Class-7 (27.11.23)

Linear D.E: If P and Q are only functions of x or constants then the D.E of the form $\frac{dy}{dx} + P.y = Q$ is called 1st order linear D.E

To solve this eqn multiply by $e^{\int P dx}$ → I.F (Integrating factor) then it becomes,

$$e^{\int P dx}\frac{dy}{dx} + Py e^{\int P dx} = Q e^{\int P dx}$$

$$\Rightarrow \frac{d}{dx}\left(y e^{\int P dx}\right) = Q e^{\int P dx}$$

Integrating both sides w.r.t to x, we get

$$y e^{\int P dx} = \int \left[Q e^{\int P dx}\right] dx + c$$

Integrating factor: An integrating factor is a function used to solve d.e.
It is a function in which an ~~ord~~ O.D.E can be multiplied to make the function integrable.

$$\frac{dy}{dx} + Py = Q$$

$$\therefore IF = e^{\int P dx}$$

Ques: Solve $x\frac{dy}{dx} - 4y = x^6 e^x$ ——— (1)

Divide (1) by x,

$$\frac{dy}{dx} - 4\frac{y}{x} = x^5 e^x \quad \text{——— (2)}$$

which is a te linear d.e of the form $\frac{dy}{dx} + Py = Q$ where $P = -\frac{4}{x}$

$$I.F = e^{\int(-\frac{4}{x})dx} = e^{-4\ln x} = e^{\ln x^{-4}} = x^{-4}$$

Multiply IF by (2),

$$x^{-4}\frac{dy}{dx} + (-\frac{4}{x})x^{-4}y = x^5 e^x x^{-4}$$

$$\Rightarrow x^{-4}\frac{dy}{dx} + (-4)x^{-5}y = xe^x$$

$$\Rightarrow \frac{d}{dx}(x^{-4}y) = xe^x$$

Integrating both sides

$$x^{-4}y = \int xe^x dx$$

$$\Rightarrow x\int e^x dx - \int(\frac{dx}{dx}\int e^x dx)dx = x^{-4}y$$

$$\Rightarrow xe^x - e^x + c = x^{-4}y$$

$$\Rightarrow y = x^4(xe^x - e^x) + c$$

Ques: Solve $(x^2+9)\frac{dy}{dx} + xy = 0$ — ①

Divide ① by x^2+9,

$$\frac{dy}{dx} + \frac{x}{x^2+9}y = 0 \quad \text{— ②}$$

which is a linear d.e of the form $\frac{dy}{dx} + Py = Q$ where $P = \frac{x}{x^2+9}$

$$I.F = e^{\int \frac{x}{x^2+9}\,dx}$$

$$= e^{\frac{1}{2}\ln(x^2+9)} = e^{\ln(x^2+9)^{\frac{1}{2}}}$$

$$\Rightarrow (x^2+9)^{\frac{1}{2}}$$

$x^2+9 = z$
$\Rightarrow 2x\,dx = dz$
$\Rightarrow x\,dx = \frac{dz}{2}$
$\int \frac{dz}{2z}$
$= \frac{1}{2}\ln z$
$= \frac{1}{2}\ln(x^2+9)$

Multiply ② by I.F,

$$\frac{dy}{dx}(x^2+9)^{\frac{1}{2}} + \frac{x}{x^2+9}y\,(x^2+9)^{\frac{1}{2}} = 0$$

$$\Rightarrow \frac{dy}{dx}(x^2+9)^{\frac{1}{2}} + \frac{xy}{\sqrt{x^2+9}} = 0$$

$$\Rightarrow \frac{d}{dx}\left\{(x^2+9)^{\frac{1}{2}}y\right\} = 0$$

Integrating both sides,

$$(x^2+9)^{\frac{1}{2}}y = c$$

$$\Rightarrow y = \frac{c}{\sqrt{x^2+9}}$$

Ques: Solve $(1+e^x)\frac{dy}{dx} + e^x y = 0$ — ①

Divide ① by $(1+e^x)$,

$$\frac{dy}{dx} + \frac{e^x}{1+e^x}y = 0 \quad \text{— ②}$$

which is a linear d.e of the form $\frac{dy}{dx} + Py = Q$ where $P = \frac{e^x}{1+e^x}$

$$I.F = e^{\int \frac{e^x}{1+e^x}dx}$$

$$= e^{\ln(1+e^x)}$$

$$= 1+e^x$$

$1+e^x = z$
$\Rightarrow e^x\,dx = dz$
$\int \frac{dz}{z}$
$= \ln z$
$= \ln(1+e^x)$

Multiply ② by I.F,

$$(1+e^x)\frac{dy}{dx} + e^x y = 0$$

Integrating both side,

$$\Rightarrow \frac{d}{dx}\{(1+e^x)y\} = 0$$

Integrating both side,

$$(1+e^x)y = c$$

$$\Rightarrow y = \frac{c}{1+e^x}$$

Bernaulli eqn: If P and Q are only functions of x, then the DE of the form $\frac{dy}{dx} + Py = Qy^n$ is called Bernaulli's D.E. Dividing the eqn by y^n it can be reduced to linear de.

Ques: Solve $\frac{dy}{dx} + \frac{y}{x} = y^2$ —— ①

Divide eqn ① by y^2,

$$\frac{1}{y^2}\frac{dy}{dx} + \frac{1}{xy} = 1 \text{ —— ②}$$

Put $v = -\frac{1}{y}$

$$\Rightarrow \frac{dv}{dx} = \frac{1}{y^2}\frac{dy}{dx}$$

Putting the values in ②,

$$\frac{dv}{dx} + \left(-\frac{1}{x}\right)v = 1$$

which is a linear eqn,

$$IF = e^{\int -\frac{1}{x}dx} = e^{-\ln x} = e^{\ln x^{-1}} = x^{-1}$$

$$x^{-1}\frac{dv}{dx} + x^{-1}\left(-\frac{1}{x}\right)v = x^{-1}$$

$$\Rightarrow x^{-1}\frac{dv}{dx} - \frac{v}{x^2} = x^{-1}$$

$$\Rightarrow \frac{d}{dx}(x^{-1}v) = x^{-1}$$

$$\Rightarrow \int \frac{d}{dx}(x^{-1}v)dx = \int x^{-1}dx$$

$$\Rightarrow x^{-1}v = \ln x + c$$

$$\Rightarrow v = x(\ln x + c)$$

$$\Rightarrow -\frac{1}{y} = x(\ln x + c)$$

$$\Rightarrow xy(\ln x + c) = -1$$

Clairauits eqn: An eqn of the form $\boxed{y = px + f(p)}$ is called Clairauits eqn.

when $p = \frac{dy}{dx}$

Complete and singular soln (S.S)

the given eqn is $y = px + f(p)$ ——— (1)

Differentiating (1) wr to x, we get,

$$\frac{dy}{dx} = p + x\frac{dp}{dx} + f'(p)\frac{dp}{dx}$$

$$p = p + x\frac{dp}{dx} + f'(p)$$

$$\left[x + f'(p)\right]\frac{dp}{dx} = 0$$

$$\frac{dp}{dx} = 0 \quad \text{——} (2)$$

$$x + f'(p) = 0 \quad \text{——} (3)$$

Now integrating ② wrt to x,
$p = c$, where c is an arbitrary constant.
Now putting the value of p in ①,

$y = cx + f(c)$ which is the required general soln of ①

Again eliminating p from ① and ③, to obtain another soln involving no arbitrary constant this soln is called the singular soln (S.S)

Solve: $y = px + \frac{a^2}{p}$ and find the singular soln (If exists)

Soln: Given eqn; $y = px + \frac{a^2}{p}$ —— ①

which is of the form ~~$y = px + f(p)$~~ ~~Elaira~~ of Clairaut's eqn.

Now diff ① w.r to x,

$$\frac{dy}{dx} = p + x\frac{dp}{dx} - \frac{a^2}{p^2}\frac{dp}{dx}$$

$$\Rightarrow p = p + x\frac{dp}{dx} - \frac{a^2}{p^2}\frac{dp}{dx}$$

$$\Rightarrow \left(x - \frac{a^2}{p^2}\right)\frac{dp}{dx} = 0$$

$$\frac{dp}{dx} = 0 \text{ —— ②}$$

$$\left(x - \frac{a^2}{p^2}\right) = 0 \text{ —— ③}$$

Now integreating ② wrt to x,
$p = c$ where c is an arbitrary constant.

Now putting the value of p in ①,

$y = cx + \frac{a^2}{c}$ which is the required general soln of ①

~~From ③,~~

Again from ③, we get,

$$x - \frac{a^2}{p^2} = 0$$

$$\Rightarrow x = \frac{a^2}{p^2}$$

$$\Rightarrow p = \frac{a}{\sqrt{x}}$$

Putting the value of p in ①,

$$y = \frac{a}{\sqrt{x}}x + \frac{a^2}{\frac{a}{\sqrt{x}}} = a\sqrt{x} + a\sqrt{x}$$

$\Rightarrow y = 2a\sqrt{x} \Rightarrow y^2 = 4a^2x$ which is the singular soln.

Solve: $(y+1)p - xp^2 + 2 = 0$

where $p = \frac{dy}{dx}$

The given eqn is,

$$(y+1)p - xp^2 + 2 = 0 \quad \text{——①}$$

$$\Rightarrow (y+1)p = xp^2 - 2$$

$$\Rightarrow y + 1 = \frac{xp^2 - 2}{p} = xp - \frac{2}{p}$$

which is the Clairaut's form

$y = px + f(p)$

Hence putting c for p, the general soln is,

$$y = xc + \left(-\frac{2}{c} - 1\right)$$

$$= cx - \left(\frac{2}{c} + 1\right)$$