Mean Value theorem

If the function $y = f(x)$ is

1) continuous in the closed interval $[a, b]$

2) differentiable in the open interval (a, b)

there exists a value of c of x between a and b such that

$$f(b) - f(a) = (b-a)\, f'(c)$$

First Mean Value Theorem (Lagrange)

Statement: Let $f(x)$ be a function such that

(I) $f(x)$ is continuous in $a \le x \le b$

II) $f'(x)$ exists in $a < x < b$, Then there exist at least one value c, $a < c < b$, such that $\frac{f(b)-f(a)}{b-a} = f'(c)$

Proof: Let $\varphi(x) = f(x) - Ax$

where A is constant which is to be chosen such that

$$\varphi(a) = \varphi(b)$$

ie $f(a) - A(a) = f(b) - A(b)$

or $$\frac{f(b)-f(a)}{b-a} = A \qquad \text{---------(1)}$$

Now $\varphi(x)$ is a function such that $\varphi(a) = \varphi(b)$ and also $\varphi(x)$ being the difference of $f(x)$ and Ax satisfies the conditions of mean value theorem for $f(x)$ and as such by Rolle's theorem $\varphi'(c) = 0$ where $a < c < b$.

Now $\varphi'(x) = f'(x) - A$

$$= f'(x) - \frac{f(b)-f(a)}{b-a} \quad [\text{by (1)}]$$

$$\therefore \varphi'(c) = f'(c) - \frac{f(b)-f(a)}{b-a}$$

or $$0 = f'(c) = \frac{f(b)-f(a)}{b-a} \quad [\varphi'(c) = 0]$$

$$f(b) - f(a) = (b-a) f'(c)$$

Test whether the Mean-value theorem is true for $f(x) = 3 + 2x - x^2$ on $(0, 1)$

Solution:

Given function $f(x) = 3 + 2x - x^2$.

$$\therefore \text{R } f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{3 + 2(x+h) - (x+h)^2 - 3 - 2x + x^2}{h}$$

$$= \lim_{h \to 0} \frac{3 + 2x + 2h - x^2 - 2xh - h^2 - 3 - 2x + x^2}{h}$$

$$= \lim_{h \to 0} \frac{2h - 2xh - h^2}{h}$$

$$= \lim_{h \to 0} (2 - 2x - h)$$

$$= 2 - 2x$$

$$\text{and } \text{L } f'(x) = \lim_{h \to 0} \frac{f(x-h) - f(x)}{(-h)}$$

$$= \lim_{h \to 0} \frac{3 + 2(x-h) - (x-h)^2 - 3 - 2x + x^2}{(-h)}$$

$$= \lim_{h \to 0} \frac{3 + 2x - 2h - x^2 + 2xh - h^2 - 3 - 2x + x^2}{(-h)}$$

$$= \lim_{h \to 0} \frac{2h - 2xh - h^2}{h}$$

$$= \lim_{h \to 0} (2 - 2x - h)$$

$$= 2 - 2x$$

$\therefore$ $Rf'(x) = Lf'(x)$ for $\forall x$

$\therefore$ $f(x)$ is differentiable on $(0,1)$.

so $f(x)$ is continuous on $[0,1]$.

since $f(x)$ satisfies all the conditions of Mean Value theorem, there exist at least one point $c \in (0,1)$ such that

$$f'(c) = \frac{f(1) - f(0)}{1 - 0} \quad \text{---------- (1)}$$

Now $f(1) = 3 + 2 - 1 = 4$

$f(0) = 3 + 0 - 0 = 3$

$f'(x) = 2 - 2x$

$\therefore$ (1) $\Rightarrow$ $2 - 2c = \frac{4 - 3}{1 - 0}$

or $2 - 2c = 1$

or $-2c = 1 - 2$

or $c = \frac{1}{2}$

$\therefore$ $c = \frac{1}{2} \in (0,1)$

Therefore, the Mean Value Theorem is applicable to the given function on $(0,1)$