EWCE 101

ANALYTICAL MECHANICS

3 Credit Hour

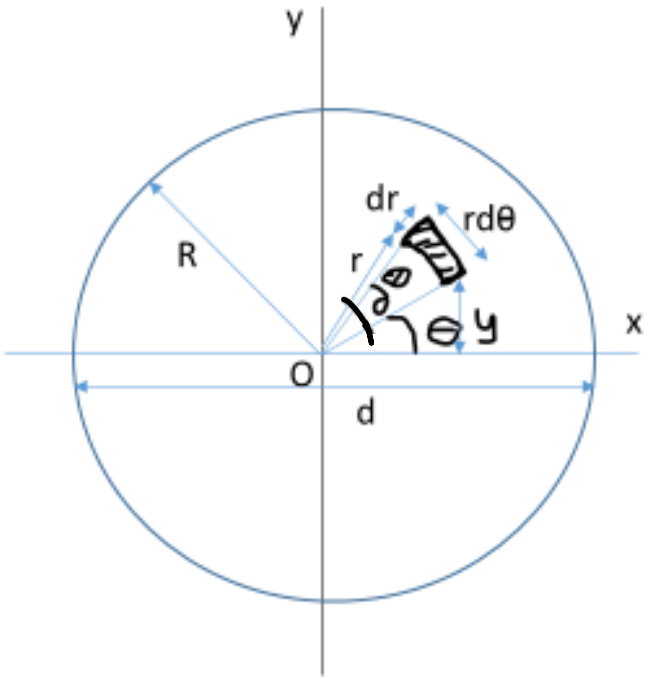
Department of Environmental, Water Resources, and Coastal Engineering (EWCE)

Military Institute of Science and Technology

Chapter
Moment of Inertia of Areas

Reference Book: Analytical Mechanics by Virgil Moring Faires

Q: Find the moment of inertia I_x of a circular area about a diameter



Area of element, $dA = r\,d\theta\,.\,dr$



Moment of inertia of element about X-X axis,

$$dI_{xx} = y^2 dA$$

$$= (r\sin\theta)^2 . r\,dr\,d\theta$$



$$\sin\theta = \frac{y}{r}$$

$$\therefore y = r\sin\theta$$

Here, r is the distance of the small element from the centre O
R is the radius of the circle
d is the diameter of the circle
y is the perpendicular distance of the small element from x axis

$$I_{xx} = \int dI_{xx} = \int_0^R \int_0^{2\pi} (r\sin\theta)^2 \cdot r\,dr\,d\theta$$

$$= \left(\int_0^{2\pi} \sin^2\theta\,d\theta\right)\left(\int_0^R r^3\,dr\right)$$

$$= \int_0^{2\pi} \left[\frac{1-\cos 2\theta}{2}\right] d\theta \cdot \left[\frac{r^4}{4}\right]_0^R$$

$$= \frac{R^4}{4} \times \frac{1}{2}\left[\int_0^{2\pi} d\theta - \int_0^{2\pi} \cos 2\theta\,d\theta\right]$$

$$= \frac{R^4}{8}\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{2\pi}$$

$$= \frac{R^4}{8}\left[2\pi - \frac{\sin 2(2\pi)}{2} - 0\right]$$

$$= \frac{2\pi R^4}{8} = \frac{\pi R^4}{4}$$

$\cos 2\theta = 2\cos^2\theta + 1$

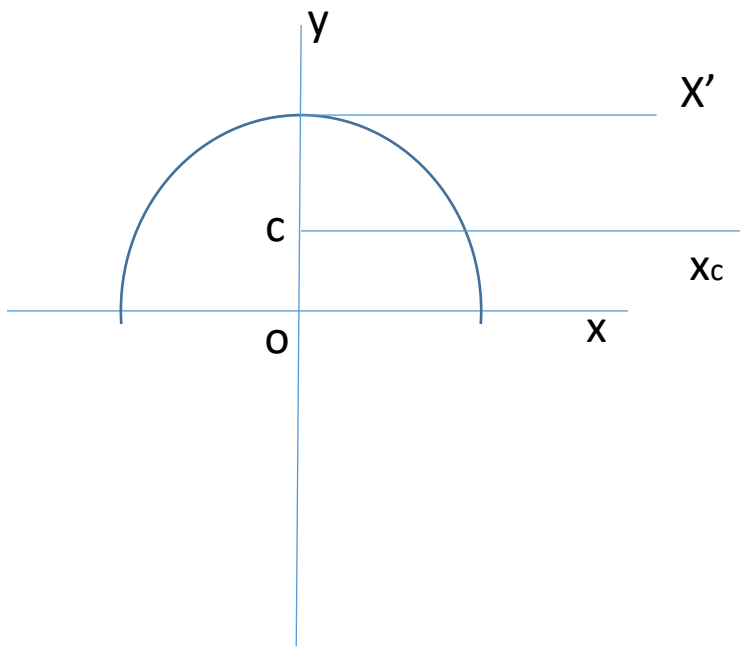
$= 1 - 2\sin^2\theta$

Now, $R = \frac{d}{2}$

$$I_{xx} = \frac{\pi}{4}\left(\frac{d}{2}\right)^4 = \frac{\pi d^4}{64}$$

$$I_{xx} = I_{yy} = \frac{\pi}{64} d^4$$

Q790: For the semicircle of radius r in figure, find the moment of inertia and radius of gyration with respect to (a) the centroidal axis x_c, (b) The x' axis and the (c) y axis. Use I for a semicircle with respect to a diameter as a starting point. (d) Also find Jo.



For circular area, $I_x = \frac{\pi D^4}{64}$

so, for semicircle, $I_x = \frac{\pi D^4}{64 \times 2}$

$= \frac{\pi \times (2r)^4}{128}$

$= 0.4r^4$

$A = \frac{\pi r^2}{2} = \frac{3.1416 r^2}{2} = 1.57 r^2$

$I_x = I_{xc} + Ad^2$

$\Rightarrow I_{xc} = I_x - Ad^2$

$= 0.4r^4 - \{1.57r^2 \times (0.42r)^2\}$

$= 0.123r^4$

Here, $\bar{y} = \frac{2r \sin\beta}{3\beta}$

$= \frac{2r \sin(\frac{\pi}{2})}{3 \times \frac{\pi}{2}}$

$= 0.42r$

$$K_{x_c} = \sqrt{\frac{I_{x_c}}{A}} = \sqrt{\frac{0.123r^4}{1.57r^2}}$$

$$= 0.279r$$

$$I_x' = I_{x_c} + Ad^2$$

$$= 0.123r^4 + \{1.57r^2 \times (r - 0.42r)^2\}$$

$$= 0.65r^4$$

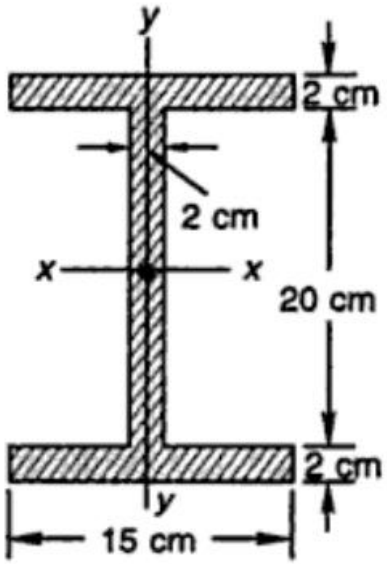
$$K_x' = \sqrt{\frac{I_x'}{A}} = \sqrt{\frac{0.65r^4}{1.57r^2}} = 0.64r$$

$$I_x = I_y = 0.4r^4$$

$$J_0 = I_x + I_Y = 0.4r^4 + 0.4r^4$$

$$= 0.8r^4$$

Q: Determine the moments of inertia with respect to the centroidal axis(X-X) of the wide flange beam section shown In figure.



By subdividing it into three rectangles; 15 cm × 2 cm at top, 15 cm × 2 cm at the bottom and 20 cm × 2 cm in the middle.

$$I_{xx} = 2 \times \left[\frac{15 \times 2^3}{12} + (15 \times 2) \times 11^2 \right]$$

(for the top and bottom rectangles)

$$= 7280 \text{ cm}^4$$

$$I_{xx} = \frac{2 \times 20^3}{12} = 1333 \text{ cm}^4$$

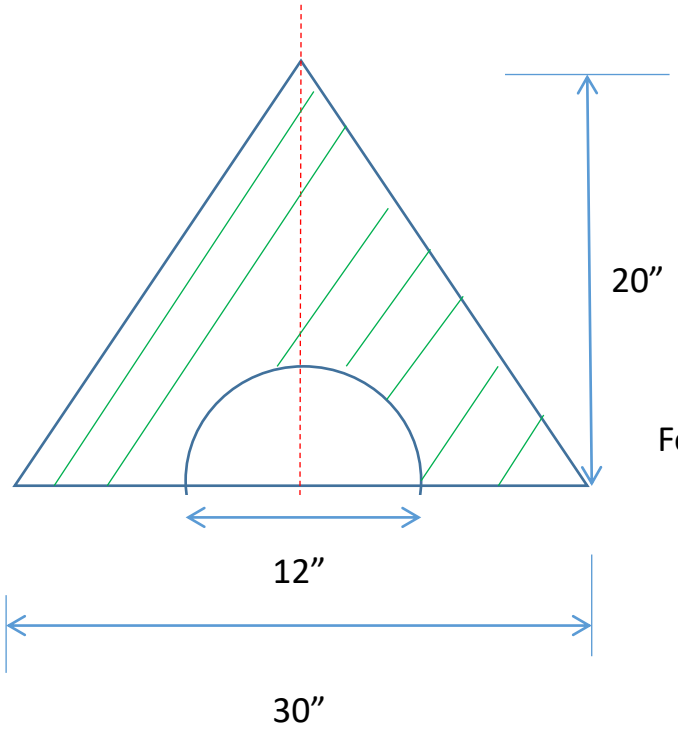
(for the middle rectangle)

I_{xx} of the composite section = 7280 + 1333 = 8613 cm^4

Try by yourself:

Determine the moment of inertia with respect to the centroidal (Y-Y) axis for the above figure.

Q: A semicircular area is removed from a triangle area as shown in Figure. Find the moment of inertia of the net composite area about the x axis.



$$I_{x_1} = \frac{bh^3}{12} = \frac{30 \times (20)^3}{12}$$

$$= 20{,}000 \text{ in}^4$$

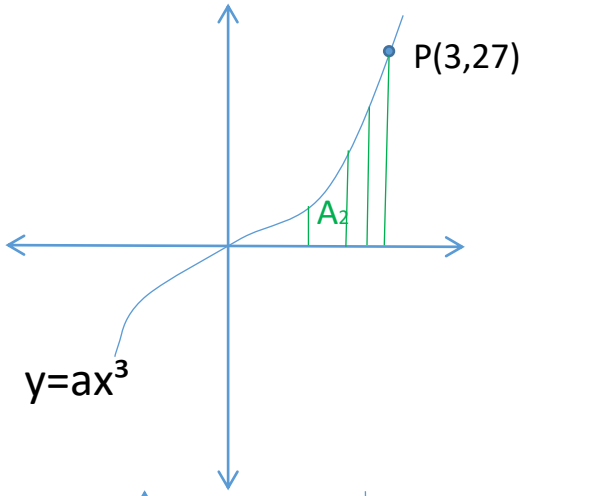
$$I_{x_2} = \frac{\pi D^4}{64 \times 2} = \frac{\pi \times (12)^4}{128}$$

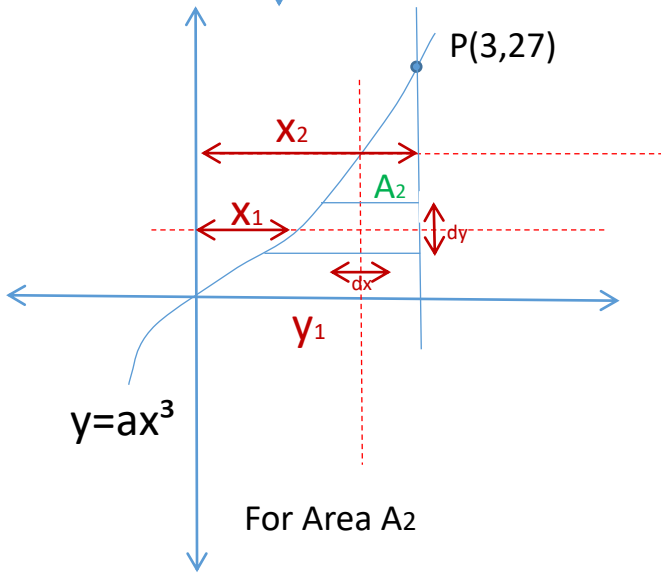
For half circle

$$= 509 \text{ in}^4$$

$$\therefore I_x = I_{x_1} - I_{x_2} = 20{,}000 - 509 = 19{,}491 \text{ in}^4$$

Q(797): If the coordinates of P in Fig are (3 in,27 in), find the radius of gyration of area A_2 with respect to the (a) x axis (b) the centroidal x axis and (c) the line y=5. The centroid of area A_2 is given at (2.4 in, 7.71 in)





For Area A_2

(a) Here, centroid (2·4, 7·71)

$y = ax^3$ | P co-ordinate is (3, 27)

$\Rightarrow y = 1 \times x^3$ | $27 = a \times 3^3$

$\therefore y = x^3$ | $\therefore a = 1$

$$A = \int dA = \int_0^{27} \left(3 - y^{\frac{1}{3}}\right) dy$$

$$= \left[3y - \frac{y^{\frac{4}{3}}}{\frac{4}{3}}\right]_0^{27}$$

$$= 20.25 \text{ in}^2$$

$$I_x = \int y^2 dA = \int_0^{27} y^2 (x_2 - x_1)\, dy$$

$$= \int_0^{27} y^2 \left(3 - y^{\frac{1}{3}}\right) dy$$

$$= \left[3 \cdot \frac{y^3}{3} - \frac{y^{\frac{10}{3}}}{\frac{10}{3}}\right]_0^{27}$$

$$= 1968.3 \text{ in}^4$$

$$\therefore k_x = \sqrt{\frac{I_x}{A}} = \sqrt{\frac{1968.3}{20.25}} = 9.86 \text{ in}$$

(b) $I_x = \bar{I}_x + Ad^2$

$\Rightarrow \bar{I}_x = I_x - Ad^2$

$= 1968.3 - \{20.25 \times (7.71)^2\}$

$= 764.55 \text{ in}^4$

$\therefore \bar{K}_x = \sqrt{\frac{\bar{I}_x}{A}} = \sqrt{\frac{764.55}{20.25}} = 6.14 \text{ in}$

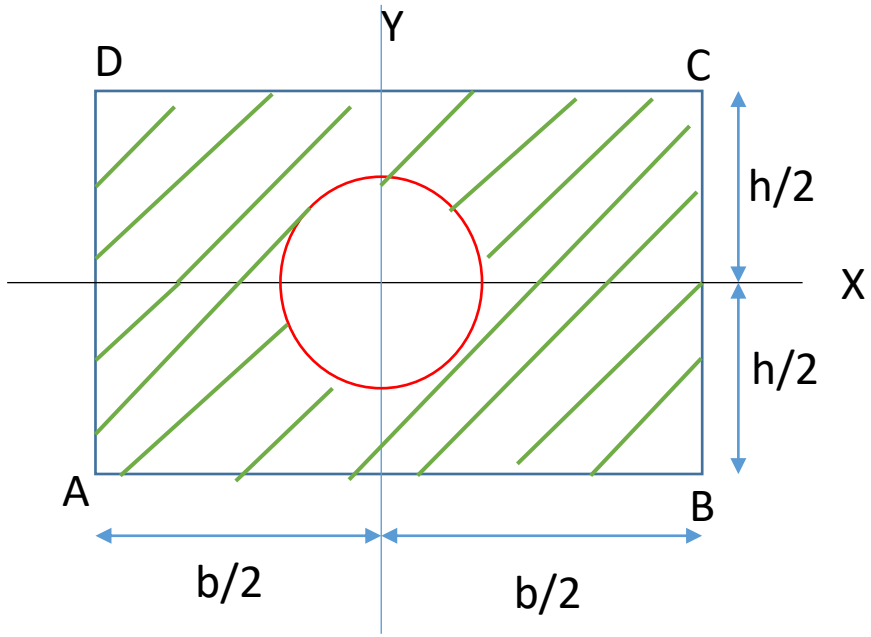
(c) $I_x' = \bar{I}_x + Ad^2$

$= 764.55 + \{20.25 \times (7.71 - 5)^2\}$

$= 913.26 \text{ in}^4$

$\therefore K_x' = \sqrt{\frac{I_x'}{A}} = \sqrt{\frac{913.26}{20.25}} = 6.71 \text{ in}$

Q 806: In the following figure, let b=10 in, h =6 in and the diameter of the hole D=2 in, find the centroidal moment of Inertia and radius of gyration of the shaded area about the axis AB.



Here,

$$I_{AB_1} = \frac{10 \times 6^3}{12} + (60 \times 3^2)$$

$$= 720 \text{ in}^4$$

$$I_{AB_2} = \frac{\pi \times 2^4}{64} + (3.14 \times 3^2)$$

$$= 29.059 \text{ in}^4$$

$$I_{AB} = I_{AB_1} - I_{AB_2}$$

$$= 720 - 29.059$$

$$= 690.94 \text{ in}^4$$

$$k_{AB} = \sqrt{\frac{I_{AB}}{A}} = \sqrt{\frac{690.940}{60 - 3.14}} = 3.49 \text{ in}$$

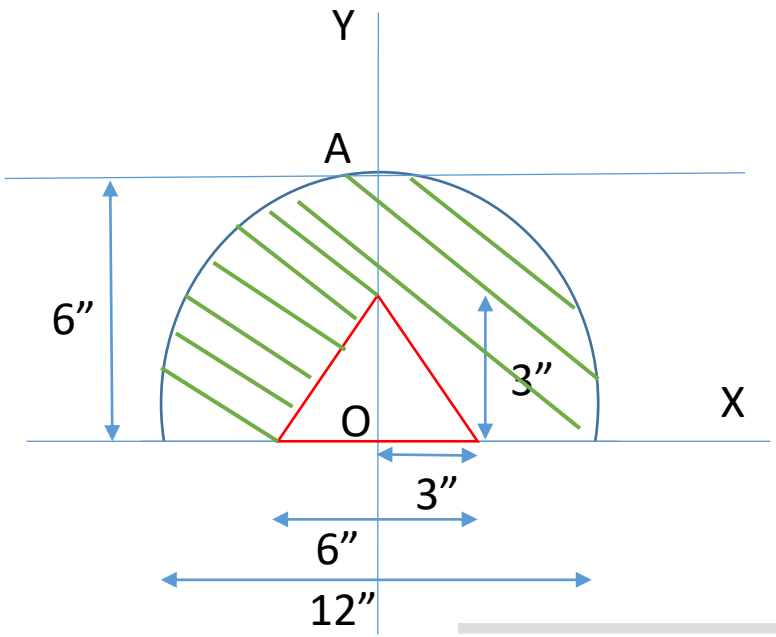
$A_1 = (6 \times 10)$

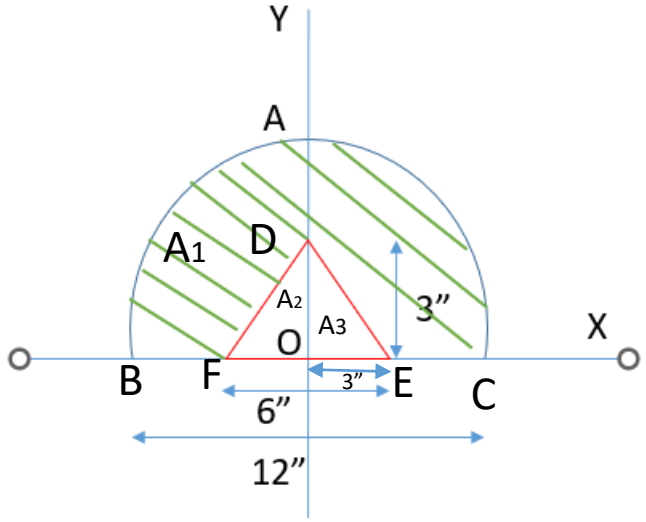
$= 60 \text{ in}^2$

$A_2 = \pi \times 1^2$

$= 3.14 \text{ in}^2$

Q 813: Determine Ix and Iy for the shaded area shown in fig, also find the polar moment of inertia and the polar radius of gyration of this area about an axis through point A.





Here, $A_1 = \frac{\pi \times 6^2}{2} = 18\pi \text{ in}^2$

$A_2 = A_3 = \left(\frac{1}{2} \times 3 \times 3\right) = 4.5 \text{ in}^2$

$\therefore A = (18\pi - 4.5 - 4.5) \text{ in}^2$

$= (18\pi - 9) \text{ in}^2$ [shaded area]

$$y_1 = \frac{2r\sin\beta}{3\beta} = \frac{2\times 6\times \sin\left(\frac{\pi}{2}\right)}{3\times\frac{\pi}{2}} = 2.596 \text{ in}$$

$$y_2 = y_3 = \frac{3}{3} = 1 \text{ in}$$

$$Ix_1 = \frac{\pi\times(12)^4}{64\times 2} = 162\pi \text{ in}^4$$

$$Ix_2 = Ix_3 = \frac{3\times 3^3}{12} = 6.75 \text{ in}^4$$

$$\therefore Ix = Ix_1 - Ix_2 - Ix_3$$

$$= 495.49 \text{ in}^4$$

$$I_{y_1} = \frac{\pi (12)^4}{64 \times 2} = 162\pi \text{ in}^4$$

$$I_{y_2} = I_{y_3} = \frac{3 \times 3^3}{12} = 6.75 \text{ in}^4$$

$$\therefore I_y = I_{y_1} - I_{y_2} - I_{y_3} = 495.44 \text{ in}^4$$

$$\therefore J = I_x + I_y = 990.88 \text{ in}^4$$

$$A\bar{y} = A_1 y_1 - A_2 y_2 - A_3 y_3$$

$$\Rightarrow \bar{y} = \frac{(18\pi \times 2.546) - (9.5 \times 1) - (9.5 \times 1)}{18\pi - 9}$$

$$= 2.83 \text{ in}$$

Formula

Polar Moment of inertia

$J = I_y + I_x$

$$J = \bar{J}_c + Ad^2$$

$$\Rightarrow \bar{J}_c = J - Ad^2$$

$$= 990.88 - \{(18\pi - 9) \times (2.83)^2\}$$

$$= 610.067 \text{ in}^4$$

$$J_A = \bar{J}_c + Ad^2$$

$$= 610.067 + \{(18\pi - 9) \times (6 - 2.83)^2\}$$

$$= 1087.87 \text{ in}^4$$

$$K_A = \sqrt{\frac{J_A}{A}} = \sqrt{\frac{1087.87}{18\pi - 9}} = 4.78 \text{ in}$$

Formula

polar radius of gyration,

$$J = k^2 A,$$