Rolle's theorem

If a function $y = f(x)$ is continuous in the closed interval $[a, b]$ and differentiable in the open interval (a, b) and $f(a) = f(b)$ then there exists at least a value c of x between a and b such that $f'(c) = 0$.

State and Prove Rolle's Theorem

Statement : Let $f(x)$ be a function (i) continuous in the closed interval $[a, b]$ ie $a \leq x \leq b$. (ii) $f(x)$ is derivable in the open interval (a, b) ie $a < x < b$ and (iii) $f(a) = f(b)$, then there exist at least one value 'c' $a < c < b$ such that $f'(c) = 0$

Proof: By virtue of the continuity of the function in the closed interval $a \leq x \leq b$, it has a greatest value M and a least value m, in the interval, so that there are two numbers c and d such that,

$$f(c) = M, \quad f(d) = m$$

Now either $M = m$ ——— (i)

or $M \neq m$ ——— (ii)

When the greatest value coincides with the least value as in the case(i), the function reduces to a constant so that the derivative is equal to 0 for every value of x and therefore the theorem is true in this case.

when M and m are unequal, as in case(ii) at least one of them must be different from the equal values $f(a)$, $f(b)$. Let $M = f(c)$ be different from them. the number c being different from a and b lies within the interval $[a, b]$ and as such belongs to the open interval (a, b). The function $f(x)$ which is derivable in the open interval (a, b) is, in particular derivable at c, so that

$$\lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$

exists and is the same when $h \to 0$ through positive or negative values.

As $f(c)$ is the greatest value of the function, we have

$$f(c+h) \leq f(c)$$

whatever positive or negative values h has. Thus

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$$\begin{cases} \dfrac{f(c+h)-f(c)}{h} \leq 0 \text{ for } h>0 \quad \text{——(III)} \\ \dfrac{f(c+h)-f(c)}{h} \geq 0 \text{ for } h<0 \quad \text{——(IV)} \end{cases}$$

Let $h \to 0$ through positive values

From (III) we get $f'(c) \leq 0$ —— (V)

Let $h \to 0$ through negative values

From (IV) we get $f'c \geq 0$ —— (VI)

the relations (V) and (VI) will both be true, if and only if $f'(c) = 0$.

The same conclusion would be similarly reached if it is least value which differs from $f(a)$ and $f(b)$.

Hence the theorem.

Verify Rolle's theorem for $f(x) = x^2$ in the interval $(-1, 1)$

solution:

Given function,

$$f(x) = x^2$$

$$\therefore f(-1) = (-1)^2 = 1$$

$$f(1) = 1^2 = 1$$

$$\therefore f(-1) = f(1)$$

Now, $Rf'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$$= \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h}$$

$$= \lim_{h \to 0} \frac{2xh + h^2}{h}$$

$$= \lim_{h \to 0} 2x + h$$

$$= 2x$$

and $Lf'(x) = \lim_{h \to 0} \frac{f(x-h) - f(x)}{(-h)}$

$$= \lim_{h \to 0} \frac{(x-h)^2 - x^2}{(-h)}$$

$$= \lim_{h \to 0} \frac{-2xh + h^2}{(-h)}$$

$$= \lim_{h \to 0} 2x - h = 2x$$

since $Rf'(x) = Lf'(x)$ $\forall x$

Therefore $f(x)$ is differentiable on $(-1,1)$

But, we know that every differentiable function is continuous, so $f(x)$ is continuous on $(-1,1)$.

Since $f(x)$ satisfies all the conditions of Rolle's theorem, there exist at least one point $c \in (-1,1)$ such that $f'(c) = 0$.

$$\Rightarrow 2c = 0 \quad [\because f'(x) = 2x]$$

$$\Rightarrow c = 0 \in (-1,1)$$

Therefore Rolle's theorem is applicable to the given function in the interval $(-1,1)$.

verify Rolle's theorem for $f(x) = |x|$ on $(-1, 1)$

Solution:

Given function,

$$f(x) = |x|$$

$\therefore$ $f(x)$ can be written as

$$f(x) = \begin{cases} x & \text{when } 0 \le x \le 1 \\ -x & \text{when } -1 < x < 0 \end{cases}$$

At the point $x = 0$

$$Rf'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{0+h-0}{h}$$

$$= \lim_{h \to 0} 1 = 1.$$

and $$Lf'(0) = \lim_{h \to 0} \frac{f(0-h) - f(0)}{-h}$$

$$= \lim_{h \to 0} \frac{-(0-h) - 0}{-h}$$

$$= \lim_{h \to 0} (-1) = -1$$

$$Rf'(0) \neq Lf'(0)$$

$f'(x)$ does not exist at the point

$x = 0 \in (-1, 1)$

Therefore Rolle's theorem is not applicable to the given function on $(-1, 1)$

Test whether Rolle's theorem is applicable for $f(x) = \sin x$ in the interval $(0, \pi)$

solution:

Given function,

$$f(x) = \sin x$$

$$f(0) = 0$$

$$f(\pi) = 0$$

$$\therefore f(0) = f(\pi)$$

$$\text{NOW } Rf'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h}$$

$$= \lim_{h \to 0} \frac{2 \cos \frac{2x+h}{2} \sin \frac{h}{2}}{h}$$

$$= \lim_{h \to 0} \cos\left(\frac{2x+h}{2}\right) \cdot \lim_{h \to 0} \frac{\sin h/2}{h/2}$$

$$= \cos\left(\frac{2x}{2}\right) \cdot 1$$

$$= \cos x$$

$$Lf'(x) = \lim_{h \to 0} \frac{f(x-h) - f(x)}{(-h)}$$

$$= \lim_{h \to 0} \cos\left(\frac{2x-h}{2}\right) \cdot \left(\frac{\sin h/2}{h/2}\right)$$

$$= \lim_{h \to 0} \cos\left(\frac{2x-h}{2}\right) \cdot \lim_{h \to 0} \left(\frac{\sin h/2}{h/2}\right)$$

$= \cos x . 1$

$= \cos x$

$\therefore \quad Rf'(x) = Lf'(x)$

$\therefore$ $f'(x)$ exists on $(0, \pi)$. But every differentiable function is continuous so $f(x)$ is continuous on $(0, \pi)$

Since $f(x)$ satisfies all the conditions of Rolle's theorem there exist at least one point $c \in (0, \pi)$ such that

$$f'(c) = 0$$

$$\Rightarrow \quad \cos c = 0 \quad [\because f'(x) = \cos x]$$

$$\Rightarrow \quad c = \frac{\pi}{2} \in (0, \pi)$$

Therefore Rolle's theorem is applicable to the given function.

Verify Rolle's theorem for $f(x) = \tan x$ on $(0, \pi)$

Solution:

Given function,

$f(x) = \tan x$

$f(0) = 0$

$f(\pi) = 0$

$\therefore \quad f(0) = f(\pi)$

But $f\left(\frac{\pi}{2}\right) = \tan\frac{\pi}{2}$ which does not exist.

when $\alpha \frac{\pi}{2} < \pi$

$\therefore$ $f(x)$ is not continuous on $(0, \pi)$

Therefore Rolle's theorem is not applicable to $f(x) = \tan x$ on $(0, \pi)$

Verify Rolle's theorem for x^2 in $(-1, 1)$

Solution: To verify Rolle's theorem the following conditions must be satisfied

(i) $f(a) = f(b)$

(ii) It is continuous in the closed interval $[a, b]$

(iii) It is derivable in the open interval (a, b)

Here let $f(x) = x^2$

$\therefore f(1) = f(-1)$ and as such condition (i) is satisfied.

As x^2 is derivable in the interval $(1, -1)$ and as such condition (iii) is satisfied.

It is also continuous in the interval satisfying the condition (ii)

Thus all the conditions of Rolle's theorem being satisfied the derivative $f'(x)$ must vanish for at least one value of x lying within $(1, -1)$

Here the derivative $2x$ vanishes for $x = 0$, a value which lies within $(-1, 1)$

Thus the theorem is verified.