Part-1 Indeterminate Form

① $\lim_{x\to 0} (\sec x)^{\tan x}$

Let, $y = \lim_{x\to 0} (\sec x)^{\tan x}$

$\Rightarrow \log y = \lim_{x\to 0} \tan x \log(\sec x)$

$= \lim_{x\to 0} \frac{\sin x \log(\sec x)}{\cos x}$ [which is the form of $\frac{0}{0}$]

$= \lim_{x\to 0} \frac{\log(\sec x)\cos x + \sin x \cdot \frac{1}{\sec x} \cdot \sec x \cdot \tan x}{-\sin x}$

$= \lim_{x\to 0} \frac{\log(\sec x)\cos x + \sin x \tan x}{-\sin x}$

$= \lim_{x\to 0} \frac{\log(\sec x)(-\sin x) + \cos x \cdot \frac{1}{\sec x} \cdot \sec x \tan x + \sin x \cdot \sec^2 x + \tan x(-\cos x)}{-\sin x}$

$= \lim_{x\to 0} \frac{-\sin x (\log(\sec x) - 1 - \sec^2 x + 1)}{-\sin x}$

$= \lim_{x\to 0} \log(\sec x) - \sec^2 x$

$= 0 - 1$

$\Rightarrow \log y = -1$

$\therefore y = e^{-1}$

$\therefore \lim_{x \to 0} (\sec x)^{\tan x} = e^{-1}$ (Ans)

② $\lim_{x \to 0} \left(\frac{\sin x}{x}\right)^{\frac{1}{x^2}}$ [which is the form of 1^∞]

Let, $y = \lim_{x \to 0} \left(\frac{\sin x}{x}\right)^{\frac{1}{x^2}}$

$\log y = \lim_{x \to 0} \frac{1}{x^2} \log\left(\frac{\sin x}{x}\right)$ [which is the form of $\infty \times 0$]

$= \lim_{x \to 0} \frac{\log\left(\frac{\sin x}{x}\right)}{x^2}$ [$\frac{0}{0}$ form]

$= \lim_{x \to 0} \frac{\frac{1}{\frac{\sin x}{x}} \cdot \frac{x\cos x - \sin x}{x^2}}{2x}$ [L' Hospital]

$= \lim_{x \to 0} \frac{x}{\sin x} \cdot \lim_{x \to 0} \frac{x\cos x - \sin x}{2x^3}$

$= 1 \cdot \lim_{x \to 0} \frac{x(-\sin x) + \cos x - \cos x}{6x^2}$

$= -\lim_{x \to 0} \frac{x \sin x}{6x^2}$

$= -\frac{1}{6} \lim_{x \to 0} \frac{\sin x}{x}$

$$= -\frac{1}{6} \times 1$$

$$\therefore \log y = -\frac{1}{6}$$

$$\therefore y = e^{-\frac{1}{6}}$$

$$\therefore \lim_{x \to 0} \left(\frac{\sin x}{x}\right)^{\frac{1}{x^2}} = e^{-\frac{1}{6}} \quad \text{(Ans)}$$

(3) $\lim_{x \to \infty} \frac{x^n}{e^x}$ [form of $\frac{\infty}{\infty}$]

$$= \lim_{x \to \infty} \frac{n x^{n-1}}{e^x} \quad [\text{L'Hospital}]$$

$$= \lim_{x \to \infty} \frac{n(n-1) x^{n-2}}{e^x}$$

Applying L'Hospital again and again, we get

$$\lim_{x \to \infty} \frac{n(n-1)(n-2) \cdots 1 \cdot x^0}{e^x} \quad [x^0 = 1]$$

$$= \lim_{x \to \infty} \frac{\text{Ln}}{e^x} = \frac{\text{Ln}}{\infty} = 0 \quad \text{(Ans)}$$

(4) $\lim_{x \to \frac{\pi}{2}} \frac{\log\left(x - \frac{\pi}{2}\right)}{\tan x}$ [Form $\frac{\infty}{\infty}$]

$= \lim_{x \to \frac{\pi}{2}} \frac{\frac{1}{\left(x - \frac{\pi}{2}\right) \cdot 1}}{\sec^2 x}$ [L'Hospital]

$= \lim_{x \to \frac{\pi}{2}} \frac{1}{\left(x - \frac{\pi}{2}\right) \sec^2 x}$

$= \lim_{x \to \frac{\pi}{2}} \frac{1}{x - \frac{\pi}{2}} \times \frac{1}{\sec^2 x}$ [$\infty \times 0$]

$= \lim_{x \to \frac{\pi}{2}} -\left(x - \frac{\pi}{2}\right)^{-2} \cdot 1 \times -2 \sec x \cdot \sec x \tan x$

$= 0$ (Ans)

(5) $\lim_{x\to 0} (\cos x)^{\cot^2 x}$

Let $y = \lim_{x\to 0} (\cos x)^{\cot^2 x}$

$\Rightarrow \log y = \lim_{x\to 0} \cot^2 x \log(\cos x)$

$= \lim_{x\to 0} \frac{\log(\cos x)}{\tan^2 x}$

$= \lim_{x\to 0} \frac{\frac{1}{\cos x}\, x - \sin x}{2\sec^2 x \tan x}$

$= \lim_{x\to 0} \frac{-\tan x}{2\sec^2 x \tan x}$

$= -\lim_{x\to 0} \frac{1}{2\sec^2 x} = -\frac{1}{2}$

$\therefore \log y = -\frac{1}{2}$

$\therefore y = e^{-\frac{1}{2}}$

$\therefore \lim_{x\to 0} (\cos x)^{\cot^2 x} = e^{-\frac{1}{2}}$

(8) $\lim_{x \to \infty} \frac{x^4}{e^x}$

$= \lim_{x \to \infty} \frac{4x^3}{e^x} \quad \left[\frac{\infty}{\infty}\right]$

$= \lim_{x \to \infty} \frac{12x^2}{e^x} \quad [\,n\,]$

$= \lim_{x \to \infty} \frac{24x}{e^x} \quad [\,n\,]$

$= \lim_{x \to \infty} \frac{24}{e^x}$

$= \frac{24}{e^{1/0}}$

$= \frac{24}{1} \times \frac{0}{1}$

$= 0 \quad (\text{Ans})$

(9) $\lim_{x \to 0} \frac{4^x - 2^x}{x}$

$= \lim_{x \to 0} \frac{4^x \cdot \ln 4 - 2^x \cdot \ln 2}{1}$

$= 4^0 \cdot \ln 4 - 2^0 \cdot \ln 2$

$$= \lim_{x\to 0} \frac{2\{x\sec^2 x + \tan x\} + 2\{x \cdot 3\tan^2 x \cdot \sec^2 x + \tan^3 x\}}{\sec^2 x + 2x\sec x \cdot \sec x \tan x + \sec^2 x}$$

$$\Rightarrow = \frac{2\times 0 + 2\times 0 + 12\times 0 + 0}{2.1 + 2\times 0}$$

$$= \frac{0}{2} = 0 \quad (\text{Ans})$$

$$\therefore \lim_{x\to 0} \left(\frac{\tan x}{x}\right)^{1/x} = e^0 = 1$$

(7) $\lim_{x\to 0} \log(\cos x)^{\frac{1}{x^2}}$

$$= \lim_{x\to 0} \frac{1}{x^2} \log(\cos x) \quad \left[\frac{0}{0}\right]$$

$$= \lim_{x\to 0} -\frac{\tan x}{2x} \quad [\text{L'Hospital}]$$

$$= -\frac{1}{2} \lim_{x\to 0} \frac{\tan x}{x} \quad \left[\frac{0}{0}\right]$$

$$= -\frac{1}{2} \lim_{x\to 0} \frac{\sec^2 x}{1}$$

$$= -\frac{1}{2} \times 1$$

$$= -\frac{1}{2}$$

(6) $\lim_{x\to 0} \left(\frac{\tan x}{x}\right)^{1/x}$

Let, $y = \lim_{x\to 0} \left(\frac{\tan x}{x}\right)^{1/x}$

$\log y = \lim_{x\to 0} \frac{1}{x} \log\left(\frac{\tan x}{x}\right)$

$= \lim_{x\to 0} \frac{1}{x} \log(\tan x) - \log(x) \left[\frac{0}{0}\right]$

$= \lim_{x\to 0} \frac{x \sec^2 x - \tan x}{x \tan x} \left[\frac{0}{0}\right]$

$= \lim_{x\to 0} \frac{x \cdot 2\sec x \cdot \sec x \tan x + \sec^2 x - \sec^2 x}{\tan x + x \cdot \sec^2 x}$

$= \lim_{x\to 0} \frac{2x(1+\tan^2 x)\tan x}{x \sec^2 x + \tan x}$

$= \lim_{x\to 0} \frac{(2x + 2x\tan^2 x)\tan x}{x \sec^2 x + \tan x}$

$= \lim_{x\to 0} \frac{2x\tan x + 2x\tan^3 x}{x \sec^2 x + \tan x}$

$= \ln 4 - \ln 2$

$= \ln\left(\frac{4}{2}\right)$

$= \ln 2$ (Ans)

(10) $\lim_{x \to 0} \frac{e^x + \sin x - 1}{\log(1+x)}$

$= \lim_{x \to 0} \frac{e^x + \cos x}{\frac{1}{1+x}}$

$= \lim_{x \to 0} (1+x)(e^x + \cos x)$

$= 2$ (Ans)

(11) $\lim_{x \to \frac{\pi}{2}} (\sin x)^{\tan x}$

Let, $y = \lim_{x \to \frac{\pi}{2}} (\sin x)^{\tan x}$

$\Rightarrow \log y = \lim_{x \to \frac{\pi}{2}} \tan x \log(\sin x)$

(8)

$$= \lim_{x \to \frac{\pi}{2}} \frac{\sin x \log(\sin x)}{\cos x} \quad \left[\frac{0}{0}\right]$$

$$= \lim_{x \to \frac{\pi}{2}} \frac{\sin x}{\sin x} \cdot \frac{\cos x + \log(\sin x)\cos x}{-\sin x}$$

$$= 0$$

$$\log y = 0$$

$$\therefore y = 1$$

$$\therefore \lim_{x \to \frac{\pi}{2}} (\sin x)^{\tan x} = 1$$

(12) $$\lim_{x \to 0} \frac{e^x + e^{-x} - 2\cos x}{x \sin x} \quad \left[\frac{0}{0}\right]$$

$$= \lim_{x \to 0} \frac{e^x - e^{-x} + 2\sin x}{x\cos x + \sin x}$$

$$= \lim_{x \to 0} \frac{e^x + e^{-x} + 2\cos x}{x(-\sin x) + \cos x + \cos x}$$

$$= \frac{4}{2}$$

$$= 2 \quad (\text{Ans})$$

(13) $\lim_{x\to 0} \frac{1-\cos \pi x}{\tan^2 \pi x}$ $\left[\frac{0}{0}\right]$

$= \lim_{x\to 0} \frac{\pi - \sin \pi x}{\pi 2 \tan \pi x \cdot \sec^2 \pi x}$

$= \lim_{x\to 0} \frac{1}{2} \cdot \sin \pi x \times \frac{\cos x \pi}{\sin x \pi x} \times \cos^2 \pi x$

$= \lim_{x\to 0} \frac{1}{2} \cos^3 \pi x$

$= \frac{1}{2}$ (Ans)

(14) $\lim_{x\to 0} \frac{e^x - e^{-x} - 2x}{x - \sin x}$ $\left[\frac{0}{0}\right]$

$= \lim_{x\to 0} \frac{e^x + e^{-x} - 2}{1 - \cos x}$

$= \lim_{x\to 0} \frac{e^x - e^{-x}}{\sin x}$

$= \lim_{x\to 0} \frac{e^x + e^{-x}}{\cos x}$

$= \frac{1+1}{1} = 2$ (Ans)

(8

=

(15) $\lim_{x \to \infty} \left(\frac{2\pi}{x} - 1 \right)$

$= \frac{2\pi}{\infty} - 1$

$= 2 - 1 = 1$ (Ans)

(16) $\lim_{x \to 0} \left(\frac{a^x + b^x}{2} \right)^{1/x}$ $[1^\infty]$

Let, $y = \lim_{x \to 0} \left(\frac{a^x + b^x}{2} \right)^{1/x}$

$\Rightarrow \log y = \lim_{x \to 0} \frac{1}{x} \log \left(\frac{a^x + b^x}{2} \right)$ $[0 \times \infty]$

$= \lim_{x \to 0} \frac{\log \left(\frac{a^x + b^x}{2} \right)}{x}$ [form $\frac{0}{0}$]

$= \lim_{x \to 0} \frac{\frac{2}{a^x + b^x} \cdot \frac{1}{2} (a^x \ln a + b^x \ln b)}{1}$

$= \lim_{x \to 0} \frac{1}{a^x + b^x} (a^x \ln a + b^x \ln b)$

$$= \frac{1}{2} \ln ab$$

$$\therefore \log y = \log \sqrt{ab}$$

$$\Rightarrow y = \sqrt{ab}$$

$$\therefore \lim_{x \to 0} \left(\frac{a^x + b^x}{2} \right)^{1/x} = \sqrt{ab} \quad \text{(Ans)}$$

(17) $$\lim_{x \to 0} \frac{(1+x)^{1/x} - e}{x}$$

$$= \lim_{x \to 0} \frac{e\left(1 - \frac{x}{2} + \frac{11}{24}x^2 \cdots\right) - e}{x}$$

$$= \lim_{x \to 0} \frac{e - \frac{ex}{2} + \frac{e11}{24}x^2 \cdots - e}{x}$$

$$= \lim_{x \to 0} \frac{-\frac{e}{2} + \frac{11e}{24}x - \cdots}{1}$$

$$= -\frac{e}{2} \quad \text{(Ans)}$$

(18) $\lim_{x\to\infty} (x+e^x)^{2/x}$

Let,

$$y = \lim_{x\to\infty} (x+e^x)^{2/x} \quad [\infty^0]$$

$$\Rightarrow \log y = \lim_{x\to\infty} \frac{2}{x} \log(x+e^x) \quad [0\times\infty \text{ form}]$$

$$= 2 \lim_{x\to\infty} \frac{\log(x+e^x)}{x} \quad \left[\frac{\infty}{\infty}\right]$$

$$= 2 \lim_{x\to\infty} \frac{\frac{1}{x+e^x}\cdot(1+e^x)}{1} \quad [\text{L' Hospital}]$$

$$= 2 \lim_{x\to\infty} \frac{1+e^x}{x+e^x} \quad \left[\frac{\infty}{\infty}\right]$$

$$= 2 \lim_{x\to\infty} \frac{e^x}{1+e^x}$$

$$= 2 \lim_{x\to\infty} \frac{e^x}{e^x}$$

$$= 2 \times 1 = 2$$

$\therefore \log y = 2$

$\therefore \lim_{x\to\infty} (x+e^x)^{2/x} = e^2$ (Ans)

$$= \lim_{x \to 0} \frac{x\cos x - \sin x}{2x \cdot \sin x}$$

$$= \lim_{x \to 0} \frac{-x\sin x + \cos x - \cos x}{2x \cdot \cos x + \sin x \, 4x}$$

$$= \lim_{x \to 0} \frac{x\cos x - \sin x}{2x \cdot \sin x + 4x\cos x + 4x\cos x + 4\sin x}$$

$$= \lim_{x \to 0} \frac{x\sin x - \cos x - \cos x}{-2x \cdot \cos x - 4x\cos x + 8\cos x - 8\sin x + 4\cos x}$$

$$= \lim_{x \to 0} \frac{x\sin x - 2\cos x}{-2x \cdot \cos x - 4x\sin x + 12\cos x - 8x\sin x}$$

$$= -\frac{2}{12} = -\frac{1}{6}$$

$$\Rightarrow \log y = -\frac{1}{6}$$

$$\therefore \lim_{x \to 0} \left(\frac{\sin x}{x}\right)^{1/x^2} = e^{-1/6}$$

(3) $\lim_{x\to 0} \left(\frac{\tan x}{x}\right)^{1/x}$

Let $y = \lim_{x\to 0} \left(\frac{\tan x}{x}\right)^{1/x}$

$\Rightarrow \log y = \lim_{x\to 0} \frac{1}{x} \log\left(\frac{\tan x}{x}\right)$

$= \lim_{x\to 0} \frac{1}{x} \log(\tan x) - \log x \quad \left[\frac{0}{0}\right]$

$= \lim_{x\to 0} \frac{\frac{\sec^2 x}{\tan x} - \frac{1}{x}}{1}$

$= \lim_{x\to 0} \frac{x\sec^2 x - \tan x}{x\tan x}$

$= \lim_{x\to 0} \frac{x.2\sec x.\sec x\tan x + \sec^2 x - \sec^2 x}{\tan x + x\sec^2 x}$

$= \lim_{x\to 0} \frac{2x\sec^2 x\tan x}{\tan x + x\sec^2 x}$

$= \lim_{x\to 0} \frac{2x(1+\tan^2 x)\tan x}{x\sec^2 x + \tan x}$

$= \lim_{x\to 0} \frac{2x\tan x + 2x\tan^3 x}{x\sec^2 x + \tan x}$

Part - 2

① $\lim_{x \to 0} \frac{1-\cos x}{x^2}$

Let,

$y = \lim_{x \to 0} \frac{1-\cos x}{x^2}$ [form $\frac{0}{0}$]

$= \lim_{x \to 0} \frac{\sin x}{2x}$ [L'Hospital]

$= \frac{1}{2} \lim_{x \to 0} \frac{\sin x}{x}$ [form $\frac{0}{0}$]

$= \frac{1}{2} \lim_{x \to 0} \frac{\cos x}{1}$

$= \frac{1}{2} \times 1 = \frac{1}{2}$ (Ans)

② $\lim_{x \to 0} \left(\frac{\sin x}{x}\right)^{1/x^2}$

Let, $y = \lim_{x \to 0} \left(\frac{\sin x}{x}\right)^{1/x^2}$

$\Rightarrow \log y = \frac{1}{x^2} \lim_{x \to 0} \frac{1}{x^2} \log\left(\frac{\sin x}{x}\right)$

$= \lim_{x \to 0} \left[\frac{\frac{1}{\sin x} \cdot \cos x - \frac{1}{x}}{2x}\right]$

$$= \lim_{x\to 0} \frac{2\{x\sec^2 x + \tan x\} + 2\{x \cdot 3\tan^2 x \cdot \sec^2 x + 2\tan x \cdot \sec^2 x + \tan^3 x\}}{\sec^2 x + 2x\sec x \cdot \sec x \tan x + \sec^2 x}$$

$$= \frac{2\times 0 + 2\times 0 + 12\times 0 + 0}{2.1 + 2\times 0}$$

$$= \frac{0}{2} = 0$$

$\therefore \log y = 0$

$\therefore \lim_{x\to 0} \left(\frac{\tan x}{x}\right)^{\frac{1}{x}} = e^0 = \underline{1}$

(4) $\lim_{x\to 0} \log(\cos x)^{\frac{1}{x^2}}$

$= \lim_{x\to 0} \frac{1}{x^2} \log(\cos x)$ $\left[\frac{0}{0}\right]$

$= \lim_{x\to 0} \frac{-\tan x}{2x}$ [L' Hospital]

$= -\frac{1}{2} \lim_{x\to 0} \frac{\tan x}{x}$ $\left[\frac{0}{0}\right]$

$= -\frac{1}{2} \lim_{x\to 0} \frac{\sec^2 x}{1} = -\frac{1}{2}$

(5) $\lim_{x \to \infty} \frac{x^4}{e^x}$

$= \lim_{x \to \infty} \frac{4x^3}{e^x} \quad \left[\frac{\infty}{\infty}\right]$

$= \lim_{x \to \infty} \frac{12x^2}{e^x} \quad [\text{''}]$

$= \lim_{x \to \infty} \frac{24x}{e^x} \quad [\text{''}]$

$= \lim_{x \to \infty} \frac{24}{e^x}$

$= \frac{24}{e^{\infty}} = \frac{24}{1} \times \frac{0}{1}$

$= 0$ (Ans)

(6) $\lim_{x \to 0} \frac{4^x - 2^x}{x}$

$= \lim_{x \to 0} \frac{4^x . \ln 4 - 2^x . \ln 2}{1}$

$= 4^0 . \ln 4 - 2^0 . \ln 2$

$= \ln 4 - \ln 2$

$= \ln\left(\frac{4}{2}\right) = \ln 2$ (Ans)

(3)

n.

(7) $\lim_{x \to \infty} \frac{x + \cos x}{1 + x}$ $\left[\frac{\infty}{\infty} \text{ form}\right]$ $\frac{x + 3x}{x}$

$= \lim_{x \to \infty} \frac{1 - \sin x}{}$ [L' Hospital]

$= 1 - [-1, 1]$ (Ans)