b) $$I_y = \int x^2 \, dA.$$

These are *rectangular* moments of inertia.

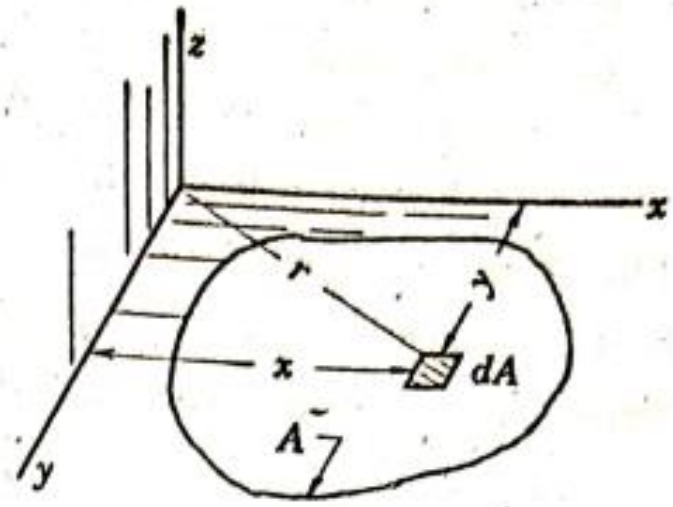


Fig. 384.

133. Polar Moment of Inertia. Since the element dA, in Fig. 384, is a distance r from the z axis and since the z axis is normal to the plane of the area, the *polar* moment of inertia about this particular axis is

(c) $$J = \int r^2\, dA.$$

Now note in Fig. 384 that $r^2 = x^2 + y^2$. Using this relation in (c), we get

$$J = \int r^2\, dA = \int (x^2 + y^2) dA = \int x^2\, dA + \int y^2\, dA,$$

or

(d) $$J = I_y + I_x$$

(d) $$J = I_y + I_x.$$

This expression (d) is an important relation which is often useful. In words, it says that the polar moment of inertia of a plane area about a particular axis perpendicular to the plane of the area is equal to the sum of two rectangular moments of inertia about axes in the plane of the area, provided that all three axes are *mutually perpendicular* and *intersect at a common point*. Since there are an infinite number of axes about which moments may be taken, there are an infinite number of moments of inertia for a particular area.

134. Radius of Gyration. The *radius of gyration* k of an area is also a mathematical conception, defined by the relation

$$k = \left(\frac{I}{A}\right)^{1/2} \quad \text{or} \quad k = \left(\frac{J}{A}\right)^{1/2} \tag{29}$$

a rectangular and a polar radius of gyration, respectively. Since, from (29),

$$I = k^2 A \quad \text{or} \quad J = k^2 A,$$

If Kx refers to the radius of gyration with respect to x axis and would be obtained from

$$k_x = \left(\frac{I_x}{A}\right)^{1/2}$$

Parallel Axis Theorem

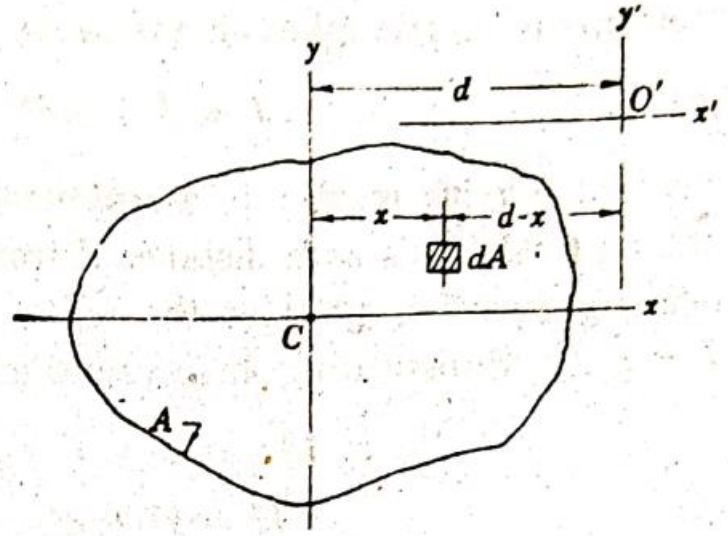


Fig. 387.

137. Transfer Formula—Parallel Axes. A simple and very useful relationship exists between the moment of inertia of an area about a given axis and the moment of inertia of the same area about a *centroidal axis parallel to the given axis.* Let C in Fig. 387 be the centroid of the area A, so that the x and y axes are centroidal axes. The x' and y' axes are any other axes in the plane of the area and parallel to the chosen centroidal axes. The moment of inertia of the differential area dA about the y' axis, for example, is $dI_{y'} = (d - x)^2\,dA$, and, with the proper limits of integration, the moment of inertia of the entire area A about the y' axis

$$I_{y'} = \int (d - x)^2\,dA$$

$$= \int x^2\,dA - 2d\int x\,dA + d^2\int dA,$$

where the terms have been rearranged after squaring $(d - x)$. In this expression, $\int x\,dA$ (which is equal to $A\bar{x}$) is zero, since the y axis passes through the *centroid*; also, $\int dA = A$. Moreover, by comparison with Fig. 387, we note that $\int x^2\,dA$ is equal to the moment of inertia of the area about the *centroidal* y axis, I_y. Hence, we find

$$I_{y'} = I_y + Ad^2,$$

or, in general terms,

(30) $$I = \bar{I} + Ad^2,$$

which is the transfer formula for parallel axes, sometimes called the ***parallel-axes theorem.*** In words, equation (30) says that the moment of inertia I of an area about any axis is equal to the moment of inertia of the area about a *parallel centroidal axis* $\bar{I}$ plus the product of the area A and the square of the distance between the axes d^2. Note particularly that

One of the axes involved in equation (30) *must be a centroidal* axis.

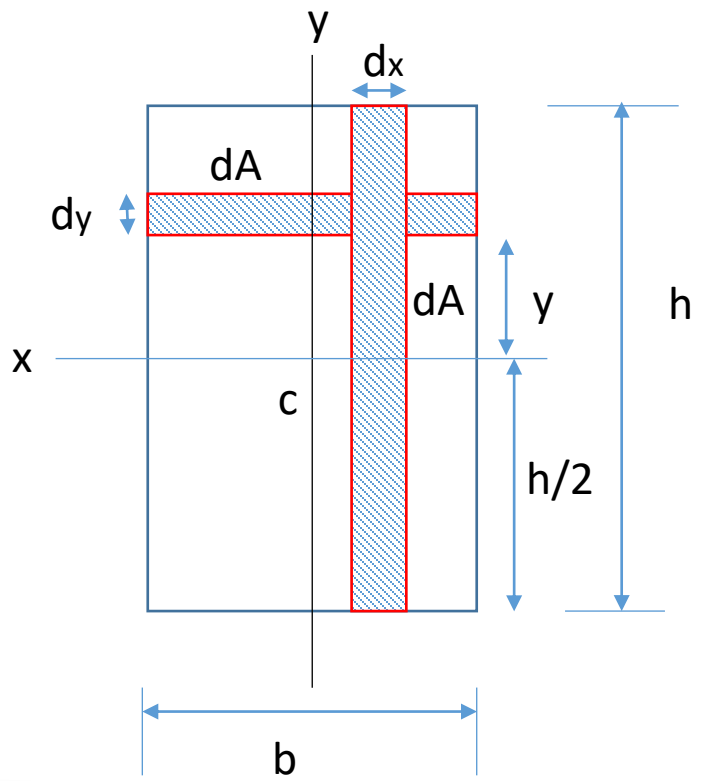
The linear units are usually inches, although of course any other linear units, such as feet, may be used.

By a similar procedure, we may show that the transfer formula for polar moments of inertia takes an analogous form

$$J = \bar{J} + Ad^2, \tag{e}$$

where $\bar{J}$ is with respect to a *centroidal* axis and J is with respect to some other parallel axis at a distance d from the centroidal axis.

Q: Derive the expression for the moment of inertia of a rectangular area about an axis through the centroid and parallel to a side. Also find the radius of gyration with respect to this axis and the polar moment of inertia with respect to an axis through the centroid.



In the rectangle shown in figure, choose a differential element of area dA such that all points in the element are the same distance from the x axis. Since all points of this element whose area is b.dy, have the same moment arm y about the x axis.



$$I_x = \int y^2 dA$$

$$= b\int_{-\frac{h}{2}}^{\frac{h}{2}} y^2 dy$$

$$= 2b\int_0^{\frac{h}{2}} y^2 dy$$

$$= 2b \cdot \left[\frac{y^3}{3}\right]_0^{\frac{h}{2}}$$

$$= 2b \cdot \left[\frac{h^3}{2^3 \times 3} - 0\right]$$

$$= 2b \times \frac{h^3}{24}$$

$$= \frac{bh^3}{12}$$

Radius of gyration, $k_x = \sqrt{\frac{I_x}{A}} = \sqrt{\frac{bh^3}{12 \cdot bh}} = \frac{h}{\sqrt{12}}$

The polar moment of inertia about the centroidal axis

$$J_o = I_x + I_y = \frac{bh^3}{12} + \frac{hb^3}{12} = \frac{hb}{12}(h^2+b^2)$$

$$= \frac{A}{12}(h^2+b^2)$$

We may find the value of I_y in the same way by integration (see next slide), using the element of area of width dx, or by analogy with the result for I_x, we may write,

$$I_y = \frac{hb^3}{12}$$

$$I_y = \int x^2 \, dA$$

$$= h \int_{-b/2}^{b/2} x^2 \, dx \qquad [dA = h \cdot dx]$$

$$= 2h \int_{0}^{b/2} x^2 \, dx$$

$$= 2h \cdot \left[\frac{x^3}{3} \right]_0^{b/2}$$

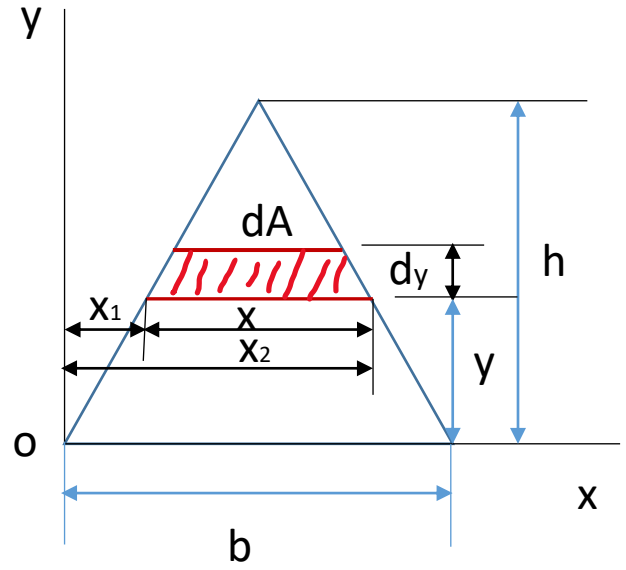
$$= 2h \cdot \left[\frac{b^3}{2^3 \times 3} - 0 \right]$$

$$= 2h \times \frac{b^3}{24}$$

$$= \frac{hb^3}{12}$$



Q: Find the moment of inertia of a triangular area about a base.



Choose an element dA, all parts of which are the same distance from the x axis, The length of the element is $x_2 - x_1 = x$; so that $dA = x.dy$; By similar triangle,

$$\frac{x}{b} = \frac{h-y}{h}$$

$$\Rightarrow x = b\left(\frac{h-y}{h}\right)$$

$$\text{Now, } dA = x.dy = b\left(\frac{h-y}{h}\right).dy$$

$$I_x = \int y^2 dA = \frac{b}{h}\int_0^h y^2(h-y)\,dy$$

$$= \frac{b}{h}\left[\int_0^h y^2 h\,dy - \int_0^h y^3 dy\right]$$

$$= \frac{b}{h}\left[\left[\frac{hy^3}{3}\right]_0^h - \left[\frac{y^4}{4}\right]_0^h\right]$$

$$= \frac{b}{h}\left[\frac{h\cdot h^3}{3} - \frac{h^4}{4}\right]$$

$$= \frac{b}{h}\left[\frac{h^4}{3} - \frac{h^4}{4}\right]$$

$$= \frac{b}{h} \times \frac{h^4}{12} = \frac{bh^3}{12} = \frac{1}{6}\left(\frac{1}{2}bh\right)h^2$$

$$= \frac{1}{6}Ah^2$$

Radius of gyration,

$$k_x = \sqrt{\frac{I_x}{A}} = \sqrt{\frac{\frac{Ah^2}{6}}{A}} = \sqrt{\left(\frac{Ah^2}{6} \times \frac{1}{A}\right)}$$

$$= \frac{h}{\sqrt{6}}$$

Q: Using the result of the previous example, find the moment of inertia of a triangular area about a line through a vertex and parallel to the opposite base.

$$I_x = \bar{I} + Ad^2$$

$$\Rightarrow I_x = I_c + Ad^2$$

$$\Rightarrow I_c = I_x - Ad^2 = \frac{bh^3}{12} - \left\{\left(\frac{1}{2}bh\right)\left(\frac{h}{3}\right)^2\right\}$$

$$= \frac{bh^3}{12} - \left(\frac{bh}{2} \times \frac{h^2}{9}\right)$$

$$= \frac{bh^3}{12} - \frac{bh^3}{18}$$

$$= \frac{bh^3}{36}$$

$$I_x' = \bar{I} + Ad^2 = \frac{bh^3}{36} + \left\{\left(\frac{1}{2}bh\right)\left(\frac{2h}{3}\right)^2\right\}$$

$$= \frac{bh^3}{36} + \left(\frac{1}{2}bh \times \frac{4h^2}{9}\right)$$

$$= \frac{bh^3}{36} + \left(\frac{2bh^3}{9}\right)$$

$$= \frac{bh^3}{4}$$

$\bar{I} = I_c$ = moment of inertia about a line through centroid

I_x = moment of inertia about a line through base

I_x' = moment of inertia about a line through vertex.

Concept: Find Ic from Ix and then find Ix' from Ic.