Homogenous Eqn:

An eqn in which the sum of the powers of x and y in every terms are equal than the eqn is called homogenous eqn.

Example: $ax^2 + 2hxy + by^2$ & is a homogenous eqn of 2^{nd} degree in x & y

Homogenous D.E: An eqn of the form $\frac{dy}{dx} = \frac{f(x,y)}{\phi(x,y)}$ in which $f(x,y)$ and $\phi(x,y)$ are homogenous functions of x and y of the same degree, then the eqn is called homogenous D.E

Working rule for solving homogenous DE:

1) Rewrite the given homogenous D.E in the form $\frac{dy}{dx} = \frac{f(x,y)}{\phi(x,y)}$ —①

2) Substitute $y = vx$

3) Differentiate $y = vx$, then we get $\frac{dy}{dx} = v + x\frac{dv}{dx}$

4) Put the values in ①

$$v + x\frac{dv}{dx} = \frac{f(v)}{\phi(v)} \Rightarrow v + x\frac{dv}{dx} = F(v)$$

where $F(v) = \frac{f(v)}{\phi(v)}$

$\Rightarrow x\frac{dv}{dx} = F(v)-v$

$\Rightarrow \frac{dv}{F(v)-v} = \frac{dx}{x}$ where variables have been ~~separa~~ seperated.

5) Then the general soln will be obtained integration.

~~Ques~~:

Ques: Solve $(x^2+y^2)dx = 2xy\,dy$

$$\frac{dy}{dx} = \frac{x^2+y^2}{2xy} \quad \text{---} (1)$$

put $y = vx$

$$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$$

Putting the values in (1),

$$v + x\frac{dv}{dx} = \frac{x^2+v^2x^2}{2x\,vx}$$ ~~$\Rightarrow v+x\,dy$~~

$$\Rightarrow v + x\frac{dv}{dx} = \frac{1+v^2}{2v}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{1+v}{2v} - v \Rightarrow x\frac{dv}{dx} = \frac{1-v^2}{2v}$$

$$\Rightarrow x\frac{dv}{dx} = -\frac{v^2-1}{2v} \Rightarrow \frac{2v\,dv}{v^2-1} = -\frac{dx}{x}$$

Integrating both sides,

$$\int\frac{2v\,dv}{v^2-1} + \int\frac{dx}{x} = 0 \Rightarrow \ln(v^2-1) + \ln x = \ln c$$

$$\Rightarrow \ln(v^2-1)x = \ln c \Rightarrow (v^2-1)x = c \Rightarrow \left(\frac{y^2}{x^2}-1\right)x = c$$

$$\Rightarrow \frac{y^2-x^2}{x}\cdot x = c \Rightarrow y^2-x^2 = cx$$

Ques: Solve $(x^2+y^2)dx + 2xy\,dy = 0$ ---(1)

$$\frac{dy}{dx} = -\frac{x^2+y^2}{2xy} \quad \text{---} (2)$$

put $y = vx$ then $\frac{dy}{dx} = v + x\frac{dv}{dx}$ ---(2)

Now put the values of y and $\frac{dy}{dx}$ in (1) we get,

$$v + x\frac{dv}{dx} = -\frac{x^2+v^2x^2}{2x\,vx} = -\frac{x^2(1+v^2)}{2x^2v} = -\frac{1+v^2}{2v}$$

$$\Rightarrow x\frac{dv}{dx} = -\frac{1+v^2}{2v} - v = \frac{-1-v^2-2v^2}{2v} = \frac{-1-3v^2}{2v} = -\frac{1+3v^2}{2v}$$

$$\Rightarrow \frac{2v}{1+3v^2}dv = -\frac{dx}{x}$$

Integrating both sides,

$$\int \frac{2v}{1+3v^2}\,dv = -\int \frac{dx}{x}$$

$1+3v^2 = z$
$\Rightarrow 6v\,dv = dz$
$\Rightarrow 2v\,dv = \frac{1}{3}dz$

$$\Rightarrow \frac{1}{3}\int \frac{dz}{z} = -\int \frac{dx}{x} \Rightarrow \frac{1}{3}\log z = -\log x + \log c$$

$$\Rightarrow \frac{1}{3}\log(1+3v^2) = -\log x + \log c$$

$$\Rightarrow \frac{1}{3}\log(1+3v^2) + \log x = \log c$$

$$\Rightarrow \log(1+3v^2)^{\frac{1}{3}} + \log x = \log c$$

$$\Rightarrow x(1+3v^2)^{\frac{1}{3}} = c$$

$$\Rightarrow x\left(1+\frac{3y^2}{x^2}\right)^{\frac{1}{3}} = c$$

Class-6 (22.11.23)

Eqn reducible to homogenous:

1) If the D.E is of the form

$$\frac{dy}{dx} = \frac{ax+by+c}{a_1x+b_1y+c_1} \quad \text{when } \frac{a}{a_1} \neq \frac{b}{b_1}$$

this type of DE can be reduced to homogenous eqn by putting $x = \alpha + h$, $y = \beta + k$ where α, β are new variables and, h, k are arbitrary constants.

Therefore putting $\beta = v\alpha$ and with the help of seperation of variables and then integrate we can find the soln.

Ques: Solve $\frac{dy}{dx} = \frac{2x+y-2}{3x+y-3}$ ——①

Here $\frac{2}{3} \neq \frac{1}{1}$

Put $x = \alpha + h$ $\quad y = \beta + k$

$dx = d\alpha$ $\quad dy = d\beta$

$$\frac{dy}{dx} = \frac{d\beta}{d\alpha} \text{ ——②}$$

Putting the values in ①,

$$\frac{d\beta}{d\alpha} = \frac{2(\alpha+h)+\beta+k-2}{3(\alpha+h)+\beta+k-3}$$

$$\Rightarrow \frac{d\beta}{d\alpha} = \frac{2\alpha+\beta+2h+k-2}{3\alpha+\beta+3h+k-3} \quad \text{——③}$$

Now choose $2h+k-2=0$

$3h+k-3=0$

$$\frac{h}{-3+2} = \frac{k}{-6+6} = \frac{1}{2-3}$$

$$\Rightarrow \frac{h}{-1} = \frac{k}{0} = \frac{1}{-1}$$

$\Rightarrow h=1$, $k=0$

From ③,

$$\frac{d\beta}{d\alpha} = \frac{2\alpha+\beta}{3\alpha+\beta} \quad \text{——④}$$

Now, put $\beta = v\alpha$

$$\frac{d\beta}{d\alpha} = v+\alpha\frac{dv}{d\alpha} \quad \text{——⑤}$$

From ④,

$$v+\alpha\frac{dv}{d\alpha} = \frac{2\alpha+v\alpha}{3\alpha+v\alpha}$$

$$\Rightarrow v+\alpha\frac{dv}{d\alpha} = \frac{2v}{3+v}$$

$$\Rightarrow \alpha\frac{dv}{d\alpha} = \frac{2+v}{3+v} - v$$

$$\Rightarrow \alpha\frac{dv}{d\alpha} = \frac{2+v-3v-v^2}{3+v}$$

$$\Rightarrow \alpha\frac{dv}{d\alpha} = \frac{-(v^2+2v-2)}{3+v}$$

$$\Rightarrow \frac{v+3}{v^2+2v-2}\,dv = -\frac{d\alpha}{\alpha}$$

$$\Rightarrow \frac{\left\{\frac{1}{2}(2v+2)+2\right\}dv}{v^2+2v-2} + \frac{d\alpha}{\alpha} = 0$$

Integrating both sides,

$$\frac{1}{2}\int\frac{2v+2}{v^2+2v-2}\,dv + 2\int\frac{dv}{v^2+2v-2} + \int\frac{d\alpha}{\alpha} = 0$$

$$\Rightarrow \quad " \quad + 2\int\frac{dv}{(v+1)^2-(\sqrt{3})^2} + " = 0$$

$$\Rightarrow \frac{1}{2}\ln(v^2+2v-2) + \frac{2}{2\sqrt{3}}\ln\frac{v+1-\sqrt{3}}{v+1+\sqrt{3}}$$
$$+\ln\alpha = \frac{1}{2}c$$

$$\Rightarrow \ln(v^2+2v-2) + \ln\alpha^2 + \frac{2}{\sqrt{3}}\ln\frac{v+1-\sqrt{3}}{v+1+\sqrt{3}} = c$$

$$\Rightarrow \ln\left(\frac{\beta^2}{\alpha^2}+\frac{2\beta}{\alpha}-2\right)\alpha^2 + \frac{2}{\sqrt{3}}\ln\frac{\beta/\alpha+1-\sqrt{3}}{\beta/\alpha+1+\sqrt{3}} = c$$

$$\Rightarrow \ln(\beta^2+2\alpha\beta-2\alpha^2) + \frac{2}{\sqrt{3}}\ln\frac{\beta+(1-\sqrt{3})\alpha}{\beta+(1+\sqrt{3})\alpha} = c$$

since $x=\alpha+1$ $\quad y=\beta+0$ so putting $\alpha=x-1$ $\quad \beta=y$ in the above eqn we get,

$$\ln\{y^2+2y(x-1)-2(x-1)^2\} + \frac{2}{\sqrt{3}}\ln\frac{y+(1-\sqrt{3})(x-1)}{y+(1+\sqrt{3})(x-1)} = c$$

2) If the de is of the form $\frac{dy}{dx} = \frac{ax+by+c}{a_1x+b_1y+c_1}$ where $\frac{a}{a_1} = \frac{b}{b_1}$

Then we put $\frac{a}{a_1} = \frac{b}{b_1} = m$

or, $a = a_1 m \quad b = b_1 m$

Now putting the values of a and b the equation is of the form

$$\frac{dy}{dx} = \frac{m(a_1x + b_1y) + c}{a_1x + b_1y + c_1}$$

Put $a_1x + b_1y = v$. Then after variables seperable, soln is obtained by integration.

Ques: Solve $(x+y+1)dx - (2x+2y+1)dy = 0$

$$\Rightarrow \frac{dy}{dx} = \frac{x+y+1}{2x+2y+1} \quad \text{---(1)}$$

$$= \frac{x+y+1}{2(x+y)+1} \quad \text{---(1)}$$

We put $x+y = v$ then,

$$1 + \frac{dy}{dx} = \frac{dv}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{dv}{dx} - 1 \quad \text{---(2)}$$

From (1) & (2),

$$\frac{dv}{dx} - 1 = \frac{v+1}{2v+1}$$

$$\Rightarrow \frac{dv}{dx} = \frac{3v+2}{2v+1}$$

$$\Rightarrow \frac{2v+1}{3v+2}\, dv = dx$$

$$\Rightarrow \frac{\frac{2}{3}(3v+2) - \frac{1}{3}}{3v+2}\, dv = dx$$

$$\Rightarrow \left\{\frac{2}{3} - \frac{1}{3(3v+2)}\right\} dv = dx$$

Integrating both sides,

$$\frac{2}{3}\int dv - \frac{1}{3}\int \frac{dv}{3v+2} = \int dx$$

$$\Rightarrow \frac{2}{3}v - \frac{1}{9}\ln(3v+2) = x + \frac{1}{9}c$$

$$\Rightarrow 6v - \ln(3v+2) = 9x + c$$

$$\Rightarrow 6(x+y) - \ln(3x+3y+2) = 9x + c$$

$$\Rightarrow 6y - 3x - \ln(3x+3y+2) = c$$

Class-7 (27.11.23)

Linear D.E: If P and Q are only functions of x or constants then the D.E of the form $\frac{dy}{dx} + P.y = Q$ is called 1st order linear D.E

To solve this eqn multiply by $e^{\int P dx}$ → I.F (Integrating factor) then it becomes,

$$e^{\int P dx}\frac{dy}{dx} + Py e^{\int P dx} = Q e^{\int P dx}$$

$$\Rightarrow \frac{d}{dx}\left(y e^{\int P dx}\right) = Q e^{\int P dx}$$

Integrating both sides w.r.t to x, we get

$$y e^{\int P dx} = \int \left[Q e^{\int P dx}\right] dx + c$$

Integrating factor: An integrating factor is a function used to solve d.e.
It is a function in which an ~~ord~~ O.D.E can be multiplied to make the function integrable.

$$\frac{dy}{dx} + Py = Q$$

$$\therefore IF = e^{\int P dx}$$

Ques: Solve $x\frac{dy}{dx} - 4y = x^6 e^x$ ——— (1)

Divide (1) by x,

$$\frac{dy}{dx} - 4\frac{y}{x} = x^5 e^x \quad \text{——— (2)}$$

which is a 1st linear d.e of the form $\frac{dy}{dx} + Py = Q$ where $P = -\frac{4}{x}$

$$I.F = e^{\int(-\frac{4}{x})dx} = e^{-4\ln x} = e^{\ln x^{-4}} = x^{-4}$$

Multiply IF by (2),

$$x^{-4}\frac{dy}{dx} + (-\frac{4}{x})x^{-4}y = x^5 e^x x^{-4}$$

$$\Rightarrow x^{-4}\frac{dy}{dx} + (-4)x^{-5}y = xe^x$$

$$\Rightarrow \frac{d}{dx}(x^{-4}y) = xe^x$$

Integrating both sides

$$x^{-4}y = \int xe^x dx$$

$$\Rightarrow x\int e^x dx - \int(\frac{dx}{dx}\int e^x dx)dx = x^{-4}y$$

$$\Rightarrow xe^x - e^x + c = x^{-4}y$$

$$\Rightarrow y = x^4(xe^x - e^x) + c$$

Ques: Solve $(x^2+9)\frac{dy}{dx} + xy = 0$ — ①

Divide ① by x^2+9,

$$\frac{dy}{dx} + \frac{x}{x^2+9}y = 0 \quad \text{—— ②}$$

which is a linear d.e of the form $\frac{dy}{dx} + Py = Q$ where $P = \frac{x}{x^2+9}$

$$I.F = e^{\int \frac{x}{x^2+9}dx}$$

$$= e^{\frac{1}{2}\ln(x^2+9)} = e^{\ln(x^2+9)^{\frac{1}{2}}}$$

$$\Rightarrow (x^2+9)^{\frac{1}{2}}$$

$x^2+9 = z$
$\Rightarrow 2x\,dx = dz$
$\Rightarrow x\,dx = \frac{dz}{2}$
$\int \frac{dz}{2z}$
$= \frac{1}{2}\ln z$
$= \frac{1}{2}\ln(x^2+9)$

Multiply ② by I.F,

$$\frac{dy}{dx}(x^2+9)^{\frac{1}{2}} + \frac{x}{x^2+9}y(x^2+9)^{\frac{1}{2}} = 0$$

$$\Rightarrow \frac{dy}{dx}(x^2+9)^{\frac{1}{2}} + \frac{xy}{\sqrt{x^2+9}} = 0$$

$$\Rightarrow \frac{d}{dx}\{(x^2+9)^{\frac{1}{2}}y\} = 0$$

Integrating both sides,

$$(x^2+9)^{\frac{1}{2}}y = c$$

$$\Rightarrow y = \frac{c}{\sqrt{x^2+9}}$$

Ques: Solve $(1+e^x)\frac{dy}{dx} + e^x y = 0$ — ①

Divide ① by $(1+e^x)$,

$$\frac{dy}{dx} + \frac{e^x}{1+e^x}y = 0 \quad \text{—— ②}$$

which is a linear d.e of the form $\frac{dy}{dx} + Py = Q$ where $P = \frac{e^x}{1+e^x}$

$$I.F = e^{\int \frac{e^x}{1+e^x}dx}$$

$$= e^{\ln(1+e^x)}$$

$$= 1+e^x$$

$1+e^x = z$
$\Rightarrow e^x\,dx = dz$
$\int \frac{dz}{z}$
$= \ln z$
$= \ln(1+e^x)$

Multiply ② by I.F,

$$(1+e^x)\frac{dy}{dx} + e^x y = 0$$

Integrating both side,

$$\Rightarrow \frac{d}{dx}\{(1+e^x)y\} = 0$$

Integrating both side,

$$(1+e^x)y = c$$

$$\Rightarrow y = \frac{c}{1+e^x}$$

Bernaulli eqn: If P and Q are only functions of x, then the DE of the form $\frac{dy}{dx} + Py = Qy^n$ is called Bernaulli's D.E. Dividing the eqn by y^n it can be reduced to linear de.

Ques: Solve $\frac{dy}{dx} + \frac{y}{x} = y^2$ —— ①

Divide eqn ① by y^2,

$$\frac{1}{y^2}\frac{dy}{dx} + \frac{1}{xy} = 1 \text{ —— ②}$$

Put $v = -\frac{1}{y}$

$$\Rightarrow \frac{dv}{dx} = \frac{1}{y^2}\frac{dy}{dx}$$

Putting the values in ②,

$$\frac{dv}{dx} + \left(-\frac{1}{x}\right)v = 1$$

which is a linear eqn,

$$IF = e^{\int -\frac{1}{x}dx} = e^{-\ln x} = e^{\ln x^{-1}} = x^{-1}$$

$$x^{-1}\frac{dv}{dx} + x^{-1}\left(-\frac{1}{x}\right)v = x^{-1}$$

$$\Rightarrow x^{-1}\frac{dv}{dx} - \frac{v}{x^2} = x^{-1}$$

$$\Rightarrow \frac{d}{dx}(x^{-1}v) = x^{-1}$$

$$\Rightarrow \int \frac{d}{dx}(x^{-1}v)dx = \int x^{-1}dx$$

$$\Rightarrow x^{-1}v = \ln x + c$$

$$\Rightarrow v = x(\ln x + c)$$

$$\Rightarrow -\frac{1}{y} = x(\ln x + c)$$

$$\Rightarrow xy(\ln x + c) = -1$$

Clairauits eqn: An eqn

of the form $\boxed{y = px + f(p)}$ is called Clairauits eqn.

when $p = \frac{dy}{dx}$

Complete and singular soln (S.S)

the given eqn is $y = px + f(p)$ ——— (1)

Differentiating (1) wr to x, we get,

$$\frac{dy}{dx} = p + x\frac{dp}{dx} + f'(p)\frac{dp}{dx}$$

$$p = p + x\frac{dp}{dx} + f'(p)$$

$$[x + f'(p)]\frac{dp}{dx} = 0$$

$$\frac{dp}{dx} = 0 \quad \text{——— (2)}$$

$$x + f'(p) = 0 \quad \text{——— (3)}$$

Now integrating (2) wrt to x, $p = c$, where c is an arbitrary constant.

Now putting the value of p in (1),

$y = cx + f(c)$ which is the required general soln of (1)

Again eliminating p from (1) and (3), to obtain another soln involving no arbitrary constant this soln is called the singular soln (S.S)

Solve: $y = px + \frac{a^2}{p}$ and find the singular soln (If exists)

Soln: Given eqn; $y = px + \frac{a^2}{p}$ —— (1)

which is of the form ~~$y = px + f(p)$~~ ~~Elaira~~ of Clairaut's eqn.

Now diff (1) w.r to x,

$$\frac{dy}{dx} = p + x\frac{dp}{dx} - \frac{a^2}{p^2}\frac{dp}{dx}$$

$$\Rightarrow p = p + x\frac{dp}{dx} - \frac{a^2}{p^2}\frac{dp}{dx}$$

$$\Rightarrow \left(x - \frac{a^2}{p^2}\right)\frac{dp}{dx} = 0$$

$$\frac{dp}{dx} = 0 \quad \text{——} (2)$$

$$\left(x - \frac{a^2}{p^2}\right) = 0 \quad \text{——} (3)$$

Now integreating (2) wrt to x, $p = c$ where c is an arbitrary constant.

Now putting the value of p in ①,

$y = cx + \frac{a^2}{c}$ which is the required general soln of ①

~~From ③,~~

Again from ③, we get,

$$x - \frac{a^2}{p^2} = 0$$

$$\Rightarrow x = \frac{a^2}{p^2}$$

$$\Rightarrow p = \frac{a}{\sqrt{x}}$$

Putting the value of p in ①,

$$y = \frac{a}{\sqrt{x}}x + \frac{a^2}{\frac{a}{\sqrt{x}}} = a\sqrt{x} + a\sqrt{x}$$

$\Rightarrow y = 2a\sqrt{x} \Rightarrow y^2 = 4a^2x$ which is the singular soln.

Solve: $(y+1)p - xp^2 + 2 = 0$

where $p = \frac{dy}{dx}$

The given eqn is,

$$(y+1)p - xp^2 + 2 = 0 \quad \text{——①}$$

$$\Rightarrow (y+1)p = xp^2 - 2$$

$$\Rightarrow y + 1 = \frac{xp^2 - 2}{p} = xp - \frac{2}{p}$$

which is the Clairaut's form

$y = px + f(p)$

Hence putting c for p, the general soln is,

$$y = xc + \left(-\frac{2}{c} - 1\right)$$

$$= cx - \left(\frac{2}{c} + 1\right)$$