VIII

MAXIMA AND MINIMA

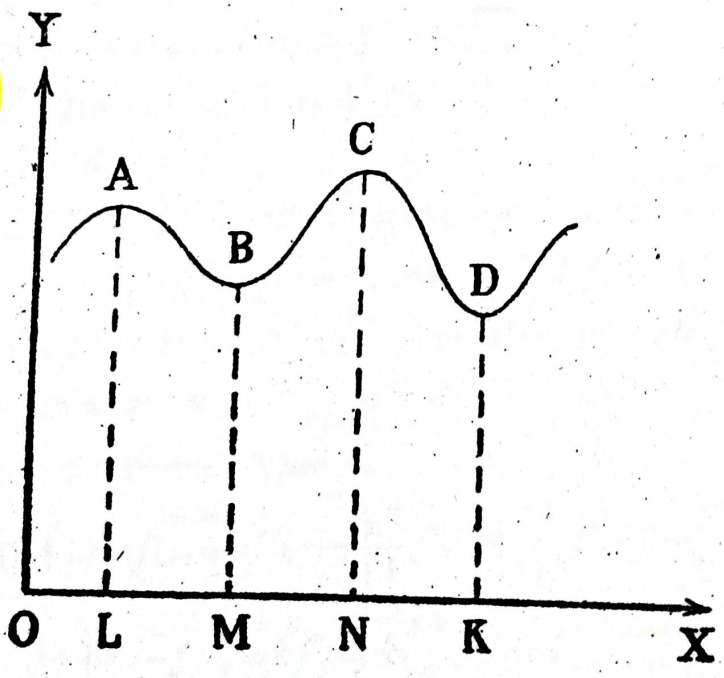
VIII-1. Definition—A function is said to have a maximum for a value a of x, if $f(a)$ is greater than any other value that the function can have in the small neighbourhood of $x=a$.

Similarly, a function $f(x)$ is said to have a minimum for a value a of x, if $f(a)$ is less than any other value that the function can have in the small neighbourhood of $x=a$.

Rigorous definition—A function $f(x)$ is said to have a maximum for a value a of x, if there exists a quantity ϵ, however small, such that $f(x)-f(a)$ is *negative* for all values of x for which $0 < |x-a| < \epsilon$.

Similarly, a function $f(x)$ is said to have a minimum for a value a of x, if there exists a quantity ϵ, however small, such that $f(x)-f(a)$ is *positive* for all values of x for which $0 < |x-a| < \epsilon$.

The following figure represents the graph of some function $f(x)$. It is seen that at A and C the ordinate ceases to increase and if at A, $x=a$, we find that $AL=f(a)$ and if it is greater than all other ordinates *i. e.* values of $f(x)$ in the neighbourhood of A *i. e.* $x=a$. Thus from the above definition $f(x)$ is said to have a maximum at A, C etc.



Moreover, it is clear from the above figure, that the ordinate at A is not the greatest. Therefore the maximum value of $f(x)$ is not

necessarily the greatest value of $f(x)$. A function may have several maximum value *i.e.*, a curve may have several maxima.

Similar is the case of minimum values of $f(x)$. In the fig. B, $D\cdots$ are points where the ordinates are less than those in the small neighbourhood of these points and as such from definition we find that $f(x)$ has minimum values at B, D,$\cdots$etc.

Moreover, it is clear from the fig. that the ordinates at B is not the smallest. **Therefore the minimum value of $f(x)$ is not necessarily the smallest value of $f(x)$** *i.e.*, a curve may have several minima.

Note. Also it is evident from the fig. that the tangents at A, B, C, $D\cdots$are parallel to x-axis *i.e.* $dy/dx=0$ and sign of dy/dx (remembering that $dy/dx=\tan\Psi$) changes from $+$ to $-$ as x passes through the values which make $f(x)$ a maximum and from $-$ to $+$ as x passes through the values which make $f(x)$ a minimum.

VIII-2. Conditions for maxima and minima. Let $f(x)$ be a given function which can be expanded in the neighbourhood of $x=a$ by Taylor's Theorem.

At $x=a$, the value of $f(x)$ is $f(a)$.

Let us consider two values of x, viz, $a+h$ and $a-h$ in the neighbourhood and on either side of $x=a$, h being very small.

Then if there is a *maximum* at $x=a$, then from definition $f(a)>f(a+h)$ and $f(a)>f(a-h)$

i.e. $f(a+h)-f(a)$ and $f(a-h)-f(a)$ are both negative for maximum at $x=a$.

And if there is a *minimum* at $x=a$, then from definition,

$$f(a)<f(a+h) \text{ and } f(a)<f(a-h)$$

i.e. $f(a+h)-f(a)$ and $f(a-h)-f(a)$ are both positive for minimum at $x=a$.

Now by Taylor's Theorem,

$$f(a+h)=f(a)+hf'(a)+\frac{h^2}{\lfloor 2}f''(a)+\frac{h^3}{\lfloor 3}f'''(a)+\frac{h^4}{\lfloor 4}f^{iv}(a)+\cdots\cdots$$

$$\text{or, } f(a+h)-f(a)=hf'(a)+\frac{h^2}{\lfloor 2}f''(a)+\frac{h^3}{\lfloor 3}f'''(a)+\frac{h^4}{\lfloor 4}f^{iv}(a)+\cdots\cdots \text{ (i)}$$

Similarly, $f(a-h)=f(a)-hf'(a)+\frac{h^2}{\lfloor 2}f''(a)-\frac{h^3}{\lfloor 3}f'''(a)+\ldots\ldots$

or, $f(a-h)-f(a)=-hf'(a)+\frac{h^2}{\lfloor 2}f''(a)-\frac{h^3}{\lfloor 3}f'''(a)+\ldots$ ... (ii)

or, If h is very small, we may neglect powers of h higher than the first and so from (i) and (ii) we get

$$f(a+h)-f(a)=hf'(a) \quad \ldots \quad \ldots \text{ (iii)}$$

and $f(a-h)-f(a)=-hf'(a)$ ... ... (iv)

Also we have seen that for maximum or minimum the sign of $f(a+h)-f(a)$ and $f(a-h)-f(a)$ must be the same. So from (iii) and (iv) we conclude that if $f(a+h)-f(a)$ and $f(a-h)-f(a)$ have the same sign, then from (iii) and (iv), $f'(a)$ should be zero, otherwise they will have different signs.

Hence the necessary condition that $f(x)$ should have a maximum at $x=a$ is $f'(a)=0$.

Now if $f'(a)=0$, we have from (i) and (ii)

$$f(a+h)-f(a)=\frac{h^2}{\lfloor 2}f''(a) \quad \ldots \quad \ldots \text{ (v)}$$

and $f(a-h)-f(a)=\frac{h^2}{\lfloor 2}f''(a)$ ... ... (vi)

neglecting powers of h higher than the second, (since h is very small.

And we find that $f(a+h)-f(a)$ and $f(a-h)-f(a)$ are both of the same sign.

Now two cases arise—

(i) When $f''(a)$ is positive. In this case both $f(a+h)-f(a)$ and $f(a-h)-f(a)$ are positive and hence there is minimum at $x=a$.

(ii) When $f''(a)$ is negative. In this case both $f(a+h)-f(a)$ and $f(a-h)-f(a)$ are negative and hence there is a maximum at $x=a$.

Hence we conclude that—

(i) the function $f(x)$ is maximum at $x=a$, if $f'(a)=0$ and $f''(a)$ is negative and (ii) the function $f(x)$ is minimum at $x=a$, if $f'(a)=0$ and $f''(a)$ is positive.

Generalisation. If $f'(a)=0$ and $f''(a)=0$, then from (i) and (ii) we get

$$f(a+h)-f(a)=\frac{h^3}{\lfloor 3} f'''(a) \quad \cdots \quad \cdots \text{ (vii)}$$

$$\text{and } f(a-h)-f(a)=-\frac{h^3}{\lfloor 3} f'''(a) \quad \cdots \quad \cdots \text{ (viii)}$$

neglecting powers of h higher than the third.

For maximum or minimum value of $f(x)$ at $x=a$, we know that $f(a+h)-f(a)$ and $f(a-h)-f(a)$ must have the same sign and for that $f'''(a)$ should be zero, otherwise they will have opposite sign as is evident from (vi) and (vii).

Hence if $f'''(a)$ is also zero, we get from (i) and (ii)

$$f(a+h)-f(a)=\frac{h^4}{\lfloor 4} f^{iv}(a)$$

and $f(a-h)-f(a)=\frac{h^4}{\lfloor 4} f^{\text{iv}}(a)$ and these are of the same sign.

$\therefore$ If $f(x)$ is maximum at $x=a$ we know both $f(a+h)-f(a)$ and $f(a-h)-f(a)$ are negative and hence $f^{iv}(a)$ must be negative.

i.e., if $f'(a)=0$, $f''(a)=0$ then function $f(x)$ is maximum at $x=a$, provided $f'''(a)=0$ and $f^{iv}(a)$ is negative.

Similarly if $f'(a)=0$, $f''(a)=0$, then function $f(x)$ is minimum at $x=a$, if $f'''(a)=0$ and $f^{iv}(a)$ is positive.

VIII-3. Working Rule :

(1) If a function $f(x)$ be given, find $f'(x)$ and equate it to zero. Solve this equation for x. Let its roots be $a_1, a_2, a_3 \cdots$

(2) Find $f''(x)$ and hence find $f''(a_1), f''(a_2), f''(a_3) \ldots$

(3) If $f''(a_1)$ is negative we have a maximum value at $x=a_1$

(4) If $f''(a_1)$ is positive we have a minimum value at $x=a_1$

(5) If $f''(a_1)=0$, find $f'''(x)$ and then $f'''(a_1)$

(6) If $f'''(a_1)\neq 0$, then there is neither maxima nor minima at $x=a$.

If $f'''(a_1)=0$, find $f^{iv}(x)$ and then $f^{iv}(a_1)$. If $f^{iv}(a_1)$ is negative then $f(x)$ is maximum at $x=a_1$ and if $f^{iv}(a_1)$ is positive then $f(x)$ is minimum at $x=a_1$

If $f^{iv}(a_1)=0$, then find $f^{v}(a_1)$ and so on.

Examples Worked Out

Ex. 1. Find the maximum and minimum values of

$$2x^3-9x^2+12x-3$$

Solution : Let $y=2x^3-9x^2+12x-3$

$$\therefore \quad \frac{dy}{dx}=6x^2-18x+2 \text{ and } \frac{d^2y}{dx^2}=12x-18.$$

Now for maxima or minima, $\frac{dy}{dx}=0$, $\therefore \quad 6x^2-18x+12=0$

or, $6(x^2-3x+2)=0$, or, $6(x-1)(x-2)=0$, whence $x=1$ or, 2.

For $x=1$, $\frac{d^2y}{dx^2}=12\times1-18=-6$, which is negative.

Hence $x=1$ corresponds to a maximum value.

The maximum value $=2.1^3-9.1^2+12.1-3=2$.

For $x=2$, $\frac{d^2y}{dx^2}=12.2-18=6$, which is positive.

Hence $x=2$ corresponds to a minimum value.

The minimum value $=2.2^3-9.2^2+12.2-3=1$.

Ex. 2. For what values of x, the following expression is maximum or minimum respectively : $2x^3-21x^2+36x-20$.

Find also the maximum and minimum values of the expression.

[Calcutta, '36]

Solution : Let $f(x)=2x^3-21x^2+36x-20$

$\therefore \quad f'(x)=6x^2-42x+36$, which exists for all values of x.

Now for maxima or minima, $f'(x)=0$

$\therefore \quad 6x^2-42x+36=0$; or, $x^2-7x+6=0$

or, $(x-1)(x-6)=0$; $\therefore \quad x=1$ or 6.

Again $f''(x)=12x-42=6(2x-7)$.

Now when $x=1$, $f''(x)=-30$ which is negative

When $x=6$, $f''(x)=30$ which is positive.

Hence $x=1$ correspond to a maximum value of the expression and $x=6$ corresponds to a minimum value.

The maximum value of $f(x)=2.1^3-21.1^2+36.1-20=-3$.

The minimum value of $f(x)=2.6^3-21.6^2+36.6-20=-128$.

Ex. 3. Investigate for what values of x $f(x)=5x^6-18x^5+15x^4-10$ is a maximum or minimum.

Solution : Let $f(x)=5x^6-18x^5+15x^4-10$

$\therefore\ f'(x)=30(x^5-3x^4+2x^3)$

Putting $f'(x)=0$, we have, $30(x^5-3x^4+2x^3)=0$

or, $x^3(x-1)(x-2)=0$; $\therefore\ x=0,\ 1,\ 2$

Again $f''(x)=30(5x^4-12x^3+6x^2)$

When $x=1$, $f''(x)$ is negative and hence $f(x)$ is a maximum for $x=1$.

When $x=2$, $f''(x)$ is positive, and hence $f(x)$ is a minimum at $x=2$.

When $x=0$, $f''(x)$ is zero, so the test fails, and we are to examine higher order derivatives.

Now $f'''(x)=120(5x^3-9x^2+3x)$; $\therefore\ f'''(x)=0$

$f^{iv}(x)=360(5x^2-6x+1)$; $\therefore\ f^{iv}(0)$ is positive

Since even order derivative is positive for $x=0$

$\therefore$ for $x=0$, $f(x)$ is minimum.

Ex. 4. Show that $x^5-5x^4+5x^3-10$ has a maximum when $x=1$, a minimum when $x=3$ and neither when $x=0$.

Solution : Let $f(x)=x^5-5x^4+5x^3-10$

$f'(x)=5x^4-20x^3+15x^2$

$f''(x)=20x^3-60x^2+30x$ ... ... (i)

and $f'''(x)=60x^2-120x+30$ ... ... (ii)

Putting $f'(x)=0$, we get $5x^4-20x^3+15x^2=0$

or, $5x^2(x^2-4x+3)=0$, or, $5x^2(x-3)(x-1)=0$

$\therefore\ x=0,\ 1,\ 3.$

When $x=3$, $f''(3)=20.3^3-60.3^2+30.3$ from (i)

$=540-540+90=$positive.

$\therefore$ When $x=3$, $f(x)$ is minimum.

When $x=1$, $f''(1)=20.1^3-60.1^2+30.1$ from (i)

$=20-60+30=$negative.

$\therefore$ when $x=1$, the function $f(x)$ is maximum.

When $x=0$, $f''(0)=0$ from (i) putting $x=0$

$\therefore\ f'''(0)=60.0-120.0+30$ from (ii)

$=30 \neq 0$

Hence there is neither a maximum or a minimum at $x=0$.

Ex. 5. Show that x^3-3x^2+3x+7 has neither a maximum nor a minimum at $x=1$.

Solution : Let $f(x)=x^3-3x^2+3x+7$

$\therefore\ f'(x)=3x^2-6x+3,\ f''(x)=6x-6$ and $f'''(x)=6$

Putting $f'(x)=0$, we get $3(x^2-2x+1)=0$

or, $3(x-1)^2=0$, or, $x=1$

Also at $x=1$, $f''(1)=6-6=0$; $f'''(1)=6 \neq 0$

Hence the given function has neither a maximum nor a minimum at $x=1$.

Ex. 6. Investigate the maxima for the curve

$$y=(x-2)^6(x-3)^5$$

Solution : $y=(x-2)^6(x-3)^5$

$$\therefore\ \frac{dy}{dx}=(x-2)^6.5.(x-3)^4+6(x-2)^5(x-3)^5$$

$$=(x-2)^5(x-3)^4[5(x-2)+6(x-3)]$$

$$=(x-2)^5(x-3)^4(11x-28) \quad \ldots \quad \text{(i)}$$

Putting $\frac{dy}{dx}=0$, we get $x=2,\ 3,\ \frac{28}{11}$

Also we know that at a maximum the sign of $\frac{dy}{dx}$ changes from $+$ to $-$ and at a minimum the sign of $\frac{dy}{dx}$ changes from $-$ to $+$.

When $x=2$,

the value of $\frac{dy}{dx}$ just after $x=2$ is obtained by putting $(2+h)$ for x and is equal to

$$\underset{x\to 2+0}{Lt}\left(\frac{dy}{dx}\right)=\underset{h\to 0}{Lt}\ (2+h\quad 2)^5(2+h-3)^4\{11(2+h)-28\}$$

$=$ negative.

And the value of $\frac{dy}{dx}$ just preceding $x=2$ is obtained by putting $(2-h)$ for x and is equal to

$$\underset{x\to 2-0}{Lt}\left(\frac{dy}{dx}\right)=\underset{h\to 0}{Lt}\,[(2-h-2)^5(2-h-3)^4\{11(2-h)-28\}]$$

$$=\text{positive.}$$

$\therefore$ dy/dx changes from positive to negative as it passes through $x=2$, so there is a maxima at $x=2$.

At $x=3$, we find $\frac{dy}{dx}$ does not change sign *i.e.*, there is neither maxima nor minima at $x=3$.

At $x=28/11$, we shall find that dy/dx changes sign from negative to positive, hence there is a minima at $x=28/11$.

Ex. 7. Obtain the maximum and minimum values of the expression $\frac{x^2-3x-4}{x-8}$.

Solution : Let $y=\frac{x^2-3x-4}{x-8}$

$$\therefore \quad \frac{dy}{dx}=\frac{(2x-3)(x-8)-(x^2-3x-4)}{(x-8)^2}$$

$$=\frac{x^2-16x+28}{(x-8)^2}=\frac{(x-2)(x-14)}{(x-8)^2}.$$

Putting $\frac{dy}{dx}=0$ $\qquad x=2$ or, 14.

Now (i) when x is a little less than 2, $\frac{dy}{dx}$ is positive, and when x is a little greater than 2, dy/dx is negative. Here dy/dx changes sign from positive to negative.

$\therefore$ y is a maxima when $x=2$.

In this case, $y=\frac{4-6-4}{2-8}=\frac{-6}{-6}=1$

(ii) When x is a little less than 14, dy/dx is negative and when x is a little greater than 14, dy/dx is positive.

Here dy/dx changes sign from negative to positive.

$\therefore$ y is a minimum when $x=14$

In this case $y=\frac{196-42-4}{14-8}=\frac{150}{6}=25.$

Ex. 8. Show that there is a point of inflexion at $x=0$ on the curve : $y=x^5-5x^4+5x^3$ and find also for what value of x, y is maximum or minimum.

Solution : Here $dy/dx=5x^4-20x^3+15x^2=5x^2(x-1)(x-3)$.

Putting $dy/dx=0$, we have $5x^2(x-1)(x-3)=0$, which gives $x=0$, $x=1$ and $x=3$.

If x be a little <0, $dy/dx=(+)(-)(-)=$positive and if x be a little >0, $dy/dx=(+)(-)(-)=$positive *i. e.* dy/dx does not change sign.

$\therefore$ $x=0$ gives a point of inflexion *i.e.* there is neither a maximum nor minimum value.

If x be a little less than 1, dy/dx is positive and if x be a little greater than 1, dy/dx is negative.

Thus dy/dx changes sign from positive to negative and $x=1$ gives a maximum value.

If x be a little less than 3, dy/dx is negative.

If x be a little greater than 3, dy/dx is positive.

Thus dy/dx changes sign from negative to positive and hence $x=3$ gives a minimum value.

Ex. 9. Find the maximum and minimum values of the polynomial $8x^5-15x^4+10x^3$.

Solution : Let $f(x)=8x^5-15x^4+10x^3$

$$\therefore \quad f'(x)=40x^4-60x^3+20x=20x(2x^3-3x^2+1)$$
$$=20x(x-1)^2(2x+1)$$

Putting $f'(x)=0$, $\quad\therefore\quad x=0,\ 1,\ -\frac{1}{2}$.

Again $f''(x)=160x^3-180x^2+20=20(8x^3-9x^2+1)$

Now $f''(-\frac{1}{2})=-45$, which is negative, so that $f(-\frac{1}{2})=\frac{21}{16}$ is a maximum value.

Again $f''(0)=20$ which is positive so that $f(0)=0$ is a minimum value.

As $f''(1)=0$, we are to test for higher derivative $f'''(1)$

Now $f'''(x)=480x^3-360x$, $\quad\therefore\quad f'''(1)=120\neq 0$.

Hence $f(1)$ is neither a maximum nor a minimum value.

Ex. 10. Examine the function $e^x+2\cos x+e^{-x}$ for maximum values.

Solution : Let $f(x)=(e^x+e^{-x})+2\cos x=2(\cosh x+\cos x)$

$\therefore\ f'(x)=2(\sinh x-\sin x)$

Putting $f'(x)=0$, we get $x=0$

Now $f''(x)=2(\cosh x-\cos x)$

At $x=0$, $f''(0)=2(1-1)=0$.

Then $f'''(x)=2(\sinh x+\sin x)$. At $x=0, f'''(0)=2.0=0$

$\therefore\ f^{iv}(x)=2(\cosh x+\cos x)$.

At $x=0$, $f^{iv}(0)=2(1+1)=4=$positive.

$\therefore$ At $x=0$ there is minima.

Ex. 11. Show that the maximum value of $x+\frac{1}{x}$ is less than the minimum value.

Solution : Let $y=x+\frac{1}{x}$, $\therefore\ \frac{dy}{dx}=1-\frac{1}{x^2}$. Now for y to be maximum or minimum $\frac{dy}{dx}=0$.

$$\therefore\ 1-\frac{1}{x^2}=0,\quad \text{or,}\quad x^2=1,\quad \therefore\ x=+1 \text{ and } -1.$$

Now $\frac{d^2y}{dx^2}=0+\frac{2}{x^3}=\frac{2}{x^3}$ $\therefore$ when $x=1$, $\frac{d^2y}{dx^2}=\frac{2}{1}=2=$positive.

Hence $x=1$ gives the function a minimum value to y

$\therefore$ the minimum value $=1+1=2$.

When $x=-1$, $\frac{d^2y}{dx^2}=\frac{-2}{1^3}=-2$ negative. Hence $x=-1$ gives y a maximum value. The maximum value $=-1-1=-2$

$\therefore$ the maximum value of $y=-2$ and the minimum value of $y=2$ and hence the maximum value of x is less than its minimum value.

Ex. 12. Find the maximum and minimum value of $a\sin\theta+b\cos\theta$.

Solution : Let $y=a\sin\theta+b\cos\theta$, $\frac{dy}{d\theta}=a\cos\theta-b\sin\theta$.

Putting $\frac{dy}{d\theta}=0$, **we get** $a\cos\theta=b\sin\theta$; *i.e*, $\tan\theta=\frac{a}{b}$.

$$\therefore \quad \frac{\sin\theta}{a}=\frac{\cos\theta}{b}=\frac{\sqrt{\sin^2\theta+\cos^2\theta}}{\sqrt{(a^2+b^2)}}=\pm\frac{1}{\sqrt{(a^2+b^2)}}$$

$$\therefore \quad \sin\theta=\frac{a}{\sqrt{(a^2+b^2)}}\ ;\ \cos\theta=\frac{b}{\sqrt{(a^2+b^2)}} \quad \ldots \quad \text{(i)}$$

$$\text{or,} \quad \sin\theta=\frac{-a}{\sqrt{(a^2+b^2)}}\ ;\ \cos\theta=\frac{-b}{\sqrt{(a^2+b^2)}} \quad \ldots \quad \text{(ii)}$$

$\therefore$ y is maximum or minimum for the values of θ given by (i) and (ii).

Now $$\frac{d^2y}{d\theta^2}=-a\sin\theta-b\cos\theta=-a\cdot\frac{a}{\sqrt{(a^2+b^2)}}-b\cdot\frac{b}{\sqrt{(a^2+b^2)}}$$

$$=-\frac{a^2+b^2}{\sqrt{(a^2+b^2)}}=-\sqrt{(a^2+b^2)} \text{ which is negative.}$$

$\therefore$ y is maximum for the values of θ given by (i) and (ii),

and $\frac{d^2y}{d\theta^2}$ = positive for the values of θ given by (ii).

$$\therefore \quad \text{maximum value of } y=a.\frac{a}{\sqrt{(a^2+b^2)}}+b.\frac{b}{\sqrt{(a^2+b^2)}}$$

$$=\sqrt{(a^2+b^2)},$$

$$\text{and minimum value of } y=a.\frac{-a}{\sqrt{(a^2+b^2)}}+b.\frac{-b}{\sqrt{(a^2+b^2)}}$$

$$=-\sqrt{(a^2+b^2)}.$$

Ex. 13. Show that x^x is minimum for $x=\frac{1}{e}$.

Solution : Let $y=x^x$, $\therefore$ $\log y=x\log x$.

Differentiating both sides w.r.t. x,

$$\frac{1}{y}\frac{dy}{dx}=1+\log x, \quad \text{or,} \quad \frac{dy}{dx}=y(1+\log x)$$

$$\frac{d^2y}{dx^2}=\frac{dy}{dx}(1+\log x)+y.\frac{1}{x}$$

$$\therefore \quad \frac{d^2y}{dx^2}=y(1+\log x)^2+\frac{y}{x}.$$

$$\therefore \quad \frac{d^2y}{dx^2}=y\left\{(1+\log x)^2+\frac{1}{x}\right\}=x^x\left\{(1+\log x)^2+\frac{1}{x}\right\}$$

Putting $\frac{dy}{dx}=0$, we have $y(1+\log x)=0$

But y cannot be zero, $\therefore \quad 1+\log x=0, \quad$ or $\quad \log x=-1$

$$\therefore \quad x=e^{-1}=\frac{1}{e}.$$

Substituting this value of x in $\frac{d^2y}{dx^2}$ we have,

$$\frac{d^2y}{dx^2}=\left(\frac{1}{e}\right)^{1/e} \text{ which is positive.}$$

Hence x^x is a minimum for $x=\frac{1}{e}$.

Ex. 14. Find the maximum value of $\sin 2x-x$.

Solution : Let $f(x)=\sin 2x-x$

$$\therefore \quad f'(x)=2\cos 2x-1 \text{ and } f''(x)=-4\sin 2x$$

Putting $f'(x)=0, \quad \therefore \quad 2\cos 2x-1=0, \quad$ or, $\quad \cos 2x=\frac{1}{2}$

$\therefore \quad 2x=2n\pi\pm\frac{\pi}{3}$ (writing down the general value)

Substituting, $n\pi+\frac{\pi}{6}$ for x in $f''(x)$, we have

$$f''(x)=-4\sin(2n\pi\pm\tfrac{\pi}{3})=-4\sin\tfrac{\pi}{3}=-4\times\frac{\sqrt{3}}{2}, \text{ which is negative.}$$

Hence the given function is a maximum for $x=n\pi+\frac{\pi}{6}$.

Substituting, $n\pi-\frac{\pi}{6}$ in $f''(x)$, we have

$$f''(x)=-4\sin(2n\pi-\tfrac{\pi}{3})=-4\times(-\sin\tfrac{\pi}{3})=4\times\frac{\sqrt{3}}{2}, \text{ which is positive.}$$

Hence the given function is a minimum for $x=n\pi-\frac{\pi}{6}$.

Ex. 15. Find the maximum value of $\frac{\log x}{x}$ in $0<x<\infty$.

Solution : Let $f(x)=\dfrac{\log x}{x}$

$$\therefore \quad f'(x)=\frac{x(\frac{1}{x})-(\log x)(1)}{x^2}=\frac{1-\log x}{x^2}$$

Putting $f'(x)=0$, we get $1-\log x=0$, or, $\log x=1=\log e$, or $x=e$.

$$\text{Again } f''(x)=\frac{x^2(-\frac{1}{x})-(1-\log x)\,2x}{x^4}$$

$$\text{and at } x=e, \quad f''(x)=\frac{-e-(1-1)2e}{e^4}=\text{negative.}$$

Hence there is a maximum at $x=e$ and the maximum values of

$$f(x)=\frac{\log e}{e}=\frac{1}{e}.$$

VIII-4. Applications to Problems.

The subject maxima and minima is one of the most important branch in Calculus, and has various practical applications to Geometry, Physics and other sciences.

In solving a problem in which the maximum or a minimum value of a quantity is needed, we should always be guided by the following important principle :

Express the quantity, of which the maximum or minimum value is required, as a function of one unknown.

In some cases it may be necessary to express the quantity in terms of two or more unknowns, and then by means of a given condition to express all these unknowns in terms of one of them.

In connection with the problems concerning *spheres, cones* and *cylinder*, the following results would be often needed and as such should be **committed to memory.**

1. *Sphere of Radius* r.

   Volume$=\frac{4}{3}\pi r^3$. Surface$=4\pi r^2$

2. *Circular cylinder of height* h *and radius of base* r.

   Volume$=\pi r^2 h$, Curved surface$=2\pi rh$.

   The area of each plane face$=\pi r^2$.

3. *Circular cone of height* h, *and radius of base* r.

   Semi-vertical angle$=\tan^{-1}(r/h)$, Slant height$=\sqrt{(r^2+h^2)}$.

   Volume$=\frac{1}{3}\pi r^2 h$, Curved surface$=\pi r\sqrt{(r^2+h^2)}$.

Here cones and cylinders are always supposed to be right circular.

Ex. 16. Divide a number 15 into two parts such that the square of one multiplied with the cube of the other is a maximum:

Solution : Let one of the parts be x, then other part is $15-x$.

Let $P=x^2(15-x)^3$, $\therefore$ $\frac{dP}{dx}=2x(15-x)^3-3x^2(15-x)^2$.