The form $\frac{\infty}{\infty}$

Evaluate: $\lim_{x \to \infty} \frac{\log(1+x)}{x}$

Soln The given function assumes the form $\frac{\infty}{\infty}$ for $x = \infty$.

But $\lim_{x \to \infty} \frac{\log(1+x)}{x}$

$= \lim_{x \to \infty} \frac{\frac{1}{1+x}}{1}$

$= \frac{1}{\infty}$

$= 0$

Evaluate: $\lim_{x \to \infty} \left(\frac{\log x}{x}\right)$

Soln $\lim_{x \to \infty} \left(\frac{\log x}{x}\right)$ $\left[\text{Form } \frac{\infty}{\infty}\right]$

$= \lim_{x \to \infty} \frac{\frac{1}{x}}{1}$

$= \frac{1}{\infty}$

$= 0$

Form $0 \times \infty$

This form can be changed either to $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

Evaluate: $\lim_{x \to 0} \sin x \log x$

Sol^n $\lim_{x \to 0} \sin x \log x$ [Form $0 \times \infty$]

$= \lim_{x \to 0} \frac{\log x}{\operatorname{cosec} x}$ [Form $\frac{\infty}{\infty}$]

$= \lim_{x \to 0} \frac{\frac{1}{x}}{-\operatorname{cosec} x \cot x}$

$= \lim_{x \to 0} \frac{-\sin x \tan x}{x}$ [Form $\frac{0}{0}$]

$= \lim_{x \to 0} \frac{-\sin x \sec^2 x + \tan x \cos x}{1}$

$= 0$

Evaluate: $\lim_{x \to \infty} 2^x \sin\left(\frac{a}{2^x}\right)$

Soln $\lim_{x \to \infty} 2^x \sin\left(\frac{a}{2^x}\right)$ [Form $0 \times \infty$]

$$= \lim_{x \to \infty} \frac{a \sin \frac{a}{2^x}}{\frac{a}{2^x}}$$

$$= \lim_{y \to 0} a \frac{\sin y}{y}, \text{ where } y = \frac{a}{2^x} \text{ and } y \to 0 \text{ as } x \to \infty$$

$$= a \times 1 \qquad \left[\lim_{y \to 0} \frac{\sin y}{y} = 1\right]$$

$$= a$$

Evaluate: $\lim_{x \to \frac{\pi}{2}} (\sec x - \tan x)$

Soln $\lim_{x \to \frac{\pi}{2}} (\sec x - \tan x)$ [Form $\infty - \infty$]

$$= \lim_{x \to \frac{\pi}{2}} \left[\frac{1}{\cos x} - \frac{\sin x}{\cos x} \right]$$

$$= \lim_{x \to \frac{\pi}{2}} \frac{1 - \sin x}{\cos x}$$ [Form $\frac{0}{0}$]

$$= \lim_{x \to \frac{\pi}{2}} \frac{-\cos x}{-\sin x}$$ [By L'Hospitals Rule]

$$= \lim_{x \to \frac{\pi}{2}} (\cot x)$$

$$= 0$$

Form $\infty - \infty$

Evaluate: $\lim_{x \to 0} \left(\frac{1}{x^2} - \frac{1}{\sin x} \right)$

Sol^n $\lim_{x \to 0} \left(\frac{1}{x^2} - \frac{1}{\sin x} \right)$ [Form $\infty - \infty$]

$= \lim_{x \to 0} \frac{\sin x - x}{x^2 \sin x}$ [Form $\frac{0}{0}$]

$= \lim_{x \to 0} \frac{\left(x - \frac{1}{6}x^3 + \cdots \right) - x^2}{x^2 \left(x - \frac{1}{6}x^3 + \cdots \right)}$ [$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$]

Evaluate: $\lim_{x \to 1} \left(\frac{x}{x-1} - \frac{1}{\log x} \right)$

Solution:

$$\lim_{x \to 1} \left(\frac{x}{x-1} - \frac{1}{\log x} \right) \quad [\text{Form } \infty - \infty]$$

$$= \lim_{x \to 1} \frac{x \log x - x + 1}{(x-1)\log x} \quad \left[\text{Form } \frac{0}{0}\right]$$

$$= \lim_{x \to 1} \frac{\log x + x \cdot \frac{1}{x} - 1}{(x-1)\cdot \frac{1}{x} + \log x} \quad [\text{By L'Hospitals Rule}]$$

$$= \lim_{x \to 1} \frac{\log x}{1 - \frac{1}{x} + \log x} \quad \left[\text{Form } \frac{0}{0}\right]$$

$$= \lim_{x \to 1} \frac{\frac{1}{x}}{0 + \frac{1}{x^2} + \frac{1}{x}}$$

$$= \frac{1}{1+1}$$

$$= \frac{1}{2}$$

Form $0^0, \infty^0, 1^\infty$

Evaluate: $\lim_{x \to 0} x^x$

Soln Let $y = \lim_{x \to 0} x^x$ [Form 0^0]

Then $\log y = \lim_{x \to 0} x \log x$ [Form $0 \times \infty$]

$$= \lim_{x \to 0} \frac{\log x}{1/x} \quad \left[\text{Form } \frac{\infty}{\infty}\right]$$

$$= \lim_{x \to 0} \frac{\frac{1}{x}}{-\frac{1}{x^2}}$$

$$= \lim_{x \to 0} (-x)$$

$$= 0$$

$\log y = 0$

or $y = e^0$

or, $y = 1$

$\therefore \lim_{x \to 0} x^x = 1.$

Evaluate $\lim_{x \to 0} (\sin x)^{\tan x}$

Solution: Let $y = \lim_{x \to 0} (\sin x)^{\tan x}$ [Form 0^0]

$$\log y = \lim_{x \to 0} \tan x \log \sin x \quad [\text{Form } 0 \times \infty]$$

$$= \lim_{x \to 0} \frac{\log \sin x}{\cot x} \quad \text{Form } \frac{\infty}{\infty}$$

$$= \lim_{x \to 0} \frac{\frac{1}{\sin x} \cos x}{-\operatorname{cosec}^2 x}$$

$$= \lim_{x \to 0} \frac{\cos x / \sin x}{-1/\sin^2 x}$$

$$= \lim_{x \to 0} -\cos x \sin x$$

or $\log y = 0$

or $\log y = \log 1$

or $y = 1$

$\therefore \lim_{x \to 0} (\sin x)^{\tan x} = 1.$

Evaluate: $\lim_{x \to \frac{\pi}{2}} (\tan x)^{\cos x}$

Soln $\lim_{x \to \frac{\pi}{2}} (\tan x)^{\cos x}$ [Form ∞^0]

Let $y = \lim_{x \to \frac{\pi}{2}} (\tan x)^{\cos x}$

or $\log y = \lim_{x \to \frac{\pi}{2}} \cos x \log \tan x$

$= \lim_{x \to \frac{\pi}{2}} \frac{\log \tan x}{1/\cos x}$ [Form $\frac{\infty}{\infty}$]

$= \lim_{x \to \frac{\pi}{2}} \frac{\log \tan x}{\sec x}$ [form $\frac{\infty}{\infty}$]

$= \lim_{x \to \frac{\pi}{2}} \frac{\sec^2 x / \tan x}{\sec x \tan x}$

$= \lim_{x \to \frac{\pi}{2}} \frac{\sec x}{\tan^2 x}$ [Form $\frac{\infty}{\infty}$]

$= \lim_{x \to \frac{\pi}{2}} \frac{\cos x}{\sin^2 x}$

$= 0$

$\log y = 0$

$y = e^0 = 1$

Hence $\lim_{x \to \frac{\pi}{2}} (\tan x)^{\cos x} = 1.$

Evaluate: $\lim_{x\to 0} (\cos x)^{\frac{1}{x^2}}$

Soln $\lim_{x\to 0} (\cos x)^{\frac{1}{x^2}}$ [From 1^∞]

Let $y = \lim_{x\to 0} (\cos x)^{\frac{1}{x^2}}$

$\therefore \log y = \lim_{x\to 0} \frac{\log(\cos x)}{x^2}$ [Form $\frac{0}{0}$]

$= \lim_{x\to 0} -\frac{\tan x}{2x}$

$= \lim_{x\to 0} -\frac{\sec^2 x}{2}$

$= -\frac{1}{2}$

$\log y = -\frac{1}{2}$

$y = e^{-1/2}$

ie $\lim_{x\to 0} (\cos x)^{\frac{1}{x^2}} = e^{-1/2}$.