1. Solve $(x^2+y^2)\,dx - 2xy\,dy = 0$

solution: Given equation is

$$(x^2+y^2)\,dx = 2xy\,dy$$

$$\text{or } \frac{dy}{dx} = \frac{x^2+y^2}{2xy} \quad \text{——— (1)}$$

Put $y = vx$ then $\frac{dy}{dx} = v + x\frac{dv}{dx}$ — (2)

Now put the value of $\frac{dy}{dx}$ in (1) we get

$$v + x\frac{dv}{dx} = \frac{x^2+v^2x^2}{2xvx}$$

$$\text{or } v + x\frac{dv}{dx} = \frac{x^2(1+v^2)}{2x^2v}$$

$$\text{or } v + x\frac{dv}{dx} = \frac{1+v^2}{2v}$$

$$\text{or } x\frac{dv}{dx} = \frac{1+v^2}{2v} - v$$

$$\text{or } x\frac{dv}{dx} = \frac{1+v^2-2v^2}{2v}$$

$$\text{or } x\frac{dv}{dx} = -\frac{(v^2-1)}{2v}$$

$$\text{or, } \frac{2v\,dv}{v^2-1} = -\frac{dx}{x}$$

Integrating both sides

$$\int \frac{2v\,dv}{v^2-1} + \int \frac{dx}{x} = 0$$

$$\text{or } \ln(v^2-1) + \ln x = \ln c$$

$$\text{or, } \ln(v^2-1)x = \ln c$$

$$\text{or } (v^2-1)x = c$$

$$\text{or } \left(\frac{y^2}{x^2}-1\right)x = c$$

$$\text{or } \frac{y^2-x^2}{x} = c$$

$$\text{or } y^2 - x^2 = cx$$

Solve: $(x^2+y^2)\,dx + 2xy\,dy = 0$.

Soln Given equation is

$$(x^2+y^2)\,dx + 2xy\,dy = 0.$$

or, $$\frac{dy}{dx} = -\frac{x^2+y^2}{2xy}. \quad \text{——— (1)}$$

Put $y = vx$ then $\frac{dy}{dx} = v + x\frac{dv}{dx}$. ——— (2)

Now put the value of y and $\frac{dy}{dx}$ in (1) we get,

$$v + x\frac{dv}{dx} = -\frac{x^2 + v^2x^2}{2x \cdot vx}$$

$$v + x\frac{dv}{dx} = -\frac{x^2(1+v^2)}{2x^2v}$$

$$\Rightarrow v + x\frac{dv}{dx} = -\frac{1+v^2}{2v}$$

$$\Rightarrow x\frac{dv}{dx} = -\frac{1+v^2}{2v} - v.$$

$$\Rightarrow x\frac{dv}{dx} = \frac{-1-v^2-2v^2}{2v}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{-1-3v^2}{2v}$$

$$\Rightarrow x\frac{dv}{dx} = -\frac{1+3v^2}{2v}$$

$$\Rightarrow \frac{2v}{1+3v^2}\,dv = -\frac{dx}{x}$$

Integrating both sides

$$\int \frac{2v}{1+3v^2}\,dv = -\int \frac{dx}{x}$$

$1+3v^2 = z$

$6v\,dv = dz$

$2v\,dv = \frac{1}{3}dz$

$$\Rightarrow \frac{1}{3}\int \frac{dz}{z} = -\int \frac{dx}{x}$$

$$\Rightarrow \frac{1}{3}\log z = -\log x + \log c$$

$$\Rightarrow \frac{1}{3}\log(1+3v^2) = -\log x + \log c$$

$$\Rightarrow \frac{1}{3}\log(1+3v^2) + \log x = \log c$$

$$\Rightarrow \log(1+3v^2)^{1/3} + \log x = \log c$$

$$\Rightarrow \log x(1+3v^2)^{1/3} = \log c$$

$$\Rightarrow x(1+3v^2)^{1/3} = c$$

$$\Rightarrow x\left(1+\frac{3y^2}{x^2}\right)^{1/3} = c.$$

Equation reducible to Homogeneous

If the differential equation is of the form

1. $\frac{dy}{dx} = \frac{ax+by+c}{a_1x+b_1y+c_1}$ when $\frac{a}{a_1} \neq \frac{b}{b_1}$

This type of differential equation can be reduced to homogeneous equation by putting $x = \alpha + h$ and $y = \beta + k$ where α, β are new variables and h, k are arbitrary constants.

Thereafter putting $\beta = v\alpha$ and with the help of separation of variables and then integrate we can find the solution.

2. If the differential equation is of the form

$$\frac{dy}{dx} = \frac{ax+by+c}{a_1x+b_1y+c_1} \quad \text{where} \quad \frac{a}{a_1} = \frac{b}{b_1}$$

then we put $\frac{a}{a_1} = \frac{b}{b_1} = m$

or $a = a_1 m$ and $b = b_1 m$

Now putting the values of a and b the equation is of the form

$$\frac{dy}{dx} = \frac{m(a_1x+b_1y)+c}{a_1x+b_1y+c_1}$$

put $a_1x + b_1y = v$. Then after variables separable, solution is obtained by Integration.

1. Solve: $\frac{dy}{dx} = \frac{2x+y-2}{3x+y-3}$

Sol^n: Given equation is

$$\frac{dy}{dx} = \frac{2x+y-2}{3x+y-3} \quad \text{——— (1)} \quad \text{Here } \frac{2}{3} \neq \frac{1}{1}$$

we put $x = \alpha + h$, $y = \beta + k$ when h, k are arbitrary constants.

$\therefore dx = d\alpha$ and $dy = d\beta$

$$\therefore \frac{dy}{dx} = \frac{d\beta}{d\alpha} \quad \text{——— (2)}$$

From (1) and (2) we get

$$\frac{d\beta}{d\alpha} = \frac{2(\alpha+h)+\beta+k-2}{3(\alpha+h)+\beta+k-3}$$

or $$\frac{d\beta}{d\alpha} = \frac{2\alpha+2h+\beta+k-2}{3\alpha+3h+\beta+k-3}$$

or $$\frac{d\beta}{d\alpha} = \frac{2\alpha+\beta+2h+k-2}{3\alpha+\beta+3h+k-3} \quad \text{——— (3)}$$

Here h and k have chosen such that

$$2h+k-2=0$$

and $$3h+k-3=0$$

$$\frac{h}{-3+2} = \frac{k}{-6+6} = \frac{1}{2-3}$$

or $$\frac{h}{-1} = \frac{k}{0} = \frac{1}{-1}$$

or $h = 1$ and $k = 0$

then from (3) we get

$$\frac{d\beta}{d\alpha} = \frac{2\alpha+\beta}{3\alpha+\beta} \quad \text{——— (4)}$$

We put $\beta = v\alpha$ Then $\frac{d\beta}{d\alpha} = v + \alpha \frac{dv}{d\alpha}$ —(5)

From(5) putting the values of β and $\frac{d\beta}{d\alpha}$ in (4) we get,

$$v + \alpha \frac{dv}{d\alpha} = \frac{2\alpha + v\alpha}{3\alpha + v\alpha}$$

$$\text{or} \quad \alpha \frac{dv}{d\alpha} = \frac{2+v}{3+v} - v$$

$$\text{or} \quad \alpha \frac{dv}{d\alpha} = \frac{2+v-3v-v^2}{3+v}$$

$$\text{or} \quad \alpha \frac{dv}{d\alpha} = \frac{-(v^2+2v-2)}{v+3}$$

$$\text{or} \quad \frac{(v+3)\,dv}{v^2+2v-2} = -\frac{d\alpha}{\alpha}$$

$$\text{or} \quad \frac{\left\{\frac{1}{2}(2v+2)+2\right\} dv}{v^2+2v-2} + \frac{d\alpha}{\alpha} = 0$$

Integrating both sides,

$$\frac{1}{2}\int \frac{(2v+2)\,dv}{v^2+2v-2} + 2\int \frac{dv}{v^2+2v-2} + \int \frac{d\alpha}{\alpha} = 0$$

$$\text{or} \quad \frac{1}{2}\int \frac{(2v+2)\,dv}{v^2+2v-2} + 2\int \frac{dv}{(v+1)^2-(\sqrt{3})^2} + \int \frac{d\alpha}{\alpha} = 0$$

$$\text{or,} \quad \frac{1}{2}\ln(v^2+2v-2) + \frac{2}{2\sqrt{3}}\ln\frac{v+1-\sqrt{3}}{v+1+\sqrt{3}} + \ln\alpha = \frac{1}{2}c$$

$$\text{or} \quad \ln(v^2+2v-2) + \ln\alpha^2 + \frac{2}{\sqrt{3}}\ln\frac{v+1-\sqrt{3}}{v+1+\sqrt{3}} = c$$

$$\text{or} \quad \ln\left(\frac{\beta^2}{\alpha^2} + \frac{2\beta}{\alpha} - 2\right)\alpha^2 + \frac{2}{\sqrt{3}}\ln\frac{\beta/\alpha + 1 - \sqrt{3}}{\beta/\alpha + 1 + \sqrt{3}} = c$$

$$\text{or,} \quad \ln\left(\beta^2 + 2\alpha\beta - 2\alpha^2\right) + \frac{2}{\sqrt{3}}\ln\left\{\frac{\beta + (1-\sqrt{3})\alpha}{\beta + (1+\sqrt{3})\alpha}\right\} = c$$

Since $x = \alpha + 1$ and $y = \beta + 0$ so

putting $\alpha = x - 1$ and $\beta = y$ in the above equation we get,

$$\ln\left\{y^2 + 2y(x-1) - 2(x-1)^2\right\} + \frac{2}{\sqrt{3}}\ln\left\{\frac{y + (1-\sqrt{3})(x-1)}{y + (1+\sqrt{3})(x-1)}\right\} = c$$

2. solve : $(x+y+1)\,dx - (2x+2y+1)\,dy = 0$

sol^n: Given equation is

$$(x+y+1)\,dx - (2x+2y+1)\,dy = 0$$

or $$(x+y+1)\,dx = (2x+2y+1)\,dy$$

or $$\frac{dy}{dx} = \frac{x+y+1}{2x+2y+1} \quad \text{——— (1)}$$

or $$\frac{dy}{dx} = \frac{x+y+1}{2(x+y)+1} \quad \text{——— (1)}$$

We put $x+y=v$ then

$$1 + \frac{dy}{dx} = \frac{dv}{dx}$$

or $$\frac{dy}{dx} = \frac{dv}{dx} - 1 \quad \text{——— (2)}$$

From (1) and (2) we get,

$$\frac{dv}{dx} - 1 = \frac{v+1}{2v+1}$$

or $$\frac{dv}{dx} = 1 + \frac{v+1}{2v+1}$$

or $$\frac{dv}{dx} = \frac{3v+2}{2v+1}$$

or $$\frac{2v+1}{3v+2}\,dv = dx$$

or $$\frac{\left\{\frac{2}{3}(3v+2) - \frac{1}{3}\right\}dv}{3v+2} = dx$$

$$\text{or } \left\{\frac{2}{3} - \frac{1}{3(3v+2)}\right\} dv = dx$$

Integrating both sides

$$\frac{2}{3}\int dv - \frac{1}{3}\int \frac{dv}{3v+2} = \int dx$$

$$\text{or } \frac{2}{3}\int dv - \frac{1}{9}\int \frac{3\,dv}{3v+2} = \int dx$$

$$\text{or } \frac{2}{3}v - \frac{1}{9}\ln(3v+2) = x + \frac{1}{9}c$$

$$\text{or } 6v - \ln(3v+2) = 9x + c$$

$$\text{or } 6(x+y) - \ln(3x+3y+2) = 9x + c$$

$$\text{or } 6y - 3x - \ln(3x+3y+2) = c.$$