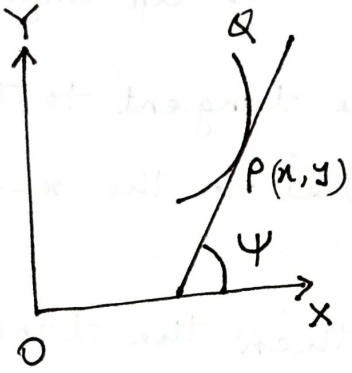
Tangent

1. the tangent at P to a given curve is defined as the limiting position of the secant PQ (when such a limit exists) as the point Q approaches P along the curve.



2. Derivative of any point on a curve is the slope of the tangent at the point (x, y) on the curve.

that is $\frac{dy}{dx} = \tan\Psi$ = slope of the tangent.

In a curve $y = f(x)$, if at a particular point (x, y)

a. $\frac{dy}{dx} = 1$, then the slope of the tangent at that point is 1, that is the tangent at that point of the curve makes an angle of 45° with the positive direction of x-axis.

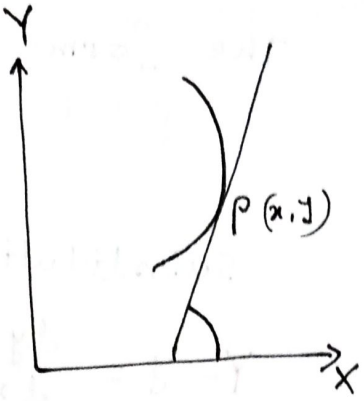
b. $\frac{dy}{dx}$ = negative, then the tangent to the curve at that point makes an obtuse angle with the positive direction of x-axis.

c. $\frac{dy}{dx} = 0$, then the tangent to the curve at that point makes an angle of 0° with the x-axis, ie the tangent to the curve at that point is parallel to the x-axis.

d. $\frac{dy}{dx} = \infty$, then the tangent to the curve at that point makes an angle of $\frac{\pi}{2}$ with the ~~angle of~~ x-axis ie the tangent to the curve at the point is perpendicular to the x-axis.

Equation of tangent at any point (x, y) of the curve $y = f(x)$

Let $P(x, y)$ be any point on the curve $y = f(x)$



The equation of any straight line passing through $P(x, y)$ is $Y - y = m(X - x)$

where $m = \tan\psi$, ψ being the angle which the straight line makes with the positive direction of the axis of x.

If this line which is inclined to the axis of x at an angle ψ, is to be a tangent to the curve at the point (x, y) then

$$\tan\psi = \frac{dy}{dx}, \text{ But } \tan\psi = m.$$

Thus m in the case of this tangent, is the value of $\frac{dy}{dx}$ at the point (x, y).

Hence the equation of the tangent at the point (x, y) is

$$Y - y = \frac{dy}{dx}(X - x)$$

Equation of the tangent at any point on the curve $f(x, y) = 0$.

If the equation of the curve be given in the form $f(x, y) = 0$ then

$$\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y}$$

Substituting this expression for $\frac{dy}{dx}$ in $Y - y = \frac{dy}{dx}(X - x)$ we have,

$$Y - y = -\frac{\partial f/\partial x}{\partial f/\partial y}(X - x)$$

or $(X - x)\frac{\partial f}{\partial x} + (Y - y)\frac{\partial f}{\partial y} = 0$ as the equation of the tangent.

Replacing $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ by f_x and f_y respectively we may write the equation of the tangent in the form

$$(X - x) f_x + (Y - y) f_y = 0$$

Students will do well to remember the following three forms of the equation of the tangent at the point (x, y) on the curve

a. If the curve be given by the equation $y = f(x)$, the equation of the tangent will be

$$Y - y = \frac{dy}{dx}(X - x)$$

b. If the curve be given by the equation $f(x, y) = 0$, the equation of the tangent will be

$$(X - x) f_x + (Y - y) f_y = 0$$

c. If the curve be given by the algebraic equation $f(x, y) = 0$ which can be made homogeneous in x, y and z, the equation of the tangent will be

$$X \cdot f_x + Y \cdot f_y + Z \cdot f_z = 0.$$

Normal and its equation.

Normal : Normal to a curve at a point is defined as the straight line through that point at right angles to the tangent to the curve at that point.

We know that tangent to a curve $y = f(x)$ at (x, y) is

$$Y - y = \frac{dy}{dx}(X - x) \quad \text{——(1)}$$

Its gradient $= \frac{dy}{dx}$

Any line through (x, y) is given by

$$Y - y = m(X - x) \quad \text{——(2)}$$

If this line is normal to the curve at (x, y) this is perpendicular to the tangent to the curve. ie

$$m \cdot \frac{dy}{dx} = -1$$

$$\therefore \quad m = -\frac{1}{dy/dx}$$

From (2) the equation of the normal at (x, y) to the curve $y = f(x)$ is

$$Y - y = -\frac{1}{dy/dx}(X - x)$$

or $$(Y - y)\frac{dy}{dx} + (X - x) = 0$$

⊛ If the equation of the curve is $f(x,y)=0$ the equation of the normal at (x,y) is

$$\frac{X-x}{f_x} = \frac{Y-y}{f_y}$$

⊛ If the equation of the curve is $x=\varphi(t)$, $y=\varphi(t)$, the equation of the normal at the point 't' is

$$\varphi'(t)\,X + \varphi'(t)\,Y = \varphi(t)\,\varphi'(t) + \varphi(t) + \varphi'(t)$$

Problem Solving

1. Find the equation of the tangent at any point of the parabola whose equation is $y^2 = 4ax$

Sol^n: From the equation of the curve we have $y^2 = 4ax$

Differentiate both sides we get

$$2y\frac{dy}{dx} = 4a$$

$$\text{or } \frac{dy}{dx} = \frac{4a}{2y}$$

$$\text{or } \frac{dy}{dx} = \frac{2a}{y}$$

Hence the equation of the tangent is

$$Y - y = \frac{2a}{y}(X - x)$$

$$\text{or } Yy - y^2 = 2aX - 2ax$$

$$\text{or } Yy - 4ax = 2aX - 2ax \quad [y^2 = 4ax]$$

$$\text{or } Yy = 2aX + 2ax$$

$$\text{or } Yy = 2a(X + x)$$

2. Find the equation of the tangent at any point (x, y) of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Soln: the equation of the given curve is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$$

thus in this case $f(x, y) = \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1$ which is an algebraic function not homogeneous of the second degree in x and y. Making it homogeneous of the second degree in x, y, z by the introduction of the linear unit z, we have

$$f(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} - z^2$$

so that $f_x = \frac{2x}{a^2}$, $f_y = \frac{2y}{b^2}$, $f_z = -2z$.

Substituting the values of f_x, f_y and f_z in the symmetrical form of equation of the tangent we have

$$X \cdot \frac{2x}{a^2} + Y \cdot \frac{2y}{b^2} + Z \cdot (-2z) = 0$$

or $\frac{Xx}{a^2} + \frac{Yy}{b^2} = 1$ which is the equation of the tangent (on replacing X and x by unity)

3. Find the equation of the tangent at any point (x, y) of the curve

$$x^{2/3} + y^{2/3} = a^{2/3}$$

Sol^n: The given equation of the curve can be written as

$$x^{2/3} + y^{2/3} - a^{2/3} = 0,$$ which when made homogeneous x, y and z becomes

$$x^{2/3} + y^{2/3} - a^{2/3} \cdot z^{2/3} = 0.$$

thus $f(x, y, z) = x^{2/3} + y^{2/3} - a^{2/3} \cdot z^{2/3}$

$\therefore f_x = \frac{2}{3} x^{-1/3}$, $f_y = \frac{2}{3} y^{-1/3}$, $f_z = -\frac{2}{3} a^{2/3} \cdot z^{-1/3}$

Hence the required equation of the tangent is

$$X \cdot \frac{2}{3} x^{-1/3} + Y \cdot \frac{2}{3} y^{-1/3} - Z\left(-\frac{2}{3} a^{2/3} z^{-1/3}\right) = 0.$$

or, $$\frac{X}{x^{1/3}} + \frac{Y}{y^{1/3}} = a^{2/3}$$ (Replacing Z and z by unity and transposing)

4. Find the equation of the tangent at the point 't' on a curve given by $x = a\cos^3 t$, $y = a\sin^3 t$.

Sol^n: Here $x = a\cos^3 t$

$$\frac{dx}{dt} = -3a\cos^2 t \cdot \sin t$$

Also $y = a\sin^3 t$

$$\frac{dy}{dt} = 3a\sin^2 t \cdot \cos t$$

Now $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = -\frac{3a\sin^2 t \cdot \cos t}{3a\cos^2 t \cdot \sin t}$

$$= -\frac{\sin t}{\cos t}$$

Substituting the values of x, y and $\frac{dy}{dx}$ in the equation $Y - y = \frac{dy}{dx}(X - x)$ we get

$$Y - a\sin^3 t = -\frac{\sin t}{\cos t}(X - a\cos^3 t)$$

or $Y\cos t - a\sin^3 t\cos t = -X\sin t + a\sin t\cos^3 t$

or $X\sin t + Y\cos t = a\sin t\cos^3 t + a\sin^3 t\cos t$

or $X\sin t + Y\cos t = a\sin t\cos t(\sin^2 t + \cos^2 t)$

or $X\sin t + Y\cos t = a\sin t\cos t$

which is the equation of the tangent at the point t.

5. Find the equation of the tangent at the point t on the cycloid given by $x = a(t+\sin t)$ and $y = a(1-\cos t)$

Sol^n: Given $x = a(t+\sin t)$

$$\frac{dx}{dt} = a(1+\cos t)$$

$$y = a(1-\cos t)$$

$$\frac{dy}{dt} = a\sin t$$

$$\therefore \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{a\sin t}{a(1+\cos t)}$$

$$= \frac{2\sin\frac{1}{2}t.\cos\frac{1}{2}t}{2\cos^2\frac{1}{2}t} = \tan\frac{1}{2}t$$

The equation of the tangent to the given curve at 't' is

$$Y - y = \frac{dy}{dx}(X - x)$$

or $Y - a(1-\cos t) = (\tan\frac{1}{2}t)\left[X - a(t+\sin t)\right]$

or $Y - 2a\sin^2\frac{1}{2}t = (\tan\frac{1}{2}t)(X - at) - a\sin t\tan\frac{1}{2}t$

or, $Y - 2a\sin^2\frac{1}{2}t = (X - at)\sin\frac{1}{2}t - 2a\sin^2\frac{1}{2}t$

or $Y = (X - at)\tan\frac{1}{2}t$

6. Find the tangent at the point 't' on the curve

$$x = a\cosh t, \quad y = b\sinh t$$

Sol^n: Given $x = a\cosh t$ $\quad y = b\sinh t$

$$\frac{dx}{dt} = a\sinh t \qquad \frac{dy}{dt} = b\cosh t$$

$$\therefore \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{b\cosh t}{a\sinh t}$$

$$= \frac{b}{a}\coth t$$

the equation of the tangent to the given curve 't' is

$$Y - y = \frac{dy}{dx}(X - x)$$

or $Y - b\sinh t = \left(\frac{b}{a}\coth t\right)(X - a\cosh t)$

or $aY\sinh t - ab\sinh^2 t = bX\cosh t - ab\cosh^2 t$

or $aY\sinh t - bX\cosh t = ab\sinh^2 t - ab\cosh^2 t$

or $bX\cosh t - aY\sinh t = ab(\cosh^2 t - \sinh^2 t)$

or $bX\cosh t - aY\sinh t = ab$

7. Find the equation of the tangent at (x, y) to the curve $\frac{x^m}{a^m} + \frac{y^m}{b^m} = 1$.

Sol^n: $\frac{x^m}{a^m} + \frac{y^m}{b^m} = 1$ ——— (1)

Differentiating,

$$\frac{m x^{m-1}}{a^m} + \frac{m y^{m-1}}{b^m} \cdot \frac{dy}{dx} = 0$$

or $$\frac{dy}{dx} = -\frac{b^m x^{m-1}}{a^m y^{m-1}}$$

∴ the equation of the tangent to (1) at (x, y) is

$$Y - y = \frac{dy}{dx}(X - x)$$

or $$Y - y = -\left(\frac{b^m x^{m-1}}{a^m y^{m-1}}\right)(X - x)$$

or, $$a^m y^{m-1} Y - a^m y^m = -b^m x^{m-1} X + b^m x^m$$

or $$b^m x^{m-1} X + a^m y^{m-1} Y = b^m x^m + a^m y^m$$

or $$\frac{x^{m-1} X}{a^m} + \frac{y^{m-1} Y}{b^m} = \frac{x^m}{a^m} + \frac{y^m}{b^m}$$

(dividing each term by $a^m b^m$)

$= 1$ from (1)

∴ the required equation is

$$\frac{x^{m-1} X}{a^m} + \frac{y^{m-1} Y}{b^m} = 1.$$

8. Find the points at which the tangent to the curve $ax^2+2hxy+by^2=1$ is (a) parallel to (b) perpendicular to the axis of x.

Sol^n: The equation of the curve is

$$ax^2+2hxy+by^2=1 \quad \text{——— (1)}$$

Differentiating w.r to x

$$2ax+2h\left(x\frac{dy}{dx}+y\right)+2by\frac{dy}{dx}=0.$$

or $$\frac{dy}{dx}=-\frac{ax+hy}{hx+by} \quad \text{——— (2)}$$

(a) At the points where the tangent to (1) is parallel to the x-axis, we get $\frac{dy}{dx}=0$.

$$\therefore \quad ax+hy=0 \quad \text{——— (3)}$$

the required points are obtained by solving (1) and (3).

(b) At the points where the tangent to (1) is perpendicular to the x-axis.

$$\frac{dy}{dx}=\tan\psi=\tan\frac{\pi}{2}=\infty.$$

or. $$\frac{dx}{dy}=0.$$

or from (2), $hx+by=0$ ——— (4)

the required points are obtained by solving (1) and (4).

9. In the curve $3b^2y = x^3 - 3ax$, find the points at which the tangent is parallel to the axis of x.

Sol^n:

10. Find the equation to the tangents to the curve $y = x^{2/3}(a+x)^{1/3}$ which are perpendicular to x-axis.

Soln: we have $y = x^{2/3}(a+x)^{1/3}$

$$\therefore \frac{dy}{dx} = \frac{2}{3}x^{-1/3}(a+x)^{1/3} + \frac{1}{3}x^{2/3}(a+x)^{-2/3}$$

$$= \frac{2}{3}x^{-1/3}(a+x)^{-2/3} + \frac{1}{3}x^{-1/3}(a+x)^{-2/3}$$

$$= \frac{2(a+x)}{3x^{1/3}(a+x)^{2/3}} + \frac{x}{3x^{1/3}(a+x)^{2/3}}$$

$$= \frac{2(a+x)+x}{3x^{1/3}(a+x)^{2/3}}$$

Now if the tangent is perpendicular to x-axis, we have $\frac{dy}{dx} = \infty$

ie $x^{1/3}(a+x)^{2/3} = 0.$

$\therefore x = 0$ or $x = -a$

$\therefore$ the tangents perpendicular to x-axis are $x = 0$ and $x = -a$.

11. Find the points on the curve $y = x/(1-x^2)$ where the tangent is inclined at angle of $\frac{\pi}{4}$ to the x-axis.

Soln: The curve is $y = \frac{x}{1-x^2}$ ——— (1)

Differentiating w.r to x

$$\frac{dy}{dx} = \frac{(1-x^2).1 - (x-2x)}{(1-x^2)^2}$$

$$= \frac{1-x^2+2x^2}{(1-x^2)^2} = \frac{1+x^2}{(1-x^2)^2} \quad \text{——— (2)}$$

At the points where the tangent to (1) is inclined at an angle of $\frac{\pi}{4}$ to the x-axis we have $\frac{dy}{dx} = \tan\psi = \tan\frac{\pi}{4} = 1$.

or $\frac{dy}{dx} = 1$ ie from (2)

$$\frac{1+x^2}{(1-x^2)^2} = 1$$

$$1+x^2 = 1 + x^4 - 2x^2$$

or $x^4 - 3x^2 = 0$

or $x^2(x^2-3) = 0$

or $x = 0, \pm\sqrt{3}$

From (1), when $x=0$, $y=0$

when $x = \sqrt{3}$ $y = \frac{\sqrt{3}}{1-3} = -\frac{1}{2}\sqrt{3}$

when $x = -\sqrt{3}$ $y = \frac{-\sqrt{3}}{1-3} = \frac{1}{2}\sqrt{3}$

$\therefore$ the required points are

$(0, 0), (\sqrt{3}, -\frac{1}{2}\sqrt{3}), (-\sqrt{3}, \frac{1}{2}\sqrt{3})$

1. find the equation of the normal at (x, y) to the parabola $y^2 = 4ax$.

Soln: the curve is $y^2 = 4ax$ ——(1)

Differentiating both sides

$$2y\frac{dy}{dx} = 4a$$

$$\text{or } \frac{dy}{dx} = \frac{2a}{y}$$

the equation of the normal to (1) is

$$Y - y = -\frac{1}{(dy/dx)}(X - x)$$

$$\text{or } Y - y = -\left(\frac{y}{2a}\right)(X - x)$$

$$\text{or } 2aY - 2ay = -yX + xy$$

$$\text{or } 2aY + yX = xy + 2ay$$

2. Find the equation of the normal at the point (x, y) on each of the following curves $\frac{x^m}{a^m} + \frac{y^m}{b^m} = 1$

Sol^n: The curve is $\frac{x^m}{a^m} + \frac{y^m}{b^m} = 1$ ——— (1)

Differentiating (1) we get

$$\frac{m x^{m-1}}{a^m} + \frac{m y^{m-1}}{b^m} \cdot \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{b^m x^{m-1}}{a^m y^{m-1}}$$

The required equation of the normal at (x, y) is

$$Y - y = -\frac{1}{dy/dx}(X - x)$$

$$\text{or } Y - y = +\frac{a^m y^{m-1}}{b^m x^{m-1}}(X - x)$$

$$\text{or } Y b^m x^{m-1} - y b^m x^{m-1} = a^m y^{m-1} X - x a^m y^{m-1}$$

$$\text{or } -\frac{Y x^{m-1}}{a^m} + \frac{y x^{m-1}}{a^m} + \frac{X y^{m-1}}{b^m} - \frac{x y^{m-1}}{b^m} = 0$$

$$\text{or } \frac{X y^{m-1}}{b^m} - \frac{Y x^{m-1}}{a^m} = \frac{x y^{m-1}}{b^m} - \frac{y x^{m-1}}{a^m}$$

3. Find the equation of the normal at (x, y) to the curve $y = a \log \sin x$

Sol^n: The curve is $y = a \log \sin x$ ——— (1)

Differentiating $\frac{dy}{dx} = \frac{a}{\sin x} \cdot \cos x$

or $\frac{dy}{dx} = a \cot x$

The equation of the normal to (1) at (x, y) is

$$Y - y = \frac{-1}{(dy/dx)} (X - x)$$

or $Y - y = -\frac{1}{a \cot x} (X - x)$

or $Y - y = -\frac{1}{a} \frac{\sin x}{\cos x} (X - x)$

or $aY \cos x - ay \cos x = -X \sin x + x \sin x$

or $X \sin x + aY \cos x = x \sin x + ay \cos x$

4. Find the equation of the normal at any point φ on the curve $x = a\cos\varphi$, $y = b\sin\varphi$

Sol^{n}.

$$x = a\cos\varphi \qquad y = b\sin\varphi$$

$$\frac{dx}{d\varphi} = -a\sin\varphi \qquad \frac{dy}{d\varphi} = b\cos\varphi$$

Now $$\frac{dy}{dx} = \frac{dy/d\varphi}{dx/d\varphi}$$

$$= -\frac{b}{a}\,\frac{\cos\varphi}{\sin\varphi}$$

Hence the required equation of the normal at $x = a\cos\varphi$, $y = b\sin\varphi$ is

$$Y - y = \frac{1}{dy/dx}(X - x)$$

or $$Y - b\sin\varphi = \frac{1}{-\frac{b}{a}\frac{\cos\varphi}{\sin\varphi}}(X - a\cos\varphi)$$

or $$(X - a\cos\varphi) + \frac{b}{a}\frac{\cos\varphi}{\sin\varphi}(Y - b\sin\varphi) = 0$$

or $$aX\sin\varphi + bY\cos\varphi = (a^2 - b^2)\sin\varphi\cos\varphi$$

or $$aX\sec\varphi + bY\operatorname{cosec}\varphi = a^2 - b^2$$

5. Find the equation of the normal at the point (x, y) to 't' on the curve $x = a(t + \sin t)$, $y = a(1 - \cos t)$

Soln:

$$x = a(t + \sin t) \qquad y = a(1 - \cos t)$$

$$\frac{dx}{dt} = a(1 + \cos t) \qquad \frac{dy}{dt} = a \sin t$$

$$\therefore \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{a \sin t}{a(1 + \cos t)}$$

$$= \frac{2 \sin \frac{1}{2}t \cos \frac{1}{2}t}{2 \cos^2 \frac{1}{2}t}$$

$$= \tan \tfrac{1}{2}t$$

The required equation of the normal is

$$Y - y = -\frac{1}{dy/dx}(X - x)$$

$$Y - a(1 - \cos t) = -\frac{1}{\tan \frac{1}{2}t} \{X - a(t + \sin t)\}$$

$$(Y - 2a \sin^2 \tfrac{1}{2}t) \tan \tfrac{1}{2}t = -X + a(t + \sin t)$$

or $$X + Y \tan \tfrac{1}{2}t = a\{t + \sin t + 2 \sin^2 \tfrac{1}{2}t \tan \tfrac{1}{2}t\}$$

Ques: Find the equation of the tangent at any point (x, y) of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Solution: The equation of the given curve is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$$

Thus in this case $f(x, y) = \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1$ which is an algebraic function not homogeneous of the second degree in x and y. Making it homogeneous of the second degree by the introduction of the linear unit z, we have

$$f(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} - z^2$$

so that, $f_x = \frac{2x}{a^2}$

$$f_y = \frac{2y}{b^2}$$

$$f_z = -2z.$$

Substituting the values of f_x, f_y and f_z in the symmetrical form of equation of the tangent, we have

$$X \cdot \frac{2x}{a^2} + Y \cdot \frac{2y}{b^2} + Z(-2z) = 0$$

or $\frac{Xx}{a^2} + \frac{Yy}{b^2} = 1$ which is the equation of the tangent (on replacing z and Z by unity)