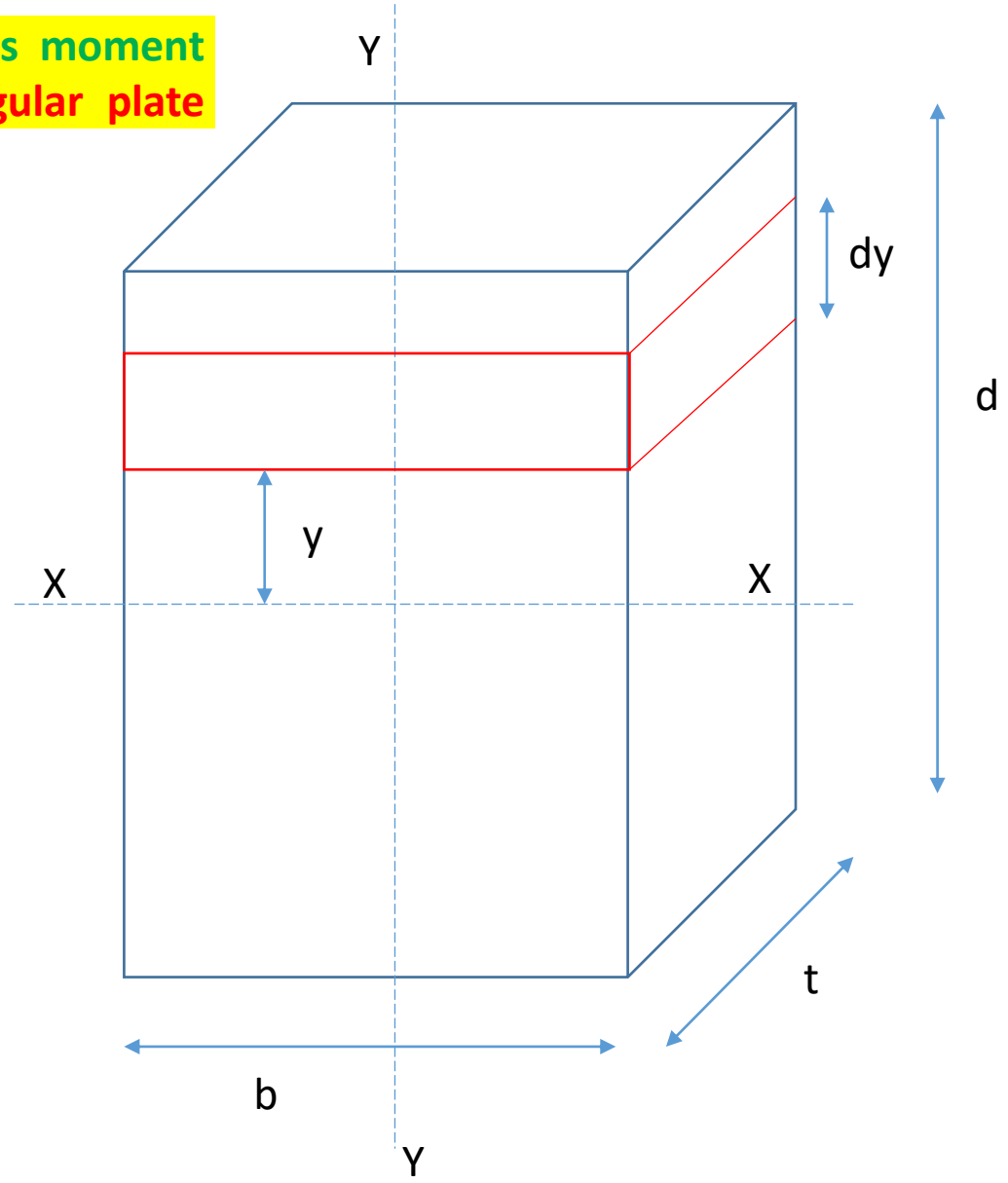
# Moment of Inertia of Masses

**Q: Determine the Mass moment of Inertia of a rectangular plate about x and y axis.**




Mass of the Rectangular body

$M$ = density * volume

$= \rho * b.d.t$

Mass of elementary plate

$dm = \rho * b.t.dy$

Mass moment of Inertia for elementary strip

$I_x = y^r dm$

For whole body,

$$I_x = \int_{-d/2}^{d/2} y^2 \, dm$$

$$= \int_{-d/2}^{d/2} y^2 \cdot (\rho * b \cdot t \, dy)$$

$$= \rho b t \int_{-d/2}^{d/2} y^2 \, dy$$

$$= \rho b t \left[\frac{y^3}{3}\right]_{-d/2}^{d/2}$$

$$= \rho b t * \frac{1}{3}\left[\frac{d^3}{8} + \frac{d^3}{8}\right]$$

$$= \rho b t \cdot \frac{d^3}{12}$$

$$= \frac{\rho b d t \cdot d^2}{12} = \frac{m d^2}{12}$$

similarly we can prove, $I_y = \frac{m b^2}{12}$

**Q: Determine the Mass moment of Inertia of disc of an axis passing through its centre and perpendicular to its plane.**

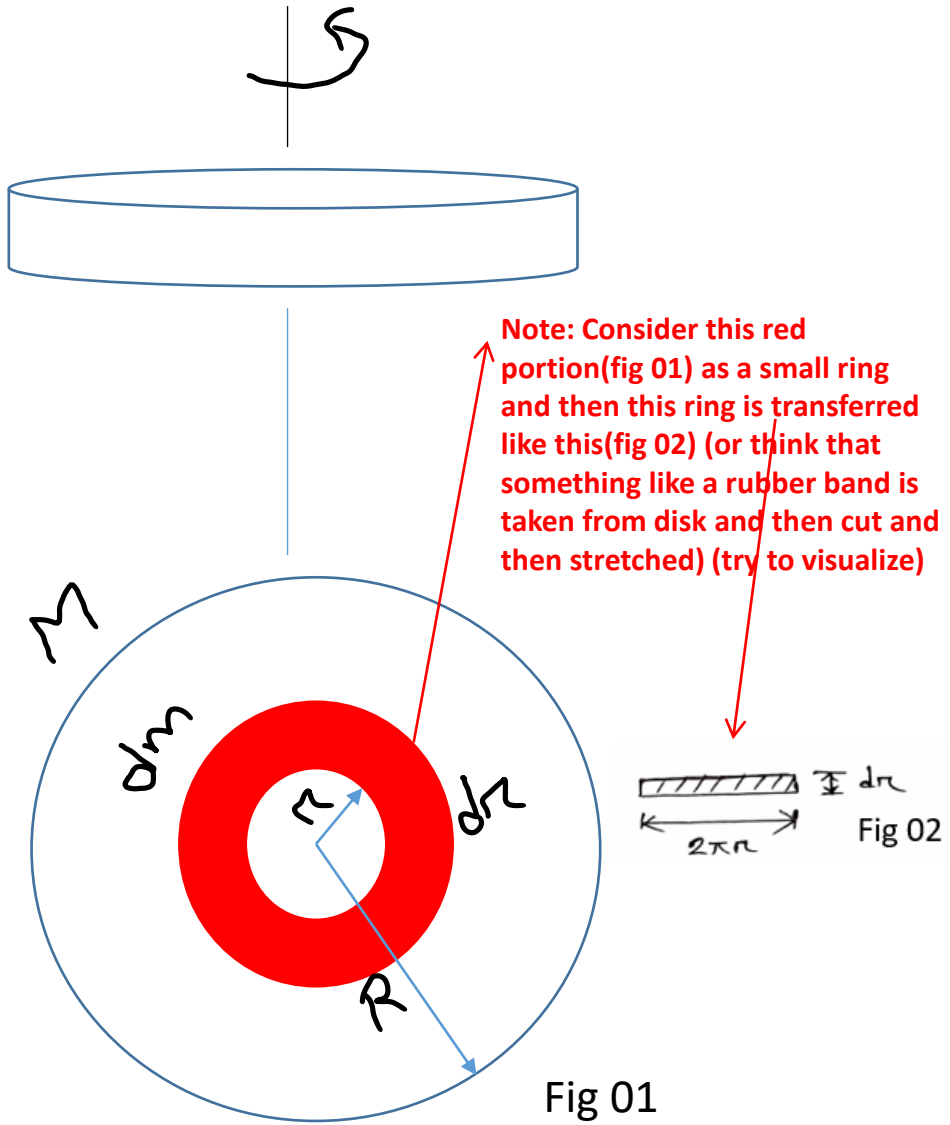



Fig 01

Note: Consider this red portion(fig 01) as a small ring and then this ring is transferred like this(fig 02) (or think that something like a rubber band is taken from disk and then cut and then stretched) (try to visualize)




Fig 02

$$\sigma = \frac{\text{mass}}{\text{Area}} = \frac{M}{\pi R^2}$$

↳ Mass per unit area

Moment of inertia of ring, $dI = dm\,r^2$

Moment of inertia of disc,

$$I = \int_0^R dm\,r^2$$

$$\sigma = \frac{dm}{2\pi r\,dr} \quad [\text{Area of ring, } dA = (2\pi r)\,dr]$$

∴ $dm = 2\pi r dr \times \sigma$

$$I = \int_0^R \sigma(2\pi r)dr\,r^2$$

$$= \sigma 2\pi \int_0^R r^3 dr$$

$$= 2\pi\sigma\left[\frac{r^4}{4}\right]_0^R$$

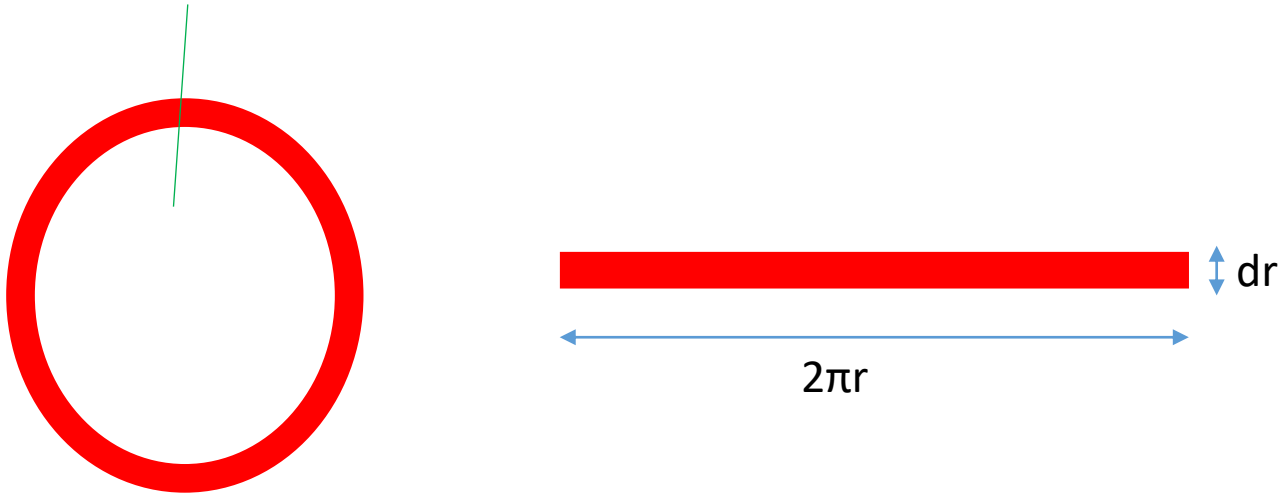
$$= 2\pi \cdot \frac{M}{\pi R^2} \times \frac{R^4}{4} = \frac{1}{2}MR^2$$

r is the distance of small portion of the disc from the centre

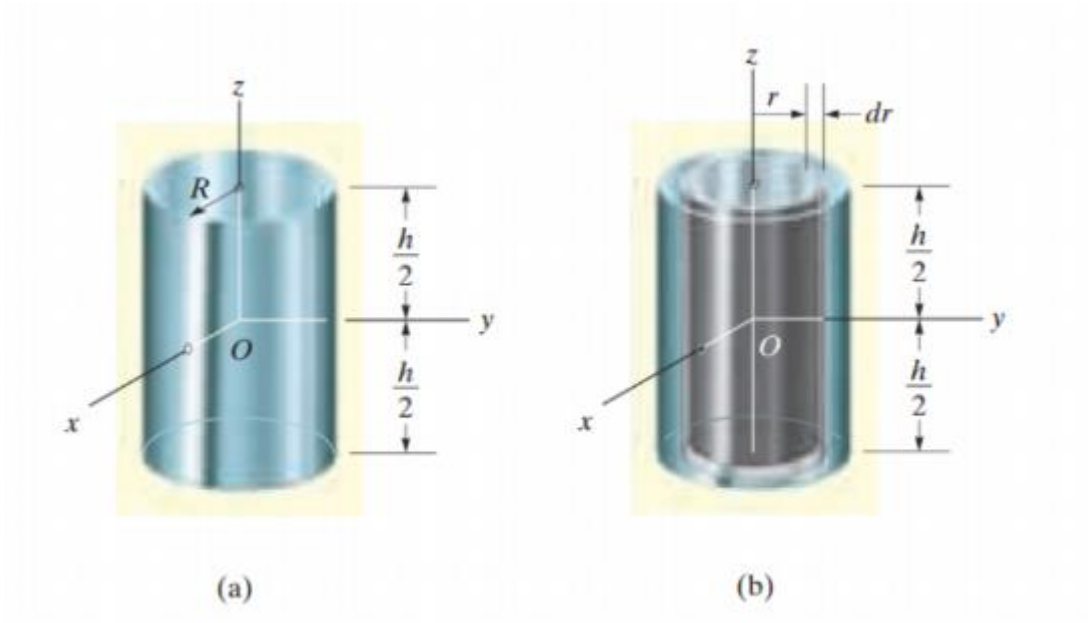
R=radius of the disc
dr= thickness of the small ring

M=mass of the total ring

Cut this rubber band of thickness dr here
and then stretch it



**Q: Determine the mass moment of inertia of a cylinder shown in fig(a) about the z axis. The density of the material is constant.**



The volume of the element is

$$dV = (2\pi r)(h)\,dr$$

$$dm = \rho\,dV = \rho(2\pi h r\,dr)$$

since the entire element lies at the same distance $r$ from the z axis, the moment of inertia of the element is,

$$dI_z = r^2\,dm = \rho\,2\pi h r^3\,dr$$

$$I_z = \int_0^R r^2\,dm = \rho\,2\pi h \int_0^R r^3\,dr$$

$$= \rho\,2\pi h \left[\frac{r^4}{4}\right]_0^R$$

$$= \rho\,2\pi h \cdot \frac{R^4}{4} = \frac{\rho\pi}{2} R^4 h$$

Here, $m = \int dm = \rho\,2\pi h \int_0^R r\,dr$

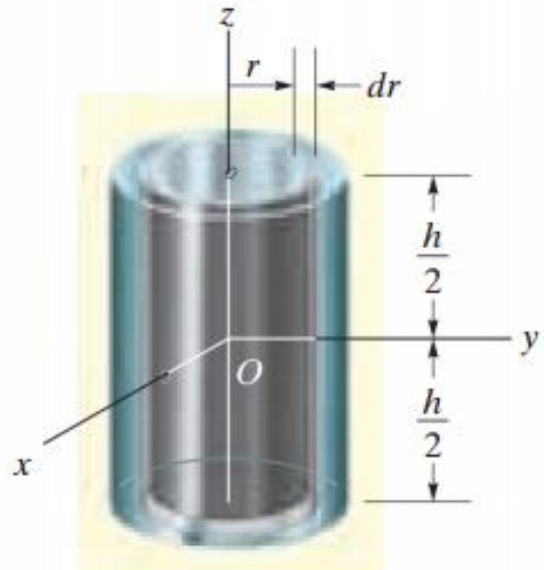
$$= \rho\,2\pi h \left[\frac{r^2}{2}\right]_0^R$$

$$= \rho\,2\pi h \cdot \frac{R^2}{2} = \rho\pi h R^2$$

$$\therefore I_z = (\rho\pi h R^2)\left(\frac{R^2}{2}\right) = m\left(\frac{R^2}{2}\right)$$

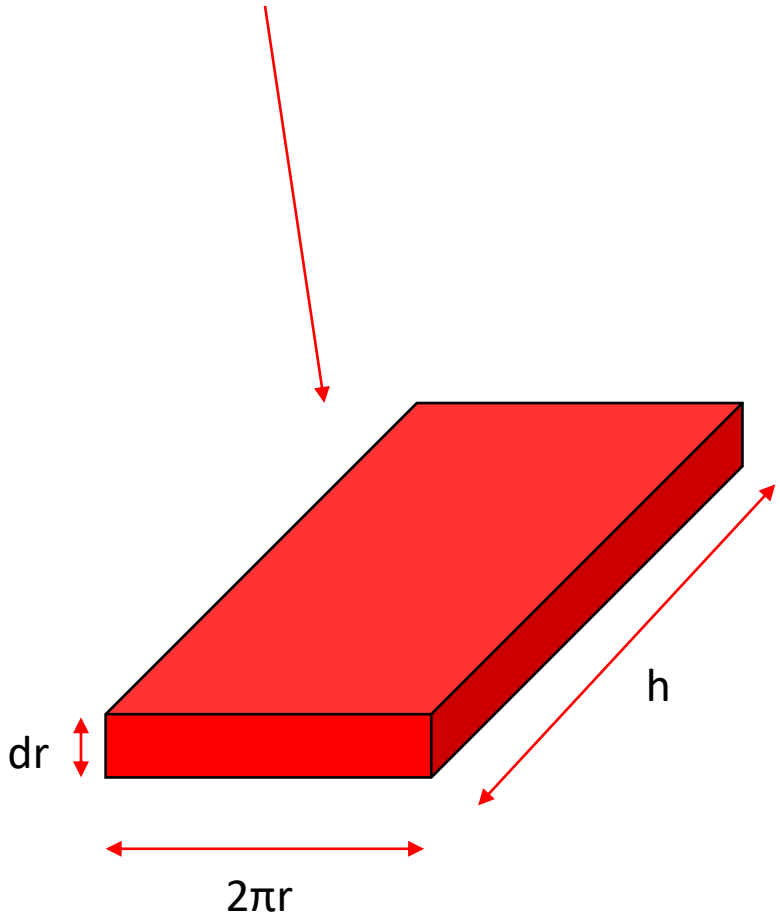
$$= \frac{1}{2} m R^2$$

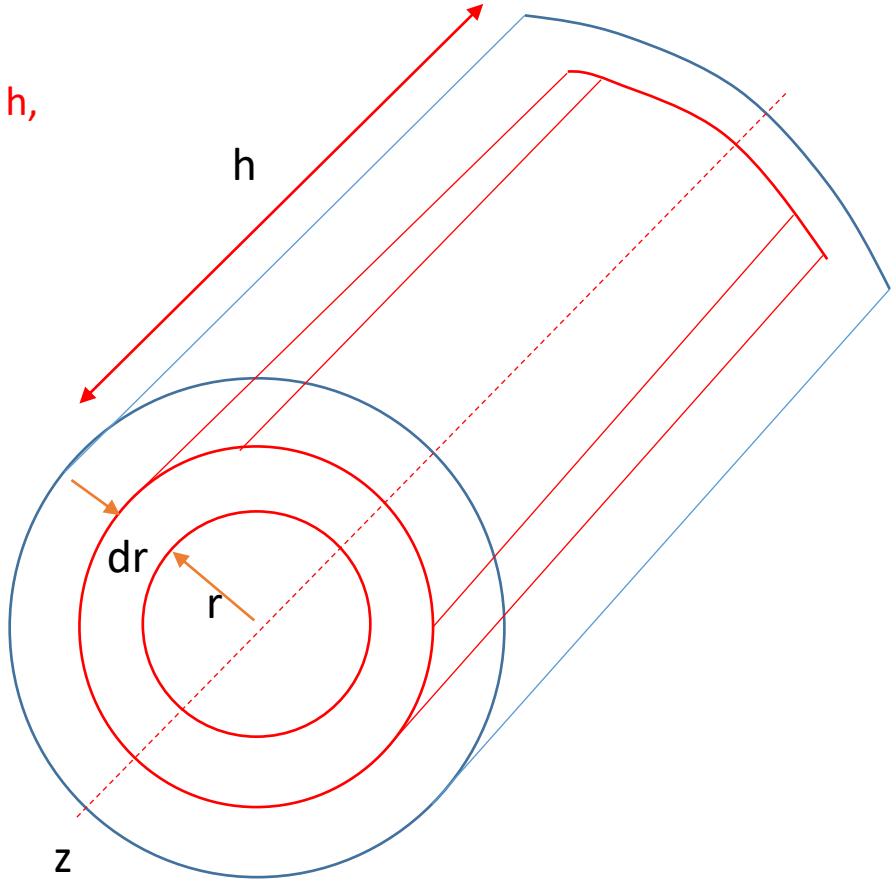
Volm=**circumference** x **height** x **thickness**
=(2 x pi x r) x h x dr




(b)

If you take an element of thickness dr at a distance r from z axis whose height is h, and then if you stretch that, it will something look like this (try to visualize)

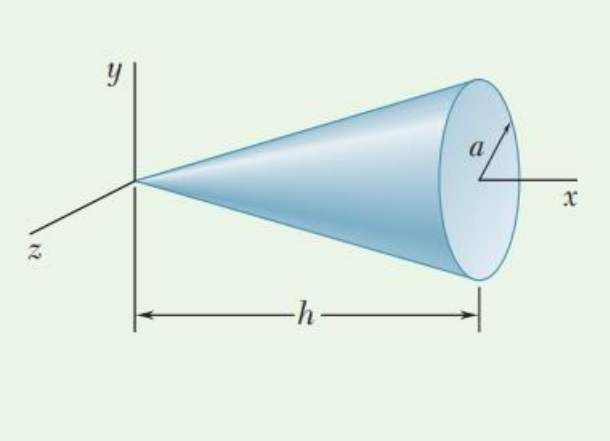



Volm = 2πr x h x dr

**This cylinder is same with the previous slide, I have just rotated it for better understanding**

**Q: Determine the mass moment of inertia of a right circular cone with respect to (a) its longitudinal axis.**



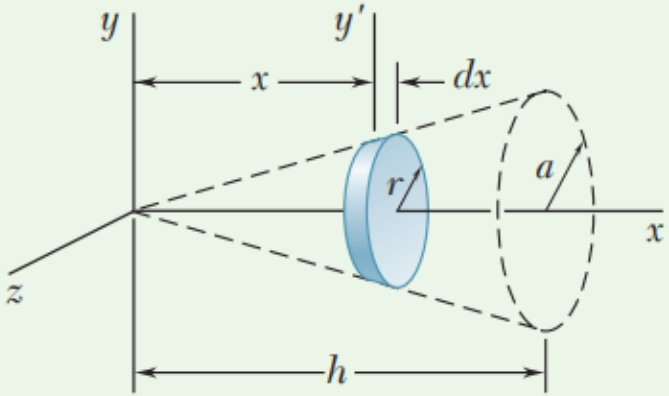



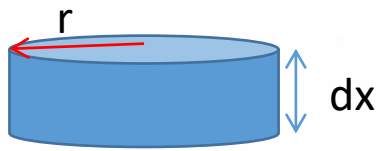

**Fig. 1** Differential element of mass.

Choose the differential element of mass shown in Figure 1. Now, Express the radius & mass of this disk as,

$$r = a \cdot \frac{x}{h}$$

Now, $dm = \rho \cdot V = \rho \cdot (\pi r^2 dx)$

$$= \rho \cdot \pi \cdot \frac{a^2}{h^2} x^2 dx$$




volm= $(\pi r^2 dx)$

$dI_x = \frac{1}{2} r^2 dm$ [concept of disk]

$$= \frac{1}{2} \left(a \cdot \frac{x}{h}\right)^2 \left(\rho \pi \cdot \frac{a^2}{h^2} x^2 dx\right)$$

$$= \frac{1}{2} \cdot \rho \pi \frac{a^4}{h^4} x^4 dx$$

$$\therefore I_x = \int dI_x = \int_0^h \frac{1}{2} \rho \pi \frac{a^4}{h^4} x^4 dx$$

$$= \frac{1}{2} \rho \pi \frac{a^4}{h^4} \left[\frac{h^5}{5}\right]$$

$$= \frac{1}{10} \rho \pi \frac{a^4}{h^4} h^5 = \frac{1}{10} \rho \pi a^4 h$$

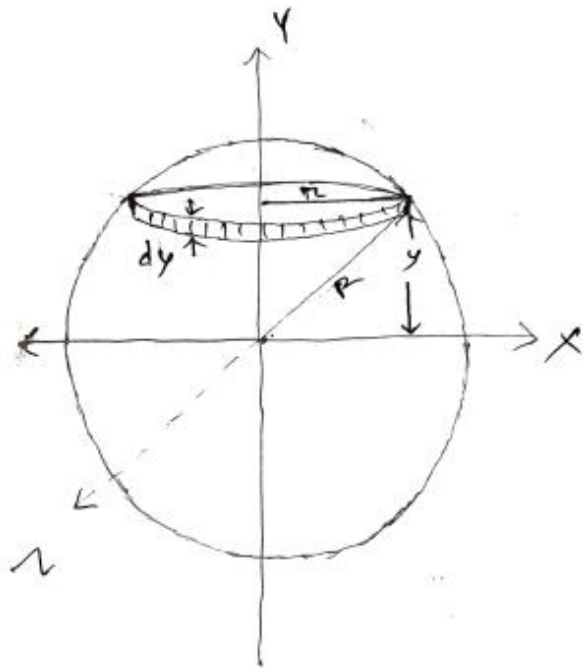
Now, $m = \rho \times \text{vol}^m = \rho \times \left(\frac{1}{3} \pi a^2 h\right)$

$$\therefore I_x = \frac{1}{10} \rho \pi a^4 h = \frac{3}{10} a^2 \left(\frac{1}{3} \rho \pi a^2 h\right)$$

$$= \frac{3}{10} m a^2$$

**Q: Determine the mass moment of inertia of solid sphere about it's diameter**

Consider an element disc of radius r and thickness dy at a distance y from the x axis.







$r^2 = R^2 - y^2$

mass of the elemental disc, $dm = \rho \times \text{volm}$

$= \rho \times \pi r^2 dy$

$(dIm)_y = \frac{1}{2} \times \text{mass} \times \text{radius}^2$

$= \frac{1}{2} \times dm \times r^2$

$= \frac{1}{2} \times \rho \pi r^2 dy \cdot r^2$

$= \frac{1}{2} \times \rho \pi r^4 \cdot dy$

$= \frac{1}{2} \times \rho \pi (R^2 - y^2)^2 dy$

$\therefore (Im)_y = \int (dIm)_y = \int_{-R}^{+R} \frac{1}{2} \rho \pi (R^2 - y^2)^2 dy$

$= \frac{\rho \pi}{2} \int_{-R}^{+R} (R^4 + y^4 - 2R^2 y^2) dy$

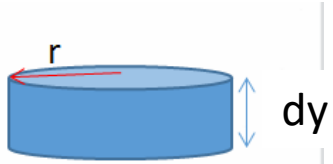
$= \frac{\rho \pi}{2} \times 2 \int_0^R (R^4 + y^4 - 2R^2 y^2) dy$

$= \rho \pi \left[ R^4 y + \frac{y^5}{5} - 2 \cdot \frac{R^2 y^3}{3} \right]_0^R$

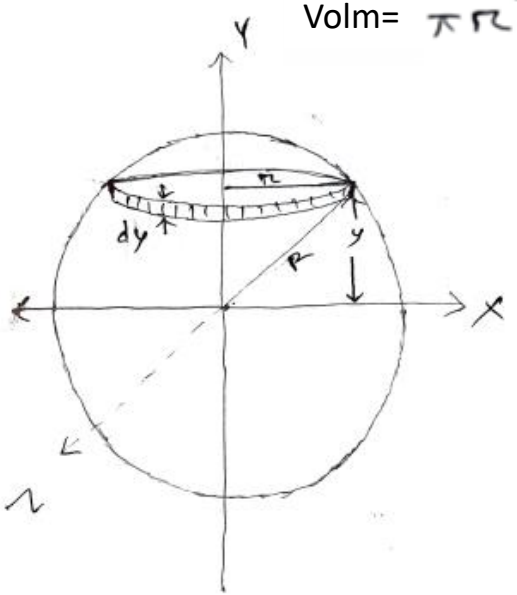
$= \rho \pi \left[ R^5 + \frac{R^5}{5} - 2\frac{R^5}{3} \right]$

$= \rho \pi R^5 \left[ 1 + \frac{1}{5} - \frac{2}{3} \right]$

$\therefore (Im)_y = \frac{8}{15} \rho \pi R^5$ .....Equation (1)




Volm= $\pi r^2 dy$







$r^2 = R^2 - y^2$

Total mass of sphere:

$$M = \int dm$$

$$= \int_{-R}^{R} \rho \pi r^2 dy$$

$$= \int_{-R}^{R} \rho \pi (R^2 - y^2) dy$$

$$= \rho \pi 2 \int_{0}^{R} (R^2 - y^2) dy$$

$$= \rho \pi 2 \left[ R^2 y - \frac{y^3}{3} \right]_0^R$$

$$= \rho \pi \cdot 2 \left[ R^3 - \frac{R^3}{3} \right]$$

$$= \rho \pi \cdot 2 \left[ \frac{2R^3}{3} \right]$$

$$= \frac{4}{3} \pi R^3 \cdot \rho$$

Or, you can calculate the mass like this

We know,

$$M = \rho \times \text{vol}^m$$

$$\Rightarrow M = \rho \times \frac{4}{3} \pi R^3$$

$$\therefore \rho \pi R^3 = \frac{3}{4} M \quad \text{...Eqn 2}$$

From Eqn 1,

$$(I_m)_y = \frac{8}{15} \rho \pi R^5$$

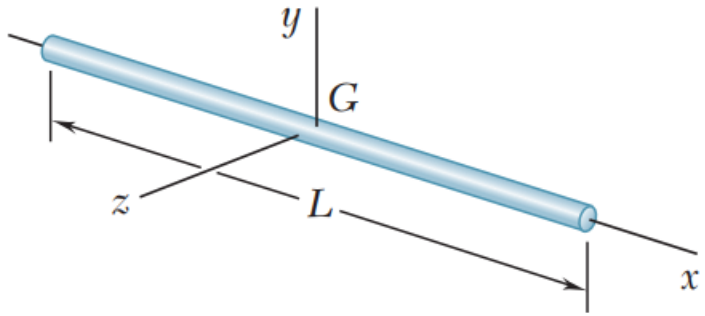
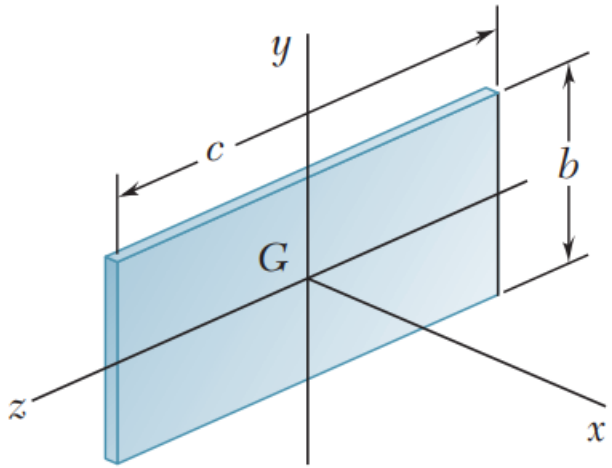
$$\therefore (I_m)_y = \frac{8}{15} \left( \frac{3}{4} m \right) R^2$$

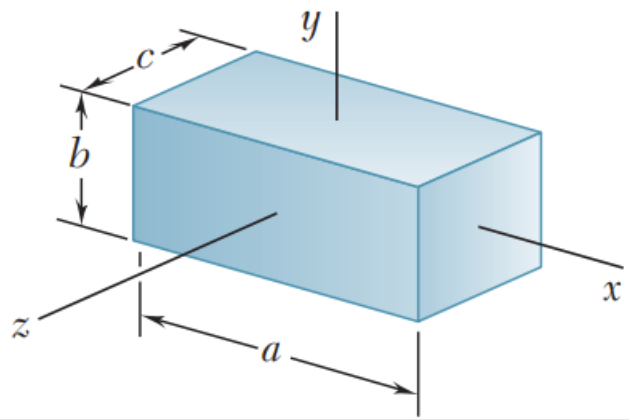
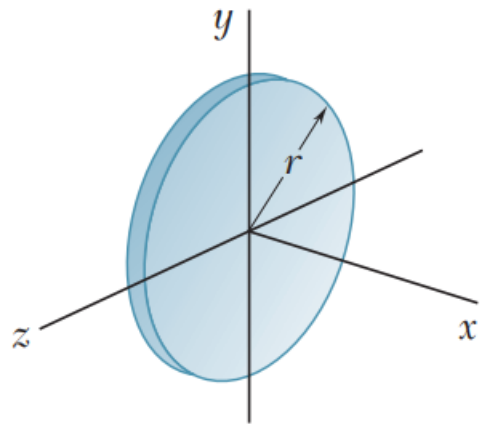
$$= \frac{2}{5} M R^2$$

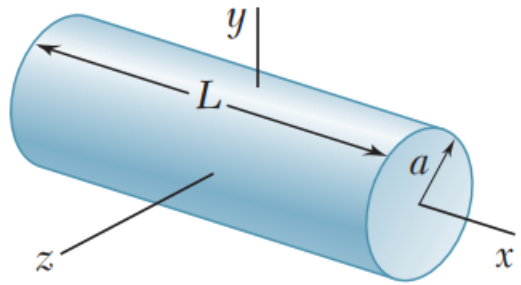
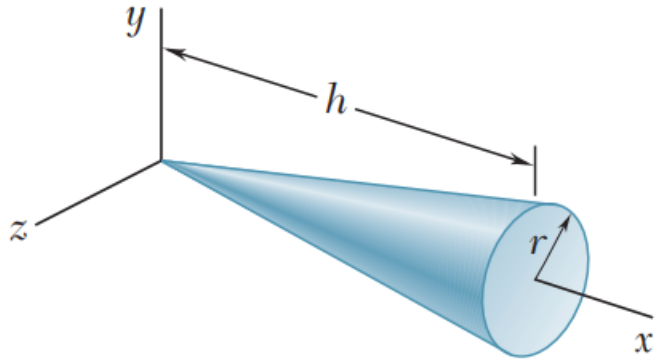
Putting the value of Eqn 2 in Eqn 1

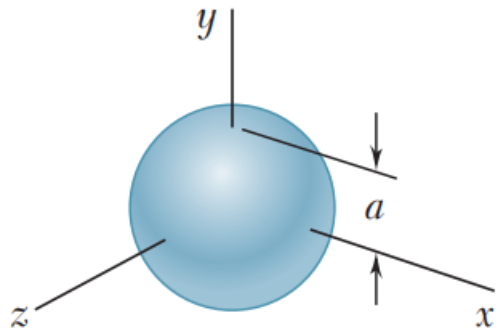
$$\rho \pi R^3 = \frac{3}{4} M$$

## Formula(very important for math solving)

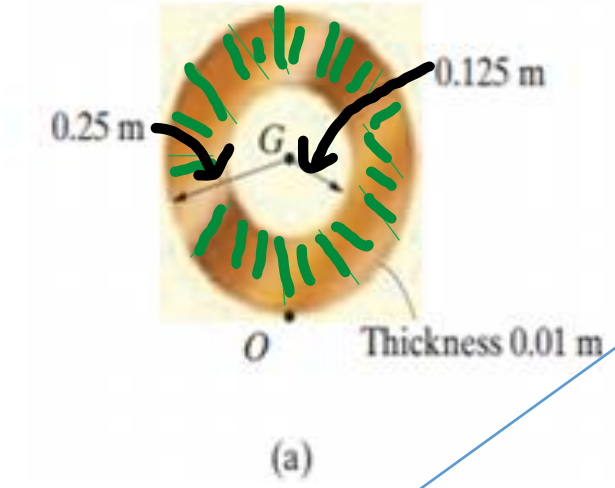
| | | |
|---|---|---|
| Slender rod |  | $I_y = I_z = \frac{1}{12} mL^2$ |
| Thin rectangular plate |  | $I_x = \frac{1}{12} m(b^2 + c^2)$<br>$I_y = \frac{1}{12} mc^2$<br>$I_z = \frac{1}{12} mb^2$ |

| | | |
|---|---|---|
| Rectangular prism |  | $I_x = \frac{1}{12}m(b^2 + c^2)$<br>$I_y = \frac{1}{12}m(c^2 + a^2)$<br>$I_z = \frac{1}{12}m(a^2 + b^2)$ |
| Thin disk |  | $I_x = \frac{1}{2}mr^2$<br>$I_y = I_z = \frac{1}{4}mr^2$ |

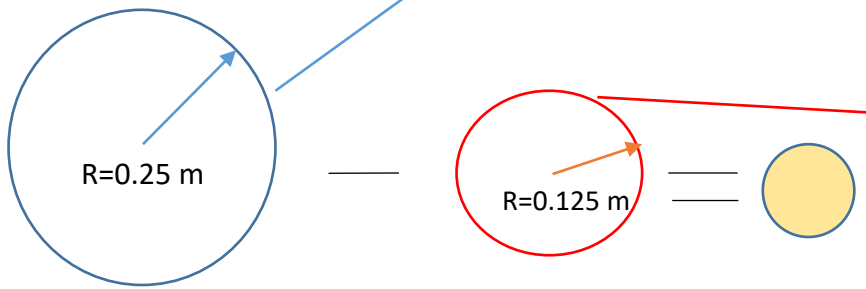
| | | |
|---|---|---|
| Circular cylinder |  | $I_x = \frac{1}{2}ma^2$<br>$I_y = I_z = \frac{1}{12}m(3a^2 + L^2)$ |
| Circular cone |  | $I_x = \frac{3}{10}ma^2$<br>$I_y = I_z = \frac{3}{5}m(\frac{1}{4}a^2 + h^2)$ |

| | | |
|---|---|---|
| Sphere |  | $I_x = I_y = I_z = \frac{2}{5}ma^2$ |

**Q: If the plate shown in Figure(a) has a density of 8000 kg/m^3 and a thickness of 10 mm, determine its mass moment of inertia(green shaded portion) about an axis perpendicular to the page and passing through the pin at O.**




(a)




**Using concept of parallel axis theorem**

$$(I_o)_d = \frac{1}{2} m_d r_d^2 + m_d d^2$$

$$= \frac{1}{2} (\rho_d V_d) r_d^2 + m_d d^2$$

$$= \frac{1}{2} \{\rho_d . (\pi r^2 . t)\} r_d^2 + m_d d^2$$

$$= \frac{1}{2} \{8000 \times \pi (0.25)^2 \times 0.01\} . (0.25)^2 + (15.71)(0.25)^2$$

$$= 1.473 \text{ kg-m}^2$$

$$m_d = \rho_d V_d = 8000 \times \pi \times (0.25)^2 \times 0.01 = 15.71 \text{ kg}$$

$$m_h = \rho_h V_h = 8000 \times \pi (0.125)^2 \times 0.01 = 3.93 \text{ kg}$$

$$\therefore (I_o)_h = \frac{1}{2} m_h r_h^2 + m_h d^2$$

$$= \frac{1}{2} \times 3.93 \times (0.125)^2 + (3.93)(0.25)^2$$

$$= 0.276 \text{ kg-m}^2$$

$$\therefore I_o = (I_o)_d - (I_o)_h$$

$$= 1.473 - 0.276 = 1.20 \text{ kg-m}^2$$

Q: Determine the **mass moment of inertia** of a slender rod of length L and mass m with respect to an axis that is perpendicular to the rod and passes through one end.

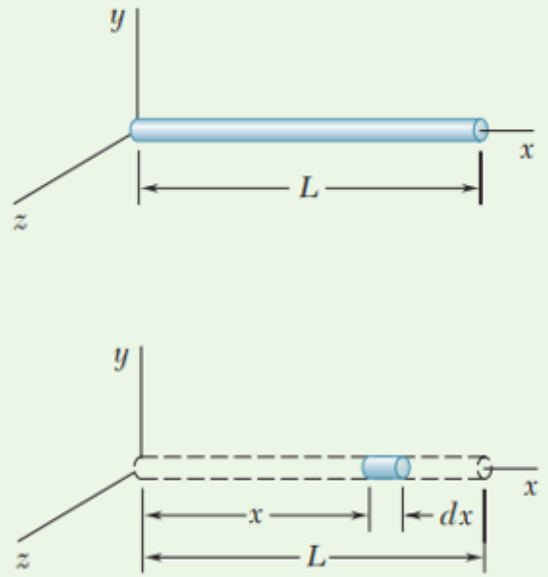



**Fig. 1** Differential element of mass.

choose the differential element of mass shown in Figure1 and express it as a mass per unit length

$$dm = \frac{m}{L}\,dx$$

$$\therefore I_y = \int x^2\,dm = \int_0^L x^2 \frac{m}{L}\,dx = \left[\frac{m}{L}\right]\left[\frac{x^3}{3}\right]_0^L$$

$$= \frac{1}{3}\cdot\frac{m}{L}\cdot L^3$$

$$= \frac{1}{3}mL^2$$

What will be the moment of inertia about it's centroidal axis? Determine by yourself