EWCE 101

ANALYTICAL MECHANICS

3 Credit Hour

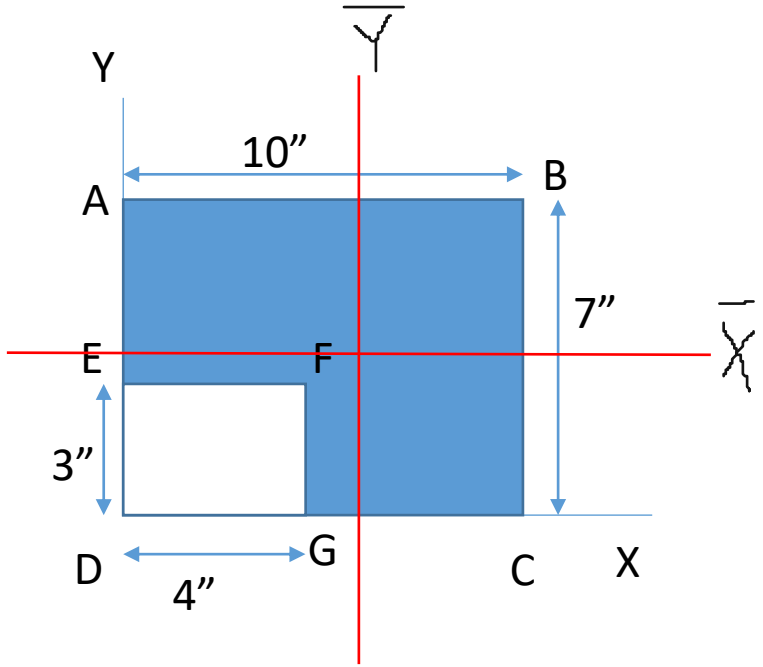
Department of Environmental, Water Resources, and Coastal Engineering (EWCE)

Military Institute of Science and Technology

Chapter
Moment of Inertia of Areas

Reference Book: Analytical Mechanics by Virgil Moring Faires

Q:For the shaded area in Fig, find $\bar{I}_x$ and $\bar{I}_y$



Here, ABCD is Area A1

EFGD is Area A2

Shaded Area is ABCGFE

$A_1 = (10 \times 7) = 70\ in^2$

$A_2 = (4 \times 3) = 12\ in^2$

$x_1 = 5\ in$

$x_2 = 2\ in$

$y_1 = 3.5\ in$

$y_2 = 1.5\ in$

$$\therefore \bar{x} = \frac{A_1 x_1 - A_2 x_2}{A_1 - A_2} = \frac{(70 \times 5) - (12 \times 2)}{70 - 12}$$

$$= 5.62\ in$$

$$\bar{y} = \frac{(70 \times 3.5) - (12 \times 1.5)}{70 - 12} = 3.9\ in$$

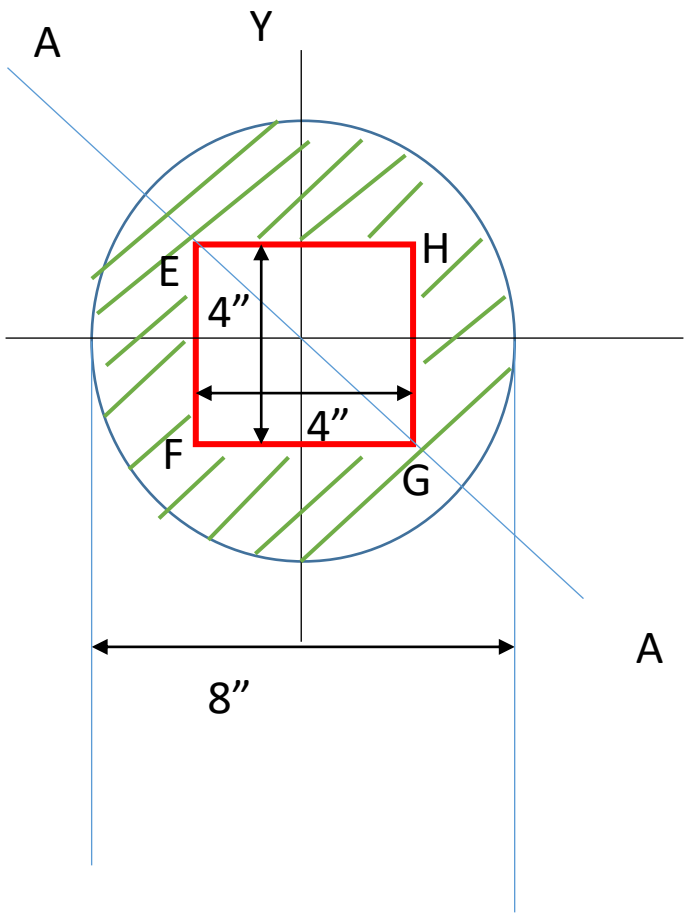
$$\bar{I}_x = \left[\frac{10 \times 7^3}{12} + \{70 \times (3.9 - 3.5)^2\}\right] - \left[\frac{4 \times 3^3}{12} + \{12 \times (3.9 - 1.5)^2\}\right]$$

$$= 218.91\ in^4$$

$$\bar{I}_y = \left[\frac{7 \times 10^3}{12} + \{70 \times (5.62 - 5)^2\}\right] - \left[\frac{3 \times 4^3}{12} + \{12 \times (5.62 - 2)^2\}\right]$$

$$= 436.99\ in^4$$

Q 808: For the shaded area in Fig, find the moment of inertia and radius of gyration about an axis parallel to AA and tangent to the circle.



Consider A’A’ is an axis parallel
To AA and tangent to circle

$A_1 = \pi \times (4)^2 = 16\pi \text{ in}^2$

$A_2 = 4 \times 4 = 16 \text{ in}^2$

$\therefore A = (16\pi - 16) \text{ in}^2$

As EFGH is square, consider two triangles as EGH and EGF, then apply concept of moment of inertia of triangle

$I_{AA_1} = \dfrac{\pi \times 8^4}{64} = 64\pi \text{ in}^4$

$I_{AA_2} = 2 \times \dfrac{\sqrt{32} \times \left(\frac{\sqrt{2} \times 4}{2}\right)^3}{12} = 21.33 \text{ in}^4$

$I_{AA} = I_{AA_1} - I_{AA_2} = (64\pi - 21.33) \text{ in}^4$

$= 179.728 \text{ in}^4$

$I_{A'A'} = I_{AA} + Ad^2 = 179.728 + \{(16\pi - 16) \times 4^2\}$

$= 727.97 \text{ in}^4$

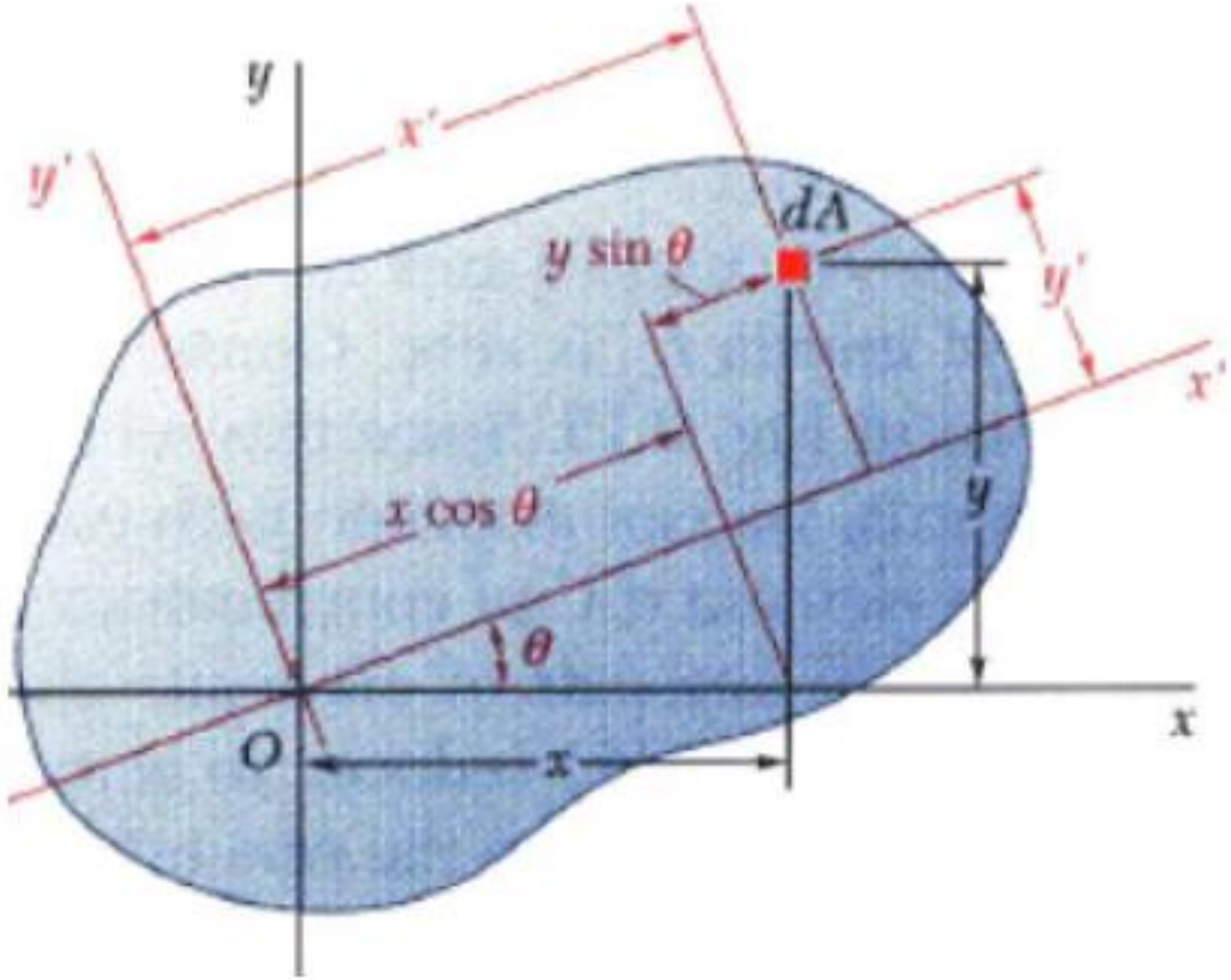
$k_{A'A'} = \left(\dfrac{I_{A'A'}}{A}\right)^{\frac{1}{2}} = \left(\dfrac{727.97}{16\pi - 16}\right)^{\frac{1}{2}} = 4.609 \text{ in}$

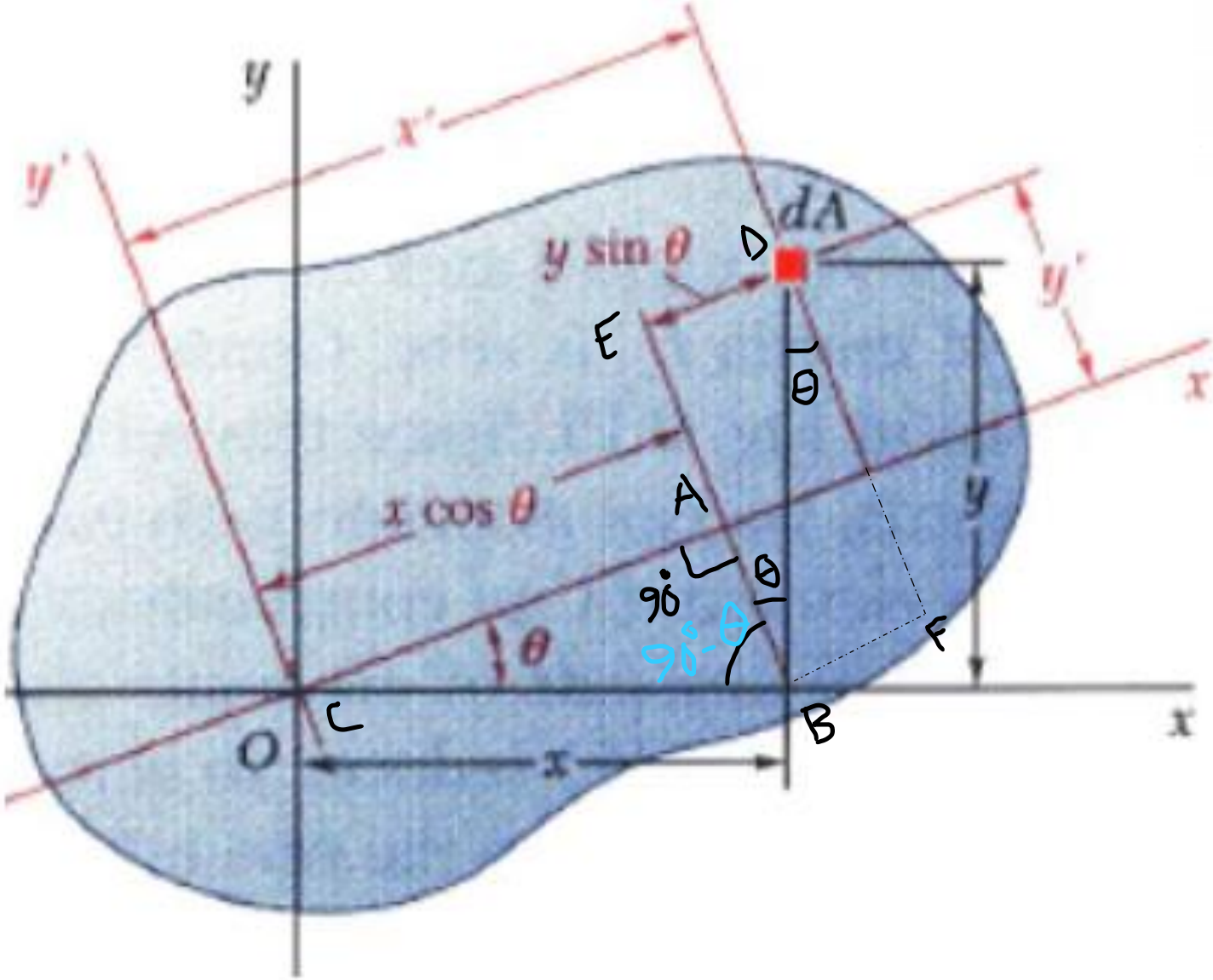
146. Product of Inertia. Another mathematical expression that appears in certain theoretical analyses is $\int xy\,dA$. This integral too has been given a name—*the product of inertia* P_{xy}.

(j) $$P_{xy} = \int xy\,dA.$$

The value of product of inertia can be positive, negative or zero

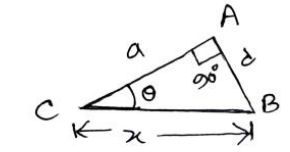
Topic: Moment of Inertia about inclined Axes





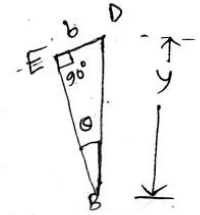
$$x' = x\cos\theta + y\sin\theta$$

$$y' = y\cos\theta - x\sin\theta$$



$\cos\theta = \frac{a}{x}$

$\Rightarrow a = x\cos\theta$

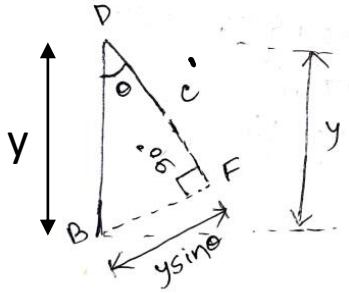


$\sin\theta = \frac{b}{y}$

$\Rightarrow b = y\sin\theta$

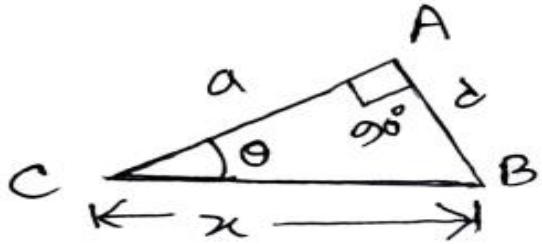
$\sin\theta = \frac{d}{x}$

$\therefore d = x\sin\theta$



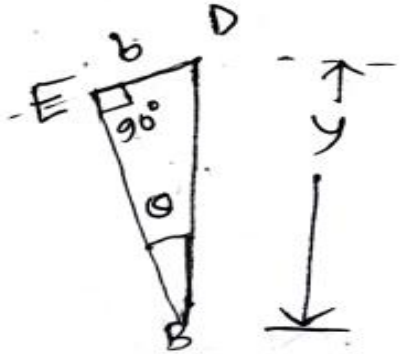
$\cos\theta = \frac{c'}{y}$

$\therefore c' = y\cos\theta$



$$\cos\theta = \frac{a}{x}$$

$$\Rightarrow a = x\cos\theta$$

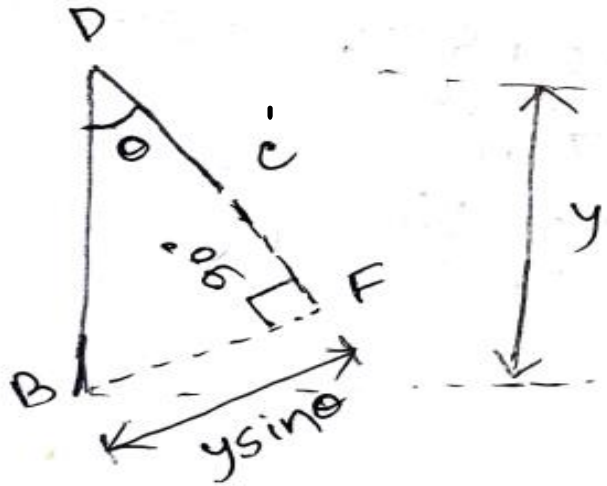


$$\sin\theta = \frac{b}{y}$$

$$\Rightarrow b = y\sin\theta$$

$$\sin\theta = \frac{d}{x}$$

$$\therefore d = x\sin\theta$$



$$\cos\theta = \frac{c'}{y}$$

$$\therefore c' = y\cos\theta$$

$$I_x = \int y^2 dA \qquad I_x' = ?$$

$$I_y = \int x^2 dA \qquad I_y' = ?$$

$$P_{xy} = \int xy \, dA \qquad P_{x'y'} = ?$$

$$I_{x'} = \int y'^2 \, dA$$

$$I_{y'} = \int x'^2 \, dA$$

$$P_{x'y'} = \int x'y' \, dA$$

$$x' = x\cos\theta + y\sin\theta$$

$$y' = y\cos\theta - x\sin\theta$$

$$I_{x'} = \int y'^2 \, dA$$

$$= \int (y\cos\theta - x\sin\theta)^2 \, dA$$

$$= \int (y^2\cos^2\theta - 2xy\sin\theta\cos\theta + x^2\sin^2\theta) \, dA$$

$$= \int (y^2 dA)\cos^2\theta + \int (x^2 dA)\sin^2\theta - \int (xy\, dA)\sin 2\theta$$

$$= I_x\cos^2\theta + I_y\sin^2\theta - P_{xy}\sin 2\theta \quad \text{—— (1)}$$

$$I_y' = \int x'^2 dA$$

$$= \int (x\cos\theta + y\sin\theta)^2 dA$$

$$= \int (x^2\cos^2\theta + y^2\sin^2\theta + 2xy\sin\theta\cos\theta)\, dA$$

$$= \int (x^2 dA)\cos^2\theta + \int (y^2 dA)\sin^2\theta + \int (xy\, dA)\sin 2\theta$$

$$= I_y \cos^2\theta + I_x \sin^2\theta + P_{xy}\sin 2\theta \quad \text{—(II)}$$

(I) + (II),

$$I_x' + I_y' = I_x(\cos^2\theta + \sin^2\theta) + I_y(\sin^2\theta + \cos^2\theta)$$

$$= I_x + I_y$$

Maximum and Minimum Moments of Inertia

$$\frac{dI_x'}{d\theta} = -2I_x \sin\theta\cos\theta - 2P_{xy}\cos 2\theta + 2I_y \cos\theta\sin\theta = 0$$

$$= -I_x \sin 2\theta - 2P_{xy}\cos 2\theta + I_y \sin 2\theta = 0$$

Now,

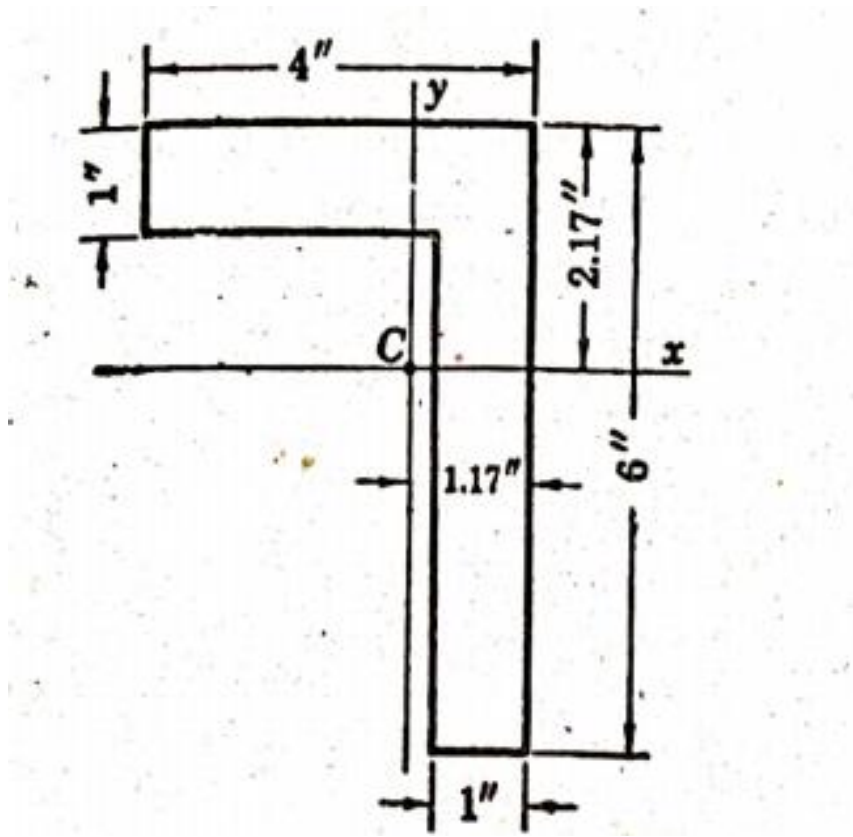
$$-I_x \sin 2\theta - 2P_{xy}\cos 2\theta + I_y \sin 2\theta = 0$$

$$\Rightarrow \sin 2\theta \left(I_y - I_x\right) = 2P_{xy}\cos 2\theta$$

$$\Rightarrow \frac{\sin 2\theta}{\cos 2\theta} = \frac{2P_{xy}}{I_y - I_x}$$

$$\therefore \tan 2\theta = \frac{2P_{xy}}{I_y - I_x}$$

Q822: For the angle section shown in figure, Ix=30.8 $inch^4$,Iy=10.8 $inch^4$, and Pxy=-10 $inch^4$, What is the moment of inertia about an axis through C making an angle of +30 degree with the x axis?

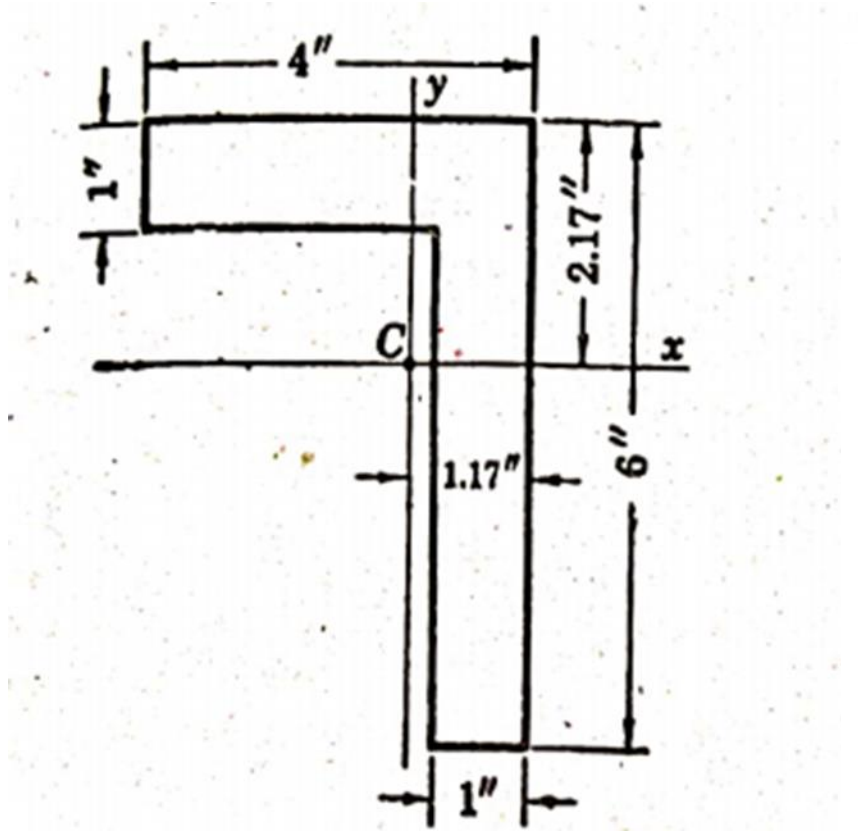


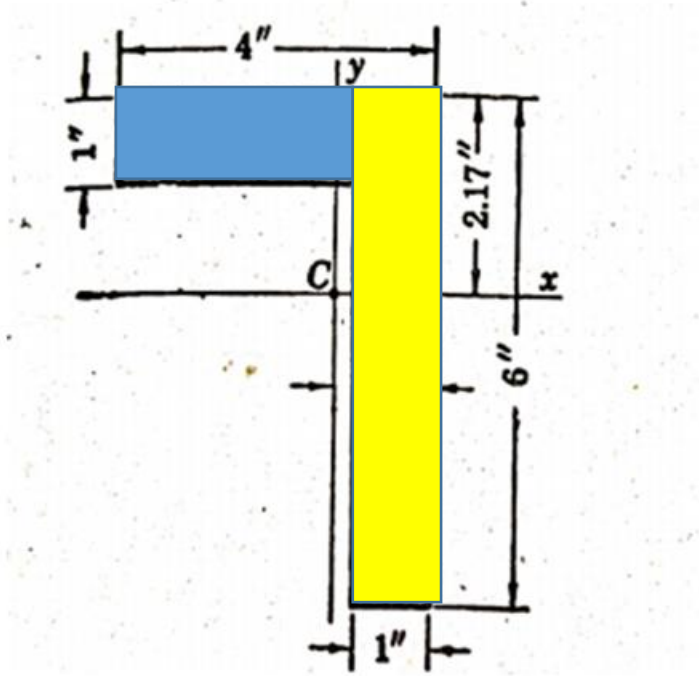
$$I_x' = I_x \cos^2\theta + I_y \sin^2\theta - P_{xy} \sin 2\theta$$

$$= \{30.8 \times (\cos 30)^2\} + \{10.8 \times (\sin 30)^2\} - \{(-10) \times \sin(2 \times 30)\}$$

$$= 34.5 \text{ in}^4$$

Q824: For the angle section shown in figure, Ix=30.8 $inch^4$,Iy=10.8 $inch^4$, and Pxy= - 10 $inch^4$, Determine the max and min centroidal moments of inertia and the minimum radius of gyration.





solution

$$\tan 2\theta = \frac{2P_{xy}}{I_y - I_x}$$

$$\Rightarrow \theta = \frac{1}{2}\tan^{-1}\left(\frac{2\times(-10)}{10.8-30.8}\right)$$

$$= 22.5^\circ$$

$$I_x' = I_x\cos^2\theta + I_y\sin^2\theta - P_{xy}\sin 2\theta$$

$$= \{30.8\times(\cos 22.5)^2\} + \{10.8\times(\sin 22.5)^2\} - \{(-10)\times\sin(2\times 22.5)\}$$

$$= 34.94 \text{ in}^4 \text{ (max)}$$

$$I_x' + I_y' = I_x + I_y$$

$$\Rightarrow I_y' = I_x + I_y - I_x'$$

$$= 30.8 + 10.8 - 34.94$$

$$= 6.66 \text{ in}^4 \text{ (min)}$$

$$K_{min} = \sqrt{\frac{I_y'}{A}}$$

$$= \sqrt{\frac{6.66}{9}}$$

$$= 0.86 \text{ in}$$

$$\text{Area} = \{(4-1)\times 1\} + (1\times 6)$$

$$= 9 \text{ in}^2$$

148. Example. Find the product of inertia of an 8 x 5 x 1-in. Z-section about the x and y axes shown in Fig. 396.

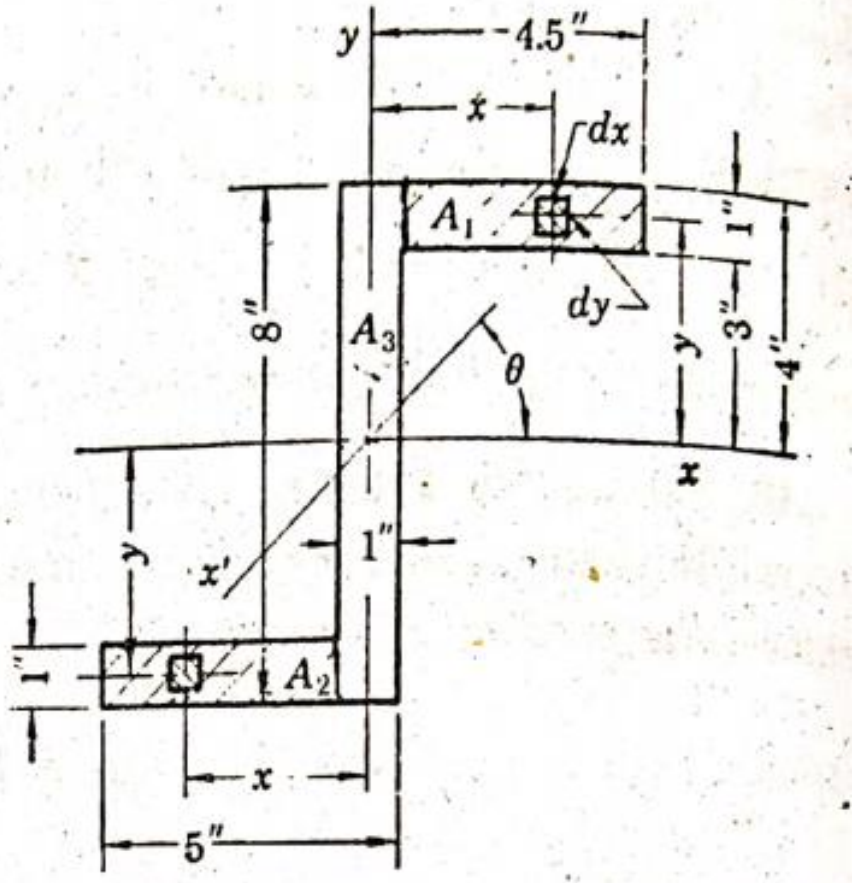


Fig. 396.

SOLUTION. Divide the Z-section into three rectangular areas, A_1, A_2, and A_3 as shown. Using a differential area $dx\,dy$ in the area A_1, we have

$$P_{xy1} = \int_3^4 \int_{0.5}^{4.5} xy\,dx\,dy = \left[\frac{y^2}{2}\right]_3^4 \left[\frac{x^2}{2}\right]_{0.5}^{4.5} = +35 \text{ in.}^4,$$

where the limits of integration for x are from 0.5 to 4.5, and for y, from 3 to 4 (Fig. 396). For the area A_2, we find

$$P_{xy2} = \int_{-4}^{-3} \int_{-4.5}^{-0.5} xy\,dx\,dy = \left[\frac{y^2}{2}\right]_{-4}^{-3} \left[\frac{x^2}{2}\right]_{-4.5}^{-0.5} = (-3.5)(-10) = +35 \text{ in.}^4,$$

where we observe that the limits are taken from left to right and from lower border to upper border, the *positive direction* in each case. For the area A_3,

$$P_{xy3} = \int_{-4}^{+4} \int_{-0.5}^{+0.5} xy\,dx\,dy = 0,$$

a result which is immediately obvious if we note that the x and y axes are axes of symmetry. Hence, the product of inertia of the Z-section with respect to xy is

$$P_{xy} = P_{xy1} + P_{xy2} + P_{xy3} = 35 + 35 = 70 \text{ in.}^4$$

Q: Find the minimum radius of gyration of the Z section of the previous example.

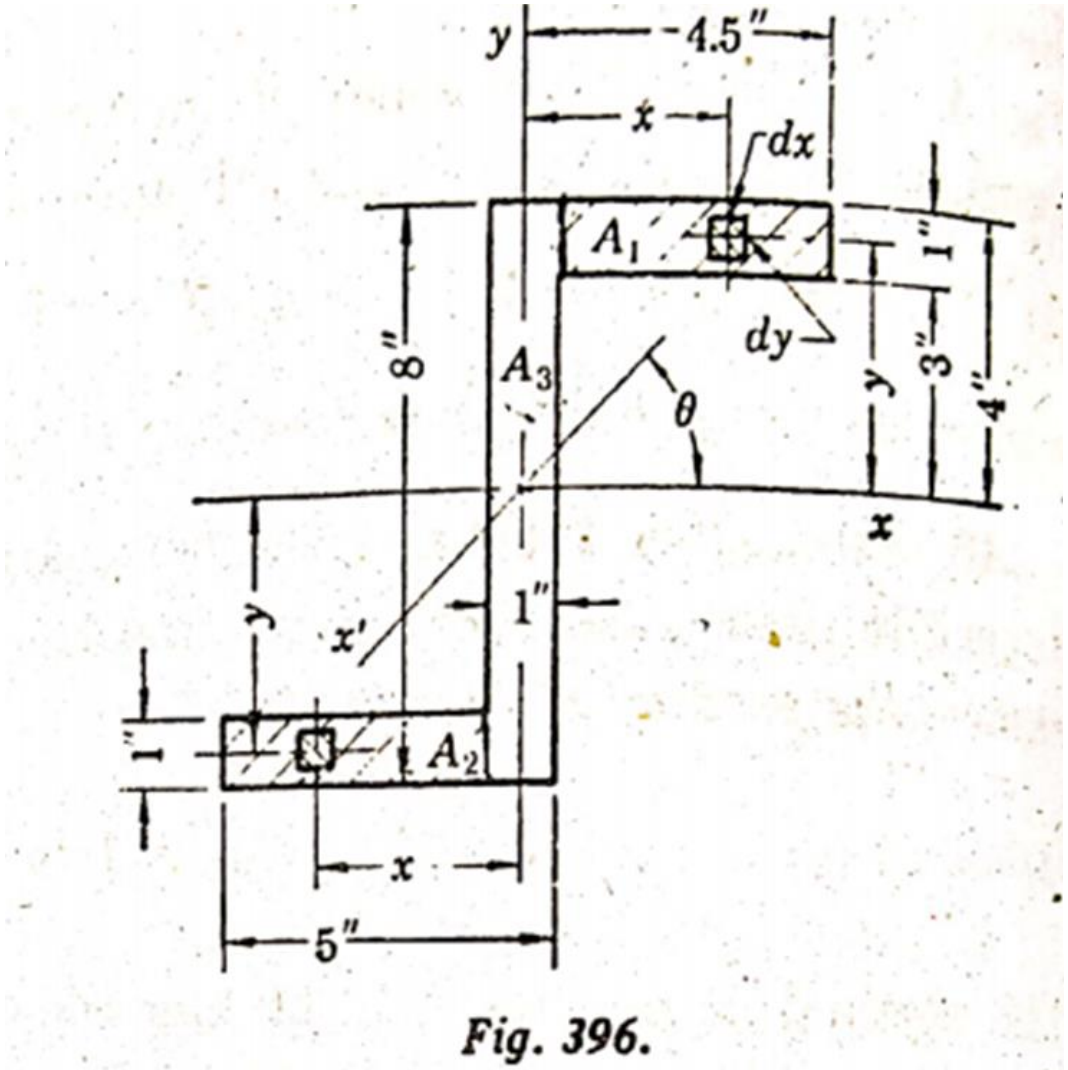
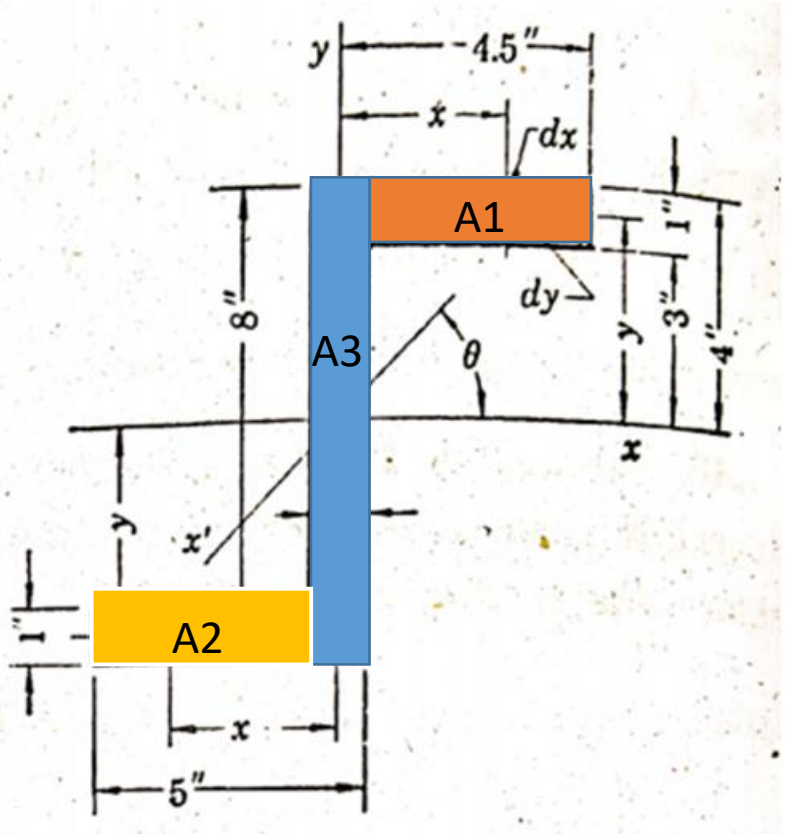
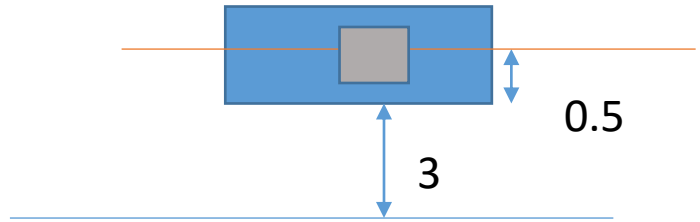


Fig. 396.



3+0.5=3.5



$$Ix_1 = \frac{4\times1^3}{12} + \{4\times1\times(3.5)^2\}$$

$$= 49.33 \text{ in}^4$$

$$Ix_2 = \frac{4\times1^3}{12} + \{4\times1\times(3.5)^2\}$$

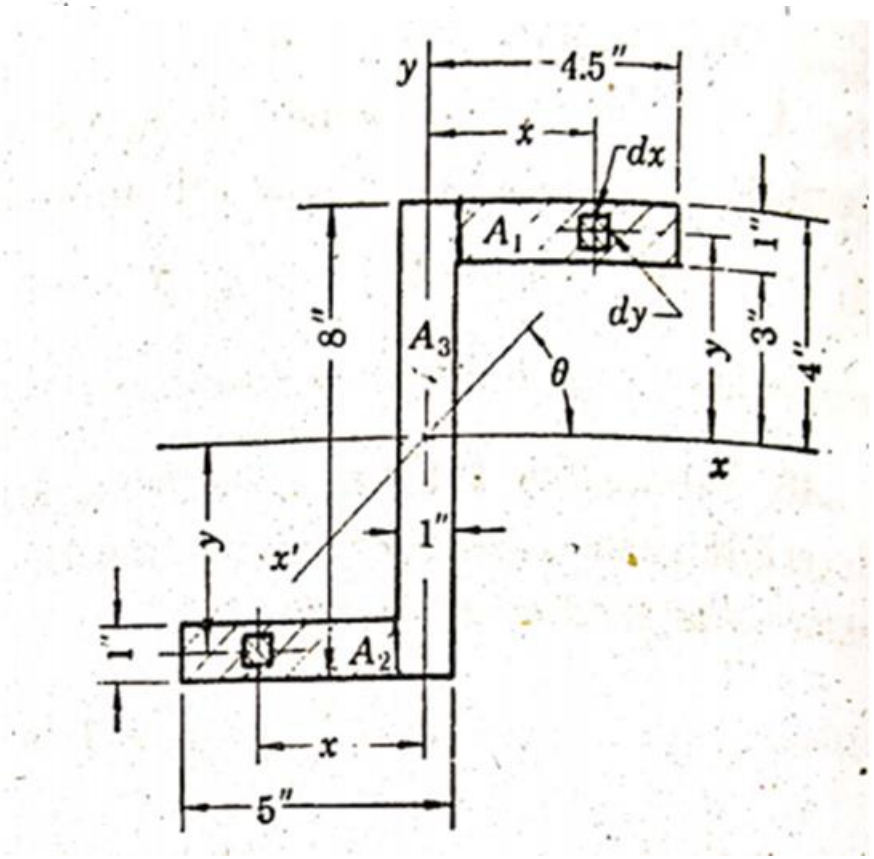
$$= 49.33 \text{ in}^4$$

$$Ix_3 = \frac{1\times8^3}{12} = 42.67 \text{ in}^4$$

$$Ix = Ix_1 + Ix_2 + Ix_3$$

$$= 49.33 + 49.33 + 42.67$$

$$= 141.33 \text{ in}^4$$



0.5 + (4/2) = 2.5

For Area A_1,

$$I_{y_1} = \frac{1 \times 4^3}{12} + \left(4 \times 1 \times (2.5)^2\right)$$

$$= 30.33 \text{ in}^4$$

Area A_2, $I_{y_2} = \frac{1 \times 4^3}{12} + \{4 \times 1 \times (2.5)^2\}$

$$= 30.33 \text{ in}^4$$

Area A_3, $I_{y_3} = \frac{8 \times 1^3}{12} = 0.67 \text{ in}^4$

$$\therefore I_y = I_{y_1} + I_{y_2} + I_{y_3}$$

$$= 61.33 \text{ in}^4$$

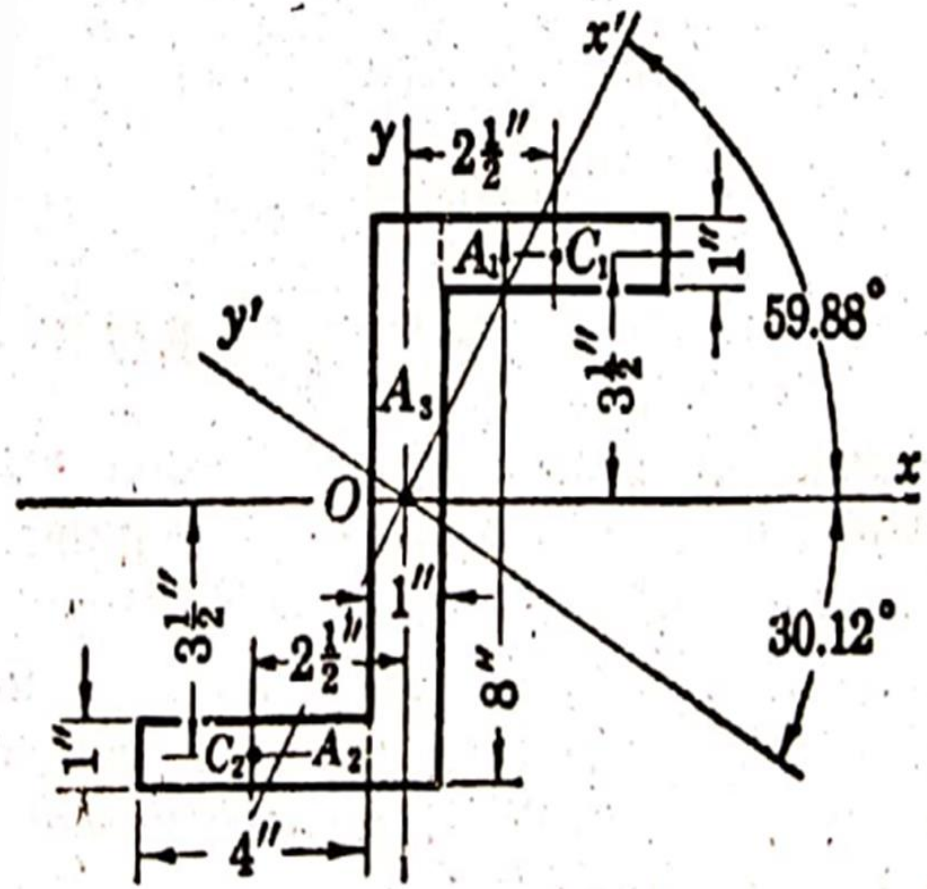


Fig. 398.

$$I_x = I_{x_1} + I_{x_2} + I_{x_3}$$

$$= 49.33 + 49.33 + 42.67$$

$$= 141.33 \text{ in}^4$$

Here, $P_{xy} = 70 \text{ in}^4$

$$\tan 2\theta = \frac{2P_{xy}}{I_y - I_x}$$

$$= \frac{2 \times 70}{61.33 - 141.33}$$

$$= -1.75$$

$$\left[\tan\theta = \tan(180^\circ + \theta)\right]$$

$\Rightarrow 2\theta = -60.25^\circ$ or 119.75°

$\therefore \theta = -30.12^\circ$ or 59.88°

$$I_x' = I_x \cos^2\theta - P_{xy} \sin 2\theta + I_y \sin^2\theta$$

$$= \{141.33 \times (\cos 59.88)^2\} - \{70 \times \sin(2 \times 59.88)\} + \{61.33 \times (\sin 59.88)^2\}$$

$$= 20.7 \text{ in}^4$$

$$I_y' = I_x \sin^2\theta + P_{xy} \sin 2\theta + I_y \cos^2\theta$$

$$= \{141.33 \times (\sin 59.88)^2\} + \{70 \times \sin(2 \times 59.88)\} + \{61.33 \times (\cos 59.88)^2\}$$

$$= 181.95 \text{ in}^4$$

or, $I_x + I_y = I_x' + I_y'$

$\Rightarrow 141.33 + 61.33 = 20.7 + I_y'$

$\therefore I_y' \approx 181.95 \text{ in}^4$

So, $I_x' < I_y'$

$\therefore k_{min} = \sqrt{\frac{I_{min}}{A}}$

$= \sqrt{\frac{20.7}{16}}$

$= 1.14$ in

$A = A_1 + A_2 + A_3$

$= \{(4.5 - 0.5) \times 1\}$

$+ \{(4.5 - 0.5) \times 1\}$

$+ (8 \times 1)$

$= 16$ in^2