51/90 ~~SB~~ SA 27/09/23

MATH 103

Differential Equation & Matrix

ODE & PDE

ODE (Ordinary Differential Equation) : Sec A
PDE & Matrix : Sec B

↳ Mon 1000 (0/7)
wed 1000 (7/7)

Gen Points

(Academics)

- Student
- Classroom
  - Ø Light, Fan, Air con.
  - Ø Seating Arrangement
  - Ø Visitors chair
  - Ø Duster, Marker
  - Ø Rub the Board
  - Ø Class info

- Lecture
- Assignment

- Discipline
- Turn out & Bearing
- Dress
- Norms > How to talk/behave
  " " " w/ mobile
- On time
- Salutation

CR
Ø Class state Ø Liasion

SA 11/10/23

D.E.

M.D. Raisinghania

Q. Define the followings:

1. Differential Equation (D.E)
2. Ordinary differential equation (ODE)
3. Partial differential equation (PDE)
4. Order of a differential equation
5. Degree of a " "

Syllabus

1. Defⁿ, Order & Degree, Formation of D.E.
2. First Order & First Degree D.E → 06 subsections parts
3. First " but higher " " → 04
4. Second and higher Order D.E. → 06
5. Homogeneous Linear D.E.
6. Series Sol", Frobenius Method
7. Bessel's function
8. Legendre polynomial

1. Differential Eq^n (D.E)

An equation involving dependent variable, independent variable, and differential co-efficient of dependent variable with respect to independent variable is called a differential equation (D.E).

The variables may or may not lie in d.e. but the differential co-efficient must be lie in d.e.

x is called independent variable and y is called dependent variable. Derivative of y with respect to x is denoted by $\frac{dy}{dx}$.

Example:

1. $\frac{dy}{dx} + (1-y^2)\tan x = 0$

2. $\frac{dy}{dn} = 1$

3. $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz$

~~There are~~

There are two types of D.E

1. ODE
2. PDE

ODE

An eqn involving one dependent variable, one independent variable and derivative of dependent variable with respect to a single independent variable is called Ordinary Differential Equation (ODE)

Example: 1. $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + by = 0$

PDE

An eqn involving one dependent variable, two or more independent variable and partial differential co-efficient with respect to more than one independent variable is called partial differential equation (PDE)

Example: 1. $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 3z$

Order of a D.E.

The order of the highest order differential co-efficient ~~co-efficient~~ involved in a d.e. is called the order of the d.e.

Example: 1. $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 0$

2. $\frac{d^4y}{dx^4} + 7\frac{d^2y}{dx^2} + 5y = \cos^2 x$

In the above, the eqn(1) is of second order and the eqn(2) is of fourth order.

Degree of a D.E.

The degree or power of the highest order derivative of a d.e. is called the degree of a d.e when the eqn has been made free from fraction.

Example:

1. $\frac{d^3y}{dx^3} + 3\left(\frac{dy}{dx}\right)^4 + y = 0$

2. $\frac{d^2y}{dx^2} = K\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{5/3}$

Since, eqn (1), the power of the highest thind order derivative is one, so the degree of eqn is one.

If so power of $\frac{dy}{dx}$ is 4 but $\frac{dy}{dx}$ is not the highest order differential co-efficient. So the degree of eqn is not 4. Hence the degree of eqn is one.

If eliminating the fractional power of eqn (2) the eqn is of the form

$$\left(\frac{d^2y}{dx^2}\right)^3 = k^3\left[1+\left(\frac{dy}{dx}\right)^2\right]^5$$

In this case, the power of highest second order derivative is 3, Hence, degree of eqn is 3.

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Q Find out the order and degree of the following with justification:

1. $\left(\frac{d^3y}{dx^3}\right)^2 + \left(\frac{dy}{dx}\right)^4 + 3y = 0$

2. $\sqrt{\left(\frac{d^2y}{dx^2}\right)^2 + \frac{dy}{dx} + 7} = \sqrt{\left(\frac{d^4y}{dx^4}\right)^3 + 7y}$

---

1. $\left(\frac{d^3y}{dx^3}\right)^2 + \left(\frac{dy}{dx}\right)^4 + 3y = 0$

In the above, the highest differential co-efficient is 3. So, order of the equation is 3 and power of the highest 3rd order differential co-efficient is 2. So, degree of the eqn is 2.

Hence,

Order = 3

Degree = 2

2.

$$\sqrt[3]{\left(\frac{d^2y}{dx^2}\right)^2 + \frac{dy}{dx} + 7} = \sqrt[5]{\left(\frac{d^4y}{dx^4}\right)^3 + 7y}$$

In the above, if eliminating the fractional power of the eqn, the eqn is of the form,

$$\left(\frac{d^2y}{dx^2} + \frac{dy}{dx} + 7\right)^5 = \left\{\left(\frac{d^4y}{dx^4}\right)^3 + 7y\right\}^3.$$

DE → 9

Order → 4.

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~~1. Formation of D.E of $f(x,y,a)=0$~~

Formation of D.E.

1. Formation of D.E of $f(x,y,a)=0$

Sol^n: Given eq^n $f(x,y,a)=0$ — (1)

Differentiating (1) w.r.t x

$$\phi\left(x, y, a, \frac{dy}{dx}\right)=0 \quad \text{—} \quad (2)$$

Eliminating a from (1) & (2) we get

$$\phi\left(x, y, \frac{dy}{dx}\right)=0$$

this is the required d.e

Q. Find the d.e from a straight line $y=mx$ or, form the d.e of all straight line passing through the origin.

Soln: Given $y = mx$ — (1) or,

Let $y = mx$ be the eqn of straight line passing through the origin.

Differentiate (1) w.r.to x

$$\frac{dy}{dx} = m \cdot 1 = m \quad \text{— (2)}$$

Putting the value of m in (1)

$$y = \frac{dy}{dx} x$$

$$y = x \frac{dy}{dx}$$

which is the required de

Again, differentiating (2) w.r.to x

$$\phi\left(x, y, a, b, \frac{dy}{dx}, \frac{d^2y}{dx^2}\right) = 0 \quad \text{— (3)}$$

Eliminating a, b from (1), (2) & (3)

$$\phi\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}\right) = 0$$

this is the required d.e.

Q. Find the d.e of the curves $y = e^x(A\cos x + B\sin x)$, where A and B are ~~arbitrary~~ arbitrary constants.

Soln: Given, $y = e^x(A\cos x + B\sin x)$ ——(1)

Differentiate (1) w.r to x

$$\frac{dy}{dx} = e^x(A\cos x + B\sin x) + e^x(-A\sin x + B\cos x)$$

$$\frac{dy}{dx} = y + e^x(-A\sin x + B\cos x) \quad ——(2)$$

Again differentiate (2) w.r to x

$$\frac{d^2y}{dx^2} = \frac{dy}{dx} + e^x(-A\sin x + B\cos x) - e^x(A\cos x + B\sin x)$$

$$\frac{d^2y}{dx^2} = \frac{dy}{dx} + \frac{d^2y}{dx^2} - y + \frac{dy}{dx} - e^x(A\cos x + B\sin x)$$

$$= \frac{dy}{dx} - y + \frac{dy}{dx} - y$$

$$\therefore \frac{d^2y}{dx^2} = 2\frac{dy}{dx} - 2y$$

This

Q. Form the d.e corresponding to

$y = ae^{2x} + be^{-3x} + ce^{x}$, where a, b, c are arbitrary constants.

Soln: Given eqn, $y = ae^{2x} + be^{-3x} + ce^{x}$ —— (1)

Differentiate (1) w.r.to x

$$\frac{dy}{dx} = 2ae^{2x} - 3be^{-3x} + ce^{x} \quad \text{—— (2)}$$

Subtract (1) from (2)

$$\frac{dy}{dx} - y = ae^{2x} - 4be^{-3x} \quad \text{—— (3)}$$

Again differentiate (3) w.r.to x

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} = 2ae^{2x} + 12be^{-3x} \quad \text{—— (4)}$$

Multiply eqn (3) by 3 and adding with eqn (4)

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} + 3\left(\frac{dy}{dx} - y\right) = 5ae^{2x} \quad \text{—— (5)}$$

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 3y = 5ae^{2x} \quad \text{—— (5)}$$

Again differentiate,

$$\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} - 3\frac{dy}{dx} = 10ae^{2x}$$

$$= 2(5ae^{2x})$$

$$\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} - 3\frac{dy}{dx} = 2\left(\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 3y\right) \quad [\text{From } 5]$$

$$\frac{d^3y}{dx^3} - 7\frac{dy}{dx} + 6y = 0$$

which is the required d.e.

Given eqn. $y = Ae^{2x} + Be^{-2x}$ — (1)

Differentiate (1) w.r.t to x

$$\frac{dy}{dx} = 2Ae^{2x} - 2Be^{-2x} \quad \text{— (2)}$$

multiply eqn (1) by 2 and subtracting from eqn (2)

$$\frac{dy}{dx} - 2y = 2Ae^{2x} - 2Ae^{2x} - 2Be^{-2x} - 2Be^{-2x}$$

$$= -4Be^{-2x}$$

08/11/23

SA

First order and first degree differential equation

A differential equation of the form $M + N\frac{dy}{dx} = 0$ or $Mdx + Ndy = 0$ is called first order and first degree differential equation where both M and N are function of x and y or constants.

Types of differential equation of first order and first degree

1. Separation of variables
2. Homogeneous equation
3. Equation reducible to homogeneous
4. Exact equation
5. Linear equation
6. Reducible to linear equation

Seperation of variables

If the de $M dx + N dy = 0$ can be expressed in the form $f(x)\,dx + \phi(y)\,dy = 0$ then if v called variables are seperable.

Rule

If d.e is of the form $f(x)\,dx + \phi(y)\,dy = 0$ i.e., function of x with dx and function of y with dy then only integrate we can find the solution.

Solve: $\frac{dy}{dx} + \frac{y^2+y+1}{x^2+x+1} = 0$

Soln:

$$\frac{dy}{dx} = -\frac{y^2+y+1}{x^2+x+1}$$

$$\Rightarrow \frac{dy}{y^2+y+1} = -\frac{dx}{x^2+x+1}$$

$$\Rightarrow \frac{dx}{x^2+x+1} + \frac{dy}{y^2+y+1} = 0$$

Integrate:

$$\int \frac{dx}{x^2+x+1} + \int \frac{dy}{y^2+y+1} = 0$$

$$\Rightarrow \int \frac{dx}{(x+1/2)^2+(\sqrt{3}/2)^2} + \int \frac{dy}{(y+1/2)^2+(\sqrt{3}/2)^2} = 0$$

$$\Rightarrow \frac{2}{\sqrt{3}} \tan^{-1}\frac{x+1/2}{\sqrt{3}/2} + \frac{2}{\sqrt{3}} \tan^{-1}\frac{y+1/2}{\sqrt{3}/2} = C$$

Rule 2

If any d.e. of the form $f(x) + \phi(y)\,dy + f(y)\,\phi(x)\,dy = 0$ then divide the equation by $\phi(x)\,\phi(y)$

The equation is of the form

$$\frac{f(x)}{\phi(x)}\,dx + \frac{f(y)}{\phi(y)}\,dy = 0$$

ie, function of x with dx and function of y with dy then only integrate we can find the solution.

Solve: $\sec^2 x \tan y\,dx + \sec^2 y \tan x\,dy = 0$ —— (1)

Divide eq^n (1) by $\tan x \tan y$

$$\frac{\sec^2 x}{\tan x}\,dx + \frac{\sec^2 y}{\tan y}\,dy = 0$$

Integrate $\int \frac{\sec^2 x}{\tan x}\,dx + \int \frac{\sec^2 y}{\tan y}\,dy = 0$

$\Rightarrow \ln(\tan x) + \ln(\tan y) = \ln c$

$\Rightarrow \ln(\tan x \tan y) = \ln c$

Solve $x\sqrt{1-y^2}\,dx + y\sqrt{1-x^2}\,dy = 0$

Ans. $\sqrt{1-x^2} + \sqrt{1-y^2} = c$

Rough / Exact

③ Solve: $(2x+3y-5)dy + (3x+2y-5)dx = 0$

which is in the form $N dy + M dx = 0$

where,

$M = (3x+2y-5)$

$N = (2x+3y-5)$

Now, $\frac{\partial M}{\partial y} = 2$ $\quad$ $\frac{\partial N}{\partial x} = 2$

$\therefore \frac{\partial M}{\partial x} = \frac{\partial N}{\partial y}$; the given differential eqⁿ is exact.

The solⁿ of the eqⁿ is,

$$\int (3x+2y-5)dx + \int (3y-5)dy = c$$

$$\Rightarrow \frac{3}{2}x^2 + 2xy - 5x + \frac{3}{2}y^2 - 5y = c$$

$$\Rightarrow 3x^2 + 3y^2 + 4xy - 10x - 10y = c$$

$\underbrace{(x+y)^2}_{M}dx - \underbrace{(y^2-2xy-x^2)}_{N}dy = 0 \Rightarrow (x+y)^2dx + (x^2+2xy-y^2)dy = 0$

$\frac{\delta M}{\delta y} = 2(x+y) \qquad \frac{\delta N}{\delta x} = -(-2y-2x)$

$= 2(x+y)$

$\int (x+y)^2 dx + \int (-y^2)dy = c$

$\Rightarrow \int (x^2+2xy+y^2)dx + \int (2xy-y^2)dy = c$

$\Rightarrow \frac{x^3}{3} + x^2y + xy^2 + xy^2 - \frac{y^3}{3} = c$

$\Rightarrow x^3 - y^3 + 3x^2y = c$

Rough / CT-2

Given $y = e^x(A\cos x + B\sin x)$

$$\frac{dy}{dx} = e^x(A\cos x + B\sin x) + e^x(-A\sin x + B\cos x)$$

$$= y + e^x(-A\sin x + B\cos x)$$

$$\frac{d^2y}{dx^2} = \frac{dy}{dx} + e^x(-A\sin x + B\cos x) + e^x(A\cos x + B\sin x)$$

$$= \frac{dy}{dx} - y + \left(\frac{dy}{dx} - y\right)$$

$$= 2\frac{dy}{dx} - 2y$$

$$y = ae^{2x} + be^{-3x} + ce^{x} \quad \text{—— (I)}$$

$$\frac{dy}{dx} = 2ae^{2x} - 3be^{-3x} + ce^{x} \quad \text{—— (II)}$$

(II) − (I) $\Rightarrow \frac{dy}{dx} - y = ae^{2x} - 4be^{-3x}$ —— (III)

$\Rightarrow \frac{d^2y}{dx^2} - \frac{dy}{dx} = 2ae^{2x} + 12be^{-3x}$ —— (IV)

(IV) − 2(III) $\Rightarrow \frac{d^2y}{dx^2} - \frac{dy}{dx} - 2\frac{dy}{dx} + 2y = 20be^{-3x}$

$\Rightarrow \frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 20be^{-3x}$

$\Rightarrow \frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = -60be^{-3x}$

$= -3(20be^{-3x})$

$\Rightarrow \frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = -3\frac{d^2y}{dx^2} + 9\frac{dy}{dx} - 6y$

$\Rightarrow \frac{d^3y}{dx^3} - 7\frac{dy}{dx} + 6y = 0$

$y = Ae^{2x} + Be^{-2x}$ — (I)

$\frac{dy}{dx} = 2Ae^{2x} - 2Be^{-2x}$ — (II)

(II) + 2(I) $\Rightarrow \frac{dy}{dx} + y = 4Ae^{2x}$

$\Rightarrow \frac{d^2y}{dx^2} + \frac{dy}{dx} = 2\frac{dy}{dx} + 2y$

$\Rightarrow \frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 0$

$C_2 = \frac{x - C_1 + y\frac{dy}{dx}}{dy/dx}$

$(x-C_1)^2 + (y-C_2)^2 = a^2$

$\Rightarrow x^2 - 2xC_1 + C_1^2 + y^2 - 2yC_2 + C_2^2 = a^2$

$\Rightarrow x^2 + y^2 - 2xC_1 - 2yC_2 + C_1^2 + C_2^2 - a^2 = 0$

$\Rightarrow 2x + 2y\frac{dy}{dx} - 2C_1 - 2C_2\frac{dy}{dx} = 0$

$\Rightarrow 2 + 2y\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 2C_2\frac{d^2y}{dx^2} = 0$

$\Rightarrow y\frac{d^2y}{dx^2} + \frac{dy}{dx} = C_2\frac{d^2y}{dx^2} - 2$

$\Rightarrow y\frac{d^3y}{dx^2} + \frac{d^2y}{dx^2} + \frac{dy}{dx} = C_2\frac{d^2y}{dx^2}$

$(x-C_1)^2 + (y-C_2)^2 = a^2$

$\Rightarrow 2(x-C_1) + 2(y-C_2)\frac{dy}{dx} = 0$

$\Rightarrow x - C_1 + y\frac{dy}{dx} - C_2\frac{dy}{dx} = 0$

$\Rightarrow 1 + y\frac{d^2y}{dx^2} + \frac{dy}{dx} - C_2\frac{d^2y}{dx^2} = 0$

$\Rightarrow$

$$(x-c_1)^2+(y-c_2)^2=a^2 \quad \text{—(i)}$$

$$\Rightarrow 2(x-c_1)+2(y-c_2)y_1'=0 \quad \text{—(ii)}$$

$$\Rightarrow 1+(y-c_2)y_2''+(y')^2=0 \quad \text{—(iii)}$$

(iii) $\Rightarrow c_2=\dfrac{(y')^2+1}{y''}+y$

(ii) $\Rightarrow (x-c_1)+\left(y-y-\dfrac{y'^2+1}{y''}\right)y'=0$

$$\Rightarrow (x-c_1)-\left(\frac{y'^2+1}{y''}\right)y'=0$$

$$\Rightarrow (x-c_1)=\frac{y'^2+1}{y''}y'$$

(i) $\Rightarrow \left\{\left(\dfrac{y'^2+1}{y''}\right)y'\right\}^2+\left(-\dfrac{y'^2+1}{y''}\right)^2=a^2$

$$\Rightarrow y'^2\left(\frac{y'^2+1}{y''}\right)^2+\left(\frac{y'^2+1}{y''}\right)^2=a^2$$

$$\Rightarrow \left(\frac{y'^2+1}{y''}\right)^2(y'^2+1)=a^2$$

$$\Rightarrow (y'^2+1)^2(y'^2+1)=a^2(y'')^2$$

$$\Rightarrow (y')^2+1)^3=(ay'')^2$$



$$\frac{dy}{dx} + \frac{y^2+y+1}{x^2+x+1} = 0$$

$$\Rightarrow \frac{dy}{dx} = -\frac{y^2+y+1}{x^2+x+1}$$

$$\Rightarrow -\frac{dx}{x^2+x+1} = \frac{dy}{y^2+y+1}$$

$$\Rightarrow \frac{1}{x^2+x+1}dx + \frac{1}{y^2+y+1}dy = 0$$

x^2+x+1

$x^2+2x\frac{1}{2}+\frac{1}{4}+\frac{3}{4}$

$\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2$

soln:

$$\int \frac{1}{x^2+x+1}dx + \int \frac{1}{y^2+y+1}dy = 0$$

$$= \int \frac{1}{(x+1/2)^2+(\sqrt{3}/2)^2}dx + \int \frac{1}{(y+1/2)^2+(\sqrt{3}/2)^2}dy = 0$$

$$= \frac{1}{\sqrt{3}/2}\tan^{-1}\frac{x+1/2}{\sqrt{3}/2} + \frac{1}{\sqrt{3}/2}\tan^{-1}\frac{y+1/2}{\sqrt{3}/2} + c = 0$$

$$\sec^2 x \tan y \, dx + \sec^2 y \tan x \, dy = 0$$

$$\Rightarrow \frac{\sec^2 x}{\tan x} dx + \frac{\sec^2 y}{\tan y} dy = 0$$

$$\int \frac{\sec^2 x}{\tan x} dx + \int \frac{\sec^2 y}{\tan y} dy = 0$$

$$\Rightarrow \ln|\tan x| + \ln|\tan y| = C$$

$$x\sqrt{1-y^2}\, dx + y\sqrt{1-x^2}\, dy = 0$$

$$\Rightarrow \frac{x}{\sqrt{1-x^2}} dx + \frac{y}{\sqrt{1-y^2}} dy = 0$$

$$\int -\frac{1}{2} \frac{-2x}{\sqrt{1-x^2}} dx + \int -\frac{1}{2} \frac{-2y}{\sqrt{1-y^2}} dy = 0$$

$$\Rightarrow -\frac{1}{2}\left[2\sqrt{1-x^2} + 2\sqrt{1-y^2}\right] = C$$

$$\Rightarrow \sqrt{1-x^2} + \sqrt{1-y^2} = C$$

$$\int \frac{-2x}{\sqrt{1-x^2}} \qquad \ln|\sqrt{1-x^2}|$$

$$\frac{1}{\sqrt{1-x^2}} \left(-2x\right)$$

$$\frac{dy}{dx} = (4x+y+1)^2$$

—

let, $z = 4x+y+1$

$$\frac{dz}{dx} = 4 + \frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{dz}{dx} - 4$$

now,

$$\frac{dz}{dx} - 4 = z^2$$

$$\Rightarrow \frac{dz}{dx} = z^2 + 4$$

$$\Rightarrow \frac{1}{z^2+4}\, dz = dx$$

integrate:

$$\int \frac{1}{z^2+2^2}\, dz = \int dx$$

$$\Rightarrow \frac{1}{2}\tan^{-1}\frac{z}{2} = x + c$$

SA 11/12/23

CT 3

Date: 17 Dec 2023

Day : Sun

Time: 0800

Syllabus

1. Homogeneous eq^n
2. Linear eq^n

   Bernoulli eq^n
3. clairauits eq^n

Clairaut's eqn

An eqn of the form $y = px + f(p)$ is called clairaut's eqn.

when $p = \frac{dy}{dx}$

Complete and singular solution (C.S)

The given eqn is $y = px + f(p) \quad -(1)$

Differentiating (1) w.r to x, we get

$$\frac{dy}{dx} = p + x\frac{dp}{dx} + f'(p)\frac{dp}{dx}$$

$$p = p + x\frac{dp}{dx} + f'(p)$$

$$[x + f'(p)]\frac{dp}{dx} = 0$$

$\frac{dp}{dx} = 0 \quad -(2)$ $\qquad$ $x + f'(p) = 0 \quad -(3)$

Now, integrating (2) w.r to x,

$p = c$, where c is an arbitrary constant

Now putting the value of p in (1)

$y = cx + f(c)$ which is the required general soln of (1).

Again, eliminating p from (1) and (3), to obtain another soln involving no arbitrary constant. this soln is called the singular soln.

Solve: $y = px + \frac{a^2}{p}$ and find the singular soln (if exists)

—

Solve: Given equation,

$$y = px + \frac{a^2}{p}$$

which is of the form of clairauits eqn.

Now differentiating (1) w.r to x

$$\frac{dy}{dx} = p + x\frac{dp}{dx} - \frac{a^2}{p^2}\frac{dp}{dx}$$

$$p = p + \left(x - \frac{a^2}{p^2}\right)\frac{dp}{dx}$$

$$\left(x - \frac{a^2}{p^2}\right)\frac{dp}{dx} = 0$$

$$\frac{dp}{dx} = 0 \quad — (2) \qquad x - \frac{a^2}{p^2} = 0 \quad — (3)$$

Now integrating (2) w.r to x

$p = c$, where c is an arbitrary const

Now, putting the value of p in (1)

$y = cx + \frac{a^2}{c}$ which is the required general soln of (1).

Again from (3) we get

$$x - \frac{a^2}{p^2} = 0$$

$$x = \frac{a^2}{p^2}$$

$$p = \frac{a}{\sqrt{x}}$$

Putting the value of p in (1)

$$y = \frac{a}{\sqrt{x}} x + \frac{a^2}{a\sqrt{x}}$$

$$y = a\sqrt{x} + a\sqrt{x}$$

$$= 2a\sqrt{x}$$

$y^2 = 4a^2x$ which is the singular soln (S.S)

Solve:

$(y+1)p = xp^2 + 2 = 0$ where $p = \frac{dy}{dx}$

Soln: Given equation,

$$(y+1)p - xp^2 + 2 = 0$$

$$\Rightarrow yp + p - xp^2 + 2 = 0$$

$$\Rightarrow yp = xp^2 - p - 2$$

$$\Rightarrow y = xp - \frac{2}{p} - 1 \quad \text{---} (1)$$

which is of the form of Clairaut's eqn

Now, differentiating (1) wr to x

$$\frac{dy}{dx} = x\frac{dp}{dx} + p + \frac{2}{p^2}\frac{dp}{dx}$$

$$\Rightarrow p = p + x\frac{dp}{dx} + \frac{2}{p^2}\frac{dp}{dx}$$

$$\Rightarrow \left(x + \frac{2}{p^2}\right)\frac{dp}{dx} = 0$$

$\frac{dP}{dx} = 0$ — (2) $\qquad (x + \frac{2}{p^2}) = 0$ — (3)

Now integrating (2) wr to x

$P = C$, where C is an arbitrary const.

Now putting the value . . .

Next. Lagranges eqn second & higher order

SA 13/12/23

Solution of Second and higher order

1. If $e^{m_1 x}, e^{m_2 x}, e^{m_3 x}, \ldots, e^{m_n x}$ be the n independent solutions of the d.e

$$P_0 \frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + P_n y = 0$$

Then its general soln is

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \cdots + c_n e^{m_n x}$$

where $c_1, c_2, \ldots, c_n$ are arbitrary constants.

2. If the d.e $D^2 y + P_1 Dy + P_2 y = 0$

has two independent solutions $y = e^{m_1 x}$ and $y = e^{m_2 x}$

then its general soln is

$$y = c_1 e^{m_1 n} + c_2 e^{m_2 n}$$

where c_1, c_2 are arbitrary constants an $D = \frac{d}{dx}$

3. Auxiliary equation (A.E)

Let the d.e be $D^2y + P_1Dy + P_2y = 0$ —(1)

Let the soln of (1) be $y = e^{mx}$, so it will satisfy (1)

$$D^2(e^{mx}) + P_1D(e^{mx}) + P_2e^{mx} = 0$$

$$m^2e^{mx} + P_1me^{mx} + P_2e^{mx} = 0$$

$$(m^2 + P_1m + P_2)e^{mx} = 0$$

Since $e^{mx} \neq 0$ so

$$m^2 + P_1m + P_2 = 0$$

this is called Auxiliary equation (A.E)

Following three cases should be remembered,

Case 1. Let the roots are real and unequal

If $m = \alpha, \beta$, the its general soln is

$$y = c_1 e^{\alpha x} + c_2 e^{\beta x}$$

Case 2. Let the roots are real and equal

If $m = \alpha, \alpha$ then it gen soln is

$$y = (c_1 + c_2 x) e^{\alpha x}$$

Case 3. Let the roots are imaginary

If $m = \alpha \pm i\beta$, then its gen soln is

$$y = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x)$$

- $y = e^{mx}$
- Auxiliary Equal
- Find Roots
- Solution

Solve: $(2D^2 - 3D + 1)y = 0$

Soln: Given, $(2D^2 - 3D + 1)y = 0$ —(1)

Let $y = e^{mx}$ be the trial soln of (1)

$$Dy = me^{mx}, \quad D^2y = m^2e^{mx}$$

Putting the values in (1)

$$(2m^2 - 3m + 1)e^{mx} = 0$$

Since, $e^{mx} \neq 0$

∴ Auxiliary eqn is

$$2m^2 - 3m + 1 = 0$$

$$2m^2 - 2m - m + 1 = 0$$

$$2m(m-1) - 1(m-1) = 0$$

$$(m-1)(2m-1) = 0$$

$$m = 1, \frac{1}{2}$$

∴ Then the gen soln is

$$y = c_1e^x + c_2e^{\frac{1}{2}x}$$

Solve: $(D^2-2D+1)y=0$

Soln: Given $(D^2-2D+1)y=0$ —— (1)

Let $y=e^{mx}$ be the trial solution of (1)

$$Dy = me^{mx}, \quad D^2y = m^2e^{mx}$$

Putting the values in (1)

$$(m^2e^{mx} - 2me^{mx} + 1)y = 0$$

$$\Rightarrow e^{mx}(m^2-2m+1) = 0$$

Since $e^{mx} \neq 0$,

$\therefore$ Auxiliary eqn is

$$m^2-2m+1=0$$

$$m^2-m-m+1=0$$

$$\Rightarrow m(m-1)-1(m-1)=0$$

$$\Rightarrow (m-1)(m-1)=0$$

$$m=1, 1$$

$\therefore$ the gen soln is $(c_1+c_2x)e^x$

Solve: $(5D^2-2D+3)y=0$

Soln: Given, $(5D^2-2D+3)y=0$ —— (1)

Let $y=e^{mx}$ be the trial soln of (1)

$$Dy = me^{mx}, \quad D^2y = m^2e^{mx}$$

Putting value in (1)

$$(5m^2-2m+3)e^{mx}=0$$

Since, $e^{mx} \neq 0$,

∴ Auxiliary eqn is

$$5m^2-2m+3=0$$

$$m = \frac{2 \pm \sqrt{4-60}}{10}$$

$$= \frac{2 \pm i2\sqrt{14}}{10}$$

$$m = \frac{1}{5} \pm \frac{i\sqrt{14}}{5}$$

$$\alpha + i\beta$$

∴ then the gen soln is $y = e^{\frac{1}{5}x}\left(C_1 \cos \frac{\sqrt{14}}{5}x + C_2 \sin \frac{\sqrt{14}}{5}x\right)$

Solve.

4. $(D^2+D-6)y=0$

5. $(5D^2-D+3)y=0$

6. $(D^3-3D+2)y=0$

7. $(D^2-4D+13)y=0$

8. $(D^4+4)y=0$

5. Soln: Given $(5D^2 - D + 3)y = 0$ — (1)

Let $y = e^{mx}$ be the soln of (1).

$$Dy = me^{mx}, \quad D^2y = m^2e^{mx}$$

Putting the values in (1),

$$(5m^2 - m + 3)e^{mx} = 0$$

Since, $e^{mx} \neq 0$,

The auxiliary eqn is

$$(5m^2 - m + 3) = 0$$

$$m = \frac{1 \pm \sqrt{1 - 4 \times 5 \times 3}}{10}$$

$$= \frac{1 \pm i\sqrt{59}}{10}$$

$$= \frac{1}{10} \pm i\frac{\sqrt{59}}{10}$$

$$\alpha + i\beta$$

∴ The gen soln is

$$y = e^{1/10}\left(C_1 \cos\frac{\sqrt{59}}{10}x + C_2 \sin\frac{\sqrt{59}}{10}x\right)$$

Rough CT-3

Homogenous D.E: An eq^n of the form $\frac{dy}{dx} = \frac{f(x,y)}{\phi(x,y)}$ where both $f(x,y)$ & $\phi(x,y)$ ~~hom~~ are homogenous func. of x, y and are of same degree.

Solve $(x^2+y^2)\,dx = 2xy\,dy$

$$\Rightarrow \frac{dy}{dx} = \frac{x^2+y^2}{2xy} \quad \text{— (1)}$$

Put $y = vx$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

Putting the values in (1),

$$v + x\frac{dv}{dx} = \frac{x^2+y^2}{2xvx}$$

$$= \frac{1+y^2}{2v}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{1+y^2}{2v} - v$$

$$v + x\frac{dv}{dx} = \frac{x^2+v^2x^2}{2xvx}$$

$$= \frac{1+v^2}{2v}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{1+v^2}{2v} - v$$

$$= \frac{1-v^2}{2v}$$

$$\Rightarrow x\frac{dv}{dx} - \frac{v^2-1}{2v} = 0$$

$$\Rightarrow$$

$$\Rightarrow \frac{x}{dx} = \frac{1-v^2}{2v}\frac{1}{dv}$$

$$\Rightarrow \frac{dx}{x} = \frac{2v\,dv}{1-v^2}$$

$$\Rightarrow \frac{dx}{x} + \frac{2v\,dv}{v^2-1} = 0$$

Integrating both sides,

$$\int \frac{dx}{x} + \int \frac{2v}{v^2-1}\, dv = 0$$

$$\Rightarrow \ln|x| + \ln|v^2-1| = 0 \ln C$$

$$\Rightarrow \ln|(v^2-1)x| = \ln C$$

$$\Rightarrow (v^2-1)x = C$$

$$\Rightarrow \left(\frac{y^2}{x^2} - 1\right) x = C$$

$$\Rightarrow \frac{y^2-x^2}{x^2} x = C$$

$$\Rightarrow y^2 - x^2 = xC$$

Solve: $(x^2+y^2)\,dx + 2xy\,dy = 0$

$\Rightarrow (x^2+y^2)\,dx = -2xy\,dy$

$\Rightarrow \frac{dy}{dx} = -\frac{x^2+y^2}{2xy}$

Let $y = vx$,

$\frac{dy}{dx} = v + x\frac{dv}{dx}$

Now,

$v + x\frac{dv}{dx} = -\frac{x^2+v^2x^2}{2xvx}$

$= \frac{1+v^2}{2v}$

$\Rightarrow x\frac{dv}{dx} = -\frac{1+v^2}{2v} - v$

$= -\frac{1+3v^2}{2v}$

$\Rightarrow \frac{x}{dx} = -\frac{1+3v^2}{2v}\frac{1}{dv}$

$\Rightarrow \frac{1}{x}dx + \frac{2v}{1+3v^2}dv = 0$

Integrate:

$\int \frac{1}{x}dx + \int \frac{2v}{1+3v^2}dv = 0$

$\Rightarrow \ln|x| + \frac{1}{3}\int \frac{du}{u} = c$

$\Rightarrow \ln|x| + \frac{1}{3}\ln|1+3v^2| = \ln c$

$\Rightarrow x(1+3v^2)^{1/3} = c$

$\Rightarrow \dots$

let,
$u = 1+3v^2$
$du = 6v\,dv$

Eq^n reducible to homogenous eq^n:

$$\frac{dy}{dx} = \frac{2x+y-2}{3x+y-3} \quad \text{— (I)}$$

Let, $x = \alpha + h$ $\quad y = \beta + k$

$dx = d\alpha$ $\quad dy = d\beta$

$$\therefore \frac{dy}{dx} = \frac{d\beta}{d\alpha}$$

—

$$\frac{d\beta}{d\alpha} = \frac{2(\alpha+h)+\beta+k-2}{3(\alpha+h)+\beta+k-3}$$

$$\Rightarrow \frac{d\beta}{d\alpha} = \frac{2\alpha+\beta+2h+k-2}{3\alpha+\beta+3h+k-3} \quad \text{— (II)}$$

—

$2h+k-2=0$; $3h+k-3=0$

$h=1$; $k=0$

—

From (II), $\frac{d\beta}{d\alpha} = \frac{2\alpha+\beta}{3\alpha+\beta}$

Let $\beta = v\alpha$

$$\frac{d\beta}{d\alpha} = v + \alpha\frac{dv}{d\alpha}$$

$$v + x\frac{dv}{dx} = \frac{2x + vx}{3x + vx}$$

$$\Rightarrow v + \frac{dv}{dx}x = \frac{2+v}{3+v}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{2+v}{3+v} - v$$

$$\Rightarrow x\frac{1}{dx} = \frac{2v+v^2}{3+v}\ \frac{2+v-3v-v^2}{3+v}\ \frac{1}{dv}$$

$$= \frac{-v^2-2v+2}{3+v}\ \frac{1}{dv}$$

$$\Rightarrow \frac{dx}{x} = \frac{(3+v)\,dv}{-(v^2+2v-2)}$$

$$\Rightarrow -\frac{dx}{x} = \frac{(3+v)\,dv}{v^2+2v-2}$$

$u = v^2+2v-2$

$du = (2v+2)dv$

$v^2+2v\cdot 1+1-3$

$(v+1)^2-(\sqrt{3})^2$

Integrate:

$$\int \frac{1}{2}\frac{2v+2}{v^2+2v-2}\,dv + 2\int \frac{dv}{v^2+2v-2}\,d + \int \frac{dx}{x} = 0$$

$$\Rightarrow \frac{1}{2}\ln\left|(v^2+1)^2 - \sqrt{3}^2\right| + 2\frac{1}{2\sqrt{3}}\ln\frac{v+1-\sqrt{3}}{\sqrt{3}+v+1} + \ln x = C$$

$$\Rightarrow \ln(v^2+2v-2) + \frac{2}{\sqrt{3}}\ln\frac{v+1-\sqrt{3}}{v+1+\sqrt{3}} + \ln x^2 = C$$

$$\Rightarrow \ln\left(\frac{B^2}{x^2} + 2\frac{B}{x} - 2\right) + \frac{2}{\sqrt{3}}\ln\frac{B+(1-\sqrt{3})x}{B+(1+\sqrt{3})x} + \ln x^2 = C$$

$(x+y+1)dx - (2x+2y+1)dy = 0$

$$\Rightarrow \frac{dy}{dx} = \frac{x+y+1}{2x+2y+1} = \frac{x+y+1}{2(x+y)+1}$$

Let, $v = x+y$

$$\frac{dv}{dx} = 1 + \frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{dv}{dx} - 1$$

$$\frac{dv}{dx} - 1 = \frac{v+1}{2v+1}$$

$$\Rightarrow \frac{dv}{dx} = \frac{v+1}{2v+1} + 1 = \frac{v+1+2v+1}{2v+1} = \frac{3v+2}{2v+1}$$

$$\Rightarrow dx = \frac{2v+1}{3v+2} dv$$

Linear DE

form: $\frac{dy}{dx} + Py = Q$; P, Q func of x or const

solve: mult. by $e^{\int P dx}$ then integrate

$$x\frac{dy}{dx} - 4y = x^6 e^x$$

$$\Rightarrow \frac{dy}{dx} - 4\frac{y}{x} = x^5 e^x \quad ; \quad P = -\frac{4}{x}$$

$$\Rightarrow$$

$$\text{I.F.} = e^{\int P dx} = e^{-4\ln x} = x^{-4}$$

Now,

$$\Rightarrow x^{-4}\frac{dy}{dx} + \left(-\frac{4}{x}\right) y x^{-4} = x e^x$$

$$\Rightarrow x^{-4}\frac{dy}{dx} - 4x^{-5}y = x e^x \quad \Rightarrow \frac{dy}{dx}x^{-4} \Rightarrow \frac{d}{dx}\left[x^{-4}y\right] = xe^x$$

Integrate

$$x^{-4}y = \int x e^x \, dx$$

$$= xe^x - \int\left[\frac{dx}{dx}\int e^x dx\right]dx = x^{-4}y$$

$$\frac{dy}{dx} + \frac{y}{x} = y^2$$

$$\Rightarrow \frac{1}{y^2}\frac{dy}{dx} + \frac{1}{xy} = 1$$

let $v = -\frac{1}{y}$

$$\frac{dv}{dx} = \frac{1}{y^2}\frac{dy}{dx}$$

—

$$\frac{dv}{dx} + \left(-\frac{1}{x}v\right) = 1 \quad ; \quad P = -\frac{1}{x}$$

I.F : $e^{\int P dx} = e^{-\ln x} = e^{\ln x^{-1}} = x^{-1}$

$$x^{-1}\frac{dv}{dx} + x^{-1}\left(-\frac{1}{x}v\right) = x^{-1}$$

$$\Rightarrow x^{-1}v = \ln x + C$$

$$\Rightarrow -\frac{1}{y}(x^{-1}) = \ln x + C$$

SA 18/12/23

The linear equation with constant co-efficient has the general form

$$(D^2 + P_1 D + P_2) y = X$$

where P_1, P_2 are constants, and X is a function of x only or constants.

working rule to find the complete solution of $f(D)y = X$

1. Firstly we have to find C.F (complementary function) i.e complete solution of the eqn $f(D)y = 0$.

2. Secondly we have to find P.I (Particular Integral) of the eqn $f(D)y = X$

$$P.I = \frac{1}{f(D)} x$$

3. The complete soln of the eqn $f(D)y = X$ is

$$y = C.F + P.I$$
$$= y_c + y_p$$

Remember the following expansions

1 $(1-D)^{-1} = 1+D+D^2+\cdots$

$(1+D)^{-1} = 1-D+D^2-\cdots$

$(1-D)^{-2} = 1+2D+3D^2+\cdots$

$(1+D)^{-2} = 1-2D+3D^2-\cdots$

$(1-D)^{-3} = 1+3D+6D^2+\cdots$

$(1+D)^{-3} = 1-3D+6D^2-\cdots$

Solve $(2D^2+5D+2)y = x^2$

Soln. Given, $(2D^2+5D+2)y = x^2$ — (I)

For complementary function CF we have,

$(2D^2+5D+2)y = 0$ — (II)

Let $y = e^{mx}$ be the trial soln of (II).

$Dy = me^{mx}$, $D^2y = m^2e^{mx}$

Putting the values in (II),

$(2m^2+5m+2)e^{mx} = 0$

Since $e^{mx} \neq 0$

A.E in $2m^2+5m+2 = 0$

$2m^2+4m+m+2 = 0$

$2m(m+2)+1(m+2) = 0$

$(m+2)(2m+1) = 0$

$m = -2, -\frac{1}{2}$

$\therefore y_c = c_1 2^{-2x} + c_2 e^{-\frac{1}{2}x}$

For P.I,

$$y_p = \frac{1}{2D^2+5D+2}\, x^2$$

$$= \frac{1}{2\left(1+\frac{2D^2+5D}{2}\right)}$$

$$= \frac{1}{2}\left(1+\frac{2D^2+5D}{2}\right)^{-1} x^2$$

$$= \frac{1}{2}\left[1 - \frac{2D^2+5D}{2} + \left(\frac{2D^2+5D}{2}\right)^2\right] x^2$$

$$= \frac{1}{2}\left[x^2 - \frac{2D^2(x^2)+5D(x^2)}{2} + \frac{4D^4(x^2) + 20D^3(x^2) + 25D^2x^2}{4}\right]$$

$$x^2 = \frac{1}{2}\left[x^2 - \frac{4+10x}{2} + \frac{50}{4}\right]$$

∴ Complete soln is

$$y = y_c + y_p$$

2 $(D^2 - 4D + 4)y = x^3$

Given, $(D^2 - 4D + 4)y = x^3$ — (1)

For complementary func CF we have, $(D^2 - 4D + 4)y = 0$ — (2)

Let, $y = e^{mx}$ be the trial soln of (2).

$$Dy = me^{mx}$$

$$D^2y = m^2e^{mx}$$

Putting the value in (2)

$$(m^2 - 4m + 4)e^{mx} = 0$$

Since, $e^{mx} \neq 0$

AE is $m^2 - 4m + 4 = 0$

$$\Rightarrow m^2 - 2m - 2m + 4 = 0$$

$$\Rightarrow m(m-2) - 2(m-2) = 0$$

$$\Rightarrow (m-2)(m-2) = 0$$

$$m = 2, 2$$

$$\therefore y_c = (c_1 + c_2)e^{2x}$$

For P.I,

$$y_p = \frac{1}{D^2 - 4D + 4} x^3$$

$$= \frac{1}{4\left(1 - D + \frac{D^2}{4}\right)} x^3$$

$$= \frac{1}{4}\left[1 - D + \frac{D^2}{4}\right]^{-1} x^3$$

$$= \frac{1}{4}\left[1 - \frac{4D + D^2}{4}\right]^{-1} x^3$$

$$= \frac{1}{4}\left[1 + \frac{4D + D^2}{4} + \left(\frac{4D + D^2}{4}\right)^2 + \left(\frac{4D + D^2}{4}\right)^3\right] x^3$$

$$= \frac{1}{4}\left[x^3 + \frac{4Dx + D^2x}{4} + \frac{(16D^2 + 8D^3 + D^4)x^3}{16} + \right.$$

Finding P.I of the d.e. ~~f(D) y = x~~ $f(D)\,y = X$ where $X = e^{ax}$ and 'a' is constant.

1. If $f(a) \neq 0$

$$P.I. = \frac{1}{f(D)} e^{ax}$$

$$= \frac{1}{f(a)} e^{ax}$$

2. If $f(a) = 0$

$$P.I. = \frac{1}{f(D)} e^{ax}$$

$$= e^{ax} \frac{1}{f(D+a)} \cdot 1$$

Solve. $(D^2+5D+6)y = e^{2x}$

Soln. Given, $(D^2+5D+6)y = 0$ — (1)

For complementary func. C.F

we have, $(D^2+5D+6)y = 0$ — (2)

Let $y = e^{mx}$ be the trial soln of (2)

$$Dy = me^{mx}$$

$$D^2y = m^2e^{mx}$$

Putting the values in (2)

$$(m^2+5m+6)e^{mx} = 0$$

Since, $e^{mx} \neq 0$

AE is $m^2+5m+6 = 0$

$$m^2+3m+2m+6 = 0$$

$$m(m+3)+2(m+3) = 0$$

$$(m+3)(m+2) = 0$$

$$m = -3, -2$$

$$y_c = c_1 e^{-3x} + c_2 e^{-2x}$$

For P.I,

$$y_p = \frac{1}{D^2+5D+6} e^{2x}$$

$$= \frac{1}{2^2+5.2+6} e^{2x}$$

$$= \frac{1}{20} e^{2x}$$

$f(D) = D^2+5D+6$

$f(a) = a^2+5a+6$

Solve $(D^2-2D+1)\,y = e^x$

$$y_p = \frac{1}{D^2-2D+1} e^x$$

$$= e^x \frac{1}{(D+1)^2-2(D+1)+1} \cdot 1$$

$$= e^x \frac{1}{D^2} \cdot 1$$

$$= e^x \frac{1}{D} \frac{1}{D}(1)$$

$$= e^x \frac{1}{D}(x)$$

$$= e^x \frac{x^2}{2}$$

Solve $(D^2-2D+1)y = e^x$

Soln. Given $(D^2-2D+1)y = e^x$ —— (1)

For complementary

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R.H.S = sinax or cosax

Determine the method of P.I

$y_p = \frac{1}{f(D^2)} \sin ax$ and

$y_p = \frac{1}{f(D^2)} \cos ax$

1. If $f(-a^2) \neq 0$ then

$\frac{1}{f(D^2)} \sin ax = \frac{1}{f(-a^2)} \sin ax$

$\frac{1}{f(D^2)} \cos ax = \frac{1}{f(-a^2)} \cos ax$

$D^2 \longrightarrow -a^2$

2. If $f(-a^2) = 0$ and $f'(-a^2) \neq 0$

then $\frac{1}{f(D^2)} \sin ax = x \frac{1}{f'(D^2)} \sin ax$

$\frac{1}{f(D^2)} \cos ax = x \frac{1}{f'(D^2)} \cos ax$

3. If $f(-a^2) = f'(-a^2) = 0$ and $f''(a) \neq 0$

then $\frac{1}{f(D^2)} \sin ax = x^2 \frac{1}{f''(D^2)} \sin ax$

$\frac{1}{f(D^2)} \cos ax = x^2 \frac{1}{f''(D^2)} \cos ax$

Solve : $(D^2 - 3D + 2)y = \sin 3x$

Sol^n : Given $(D^2 - 3D + 2)y = \sin 3x$ — (1)

For complementary func. C.F we have

$(D^2 - 3D + 2)y = 0$ — (2)

Let $y = e^{mx}$ be the trial sol^n of (2)

$Dy = me^{mx}$ $\qquad D^2y = m^2e^{mx}$

Putting the values in (2)

$(m^2 - 3m + 2)e^{mx} = 0$

Since $e^{mx} \neq 0$

A.E. is, $m^2 - 3m + 2 = 0$

$m^2 - 2m - m + 2 = 0$

$m(m-2) - 1(m-2) = 0$

$(m-1)(m-2) = 0$

$m = 1, 2$

$\therefore y_c = c_1e^{x} + c_2e^{2x}$

For P.I

$$y_p = \frac{1}{D^2-3D+2} \sin 3x$$

$D^2 \to -a^2$
$f(D^2)$

$$= \frac{1}{-3^2-3D+2} \sin 3x$$

$$= \frac{1}{-9-3D+2} \sin 3x$$

$$= -\frac{1}{3D+7} \sin 3x$$

$$= -\frac{(3D-7)}{(3D+7)(3D-7)} \sin 3x$$

$$= -\frac{3D-7}{9(-3)^2-49} \sin 3x$$

$$= \frac{1}{130}(3D-7)\sin 3x$$

$$= \frac{1}{130}[3D(\sin 3x) - 7\sin 3x]$$

$$= \frac{1}{130}(9\cos 3x - 7\sin 3x)$$

∴ General soln $y = y_c + y_p$

2. $(D^2-2D+3)y = \cos x + x^2$

Soln: Given $(D^2-2D+3)y = \cos x + x^2$ —— (1)

For complementary func. C.F we have

$(D^2-2D+3)y = 0$ —— (2)

Let $y = e^{mx}$ be the trial soln of (2)

$Dy = me^{mx}$ $\quad D^2y = m^2e^{mx}$

Putting the values in (2)

$(m^2-2m+3)e^{mx} = 0$

Since $e^{mx} \neq 0$

A.E is, $m^2-2m+3 = 0$

$$m = \frac{2 \pm \sqrt{4-4\times1\times3}}{2}$$

$$= \frac{2 \pm \sqrt{-8}}{2}$$

$$= \frac{2 \pm i\sqrt{8}}{2}$$

$$m_1 = \frac{2+i2\sqrt{2}}{2} \qquad m_2 = \frac{2-i2\sqrt{2}}{2}$$

$$= 1+i\sqrt{2} \qquad = 1-i\sqrt{2}$$

$\therefore y_c = e^x(c_1 \cos\sqrt{2}x + c_2 \sin\sqrt{2}x)$

Next
R.H.S $= e^{ax} \cdot v$

For P.I,

$y_p =$

R.H.S/ $R(x) = e^{ax} \cdot V$

Where $V = \sin ax$ or $\cos ax$ or x^m

Rule 1 If $y_p = \frac{1}{f(D)} e^{ax} \cdot V$ then put e^{ax} left side of $\frac{1}{f(D)}$ and replace D by D+a.

$$y_p = \frac{1}{f(D)} e^{ax} \cdot V$$

$$= a e^{ax} \frac{1}{f(D+a)} V$$

Rule 2 If $y_p = \frac{1}{f(D)} x^n \cos ax$ or

$$y_p = \frac{1}{f(D)} x^n \sin ax$$

then put $X = \frac{1}{f(D)} x^n \cos ax$

$$Y = \frac{1}{f(D)} x^n \sin ax$$

$$X + iY = \frac{1}{f(D)} x^n (\cos ax + i \sin ax)$$

$$= \frac{1}{f(D)} x^n e^{iax}$$

$$= e^{iax} \frac{1}{f(D+ia)} x^n .$$

Rule 3

$$\phi = \frac{1}{f(D)} x v$$

$$= x \frac{1}{f(D)} v - \frac{f'(D)}{\{f(D)\}^2} v$$

Solve: $(D^2-9)y = 2xe^{3x}$

Soln: Given, $(D^2-9)y = 2xe^{3x}$ —— (1)

For C.F. we have,

$$(D^2-9)y = 0 \quad \text{—— (2)}$$

Let $y = e^{mx}$ be the trial solution of (2)

$$Dy = me^{mx}, \qquad D^2y = m^2e^{mx}$$

Putting the values in (2),

$$(m^2-9)e^{mx} = 0$$

Since, $e^{mx} \neq 0$

A.E is

$$m^2-9 = 0$$

$$(m-3)(m+3) = 0$$

$$m = 3, -3$$

$$\therefore y_c = c_1e^{3x} + c_2e^{-3x}$$

$$y_p = \frac{1}{D^2-9} 2xe^{3x}$$

$$= e^{3x} \frac{1}{(D+3)^2-9} 2x$$

$$= e^{3x} \frac{1}{D^2+6D} 2x$$

$$= e^{3x} \frac{1}{6D(1+\frac{D}{6})} 2x$$

$$= \frac{2e^{3x}}{6} \cdot \frac{1}{D}\left(1+\frac{D}{6}\right)^{-1} x$$

$$= \frac{e^{3x}}{3} \frac{1}{D}\left[1-\frac{D}{6}+\frac{D^2}{36}-\cdots\right]$$

$$= \frac{e^{3x}}{3} \frac{1}{D}\left[x-\frac{D(x)}{6}+\frac{D^2(x)}{36}-\cdots\right]$$

$$= \frac{e^{3x}}{3} \frac{1}{D}\left[x-\frac{1}{6}\right]$$

$$= \frac{e^{3x}}{2}\left[\frac{x^2}{2}-\frac{x}{6}\right]$$

$(D^2-2D+1)y = e^x \sin x$

Soln Given, ~~$(D^2-9)y = 2xe^{3x}$~~

$$(D^2-2D+1)y = e^x \sin x \quad \text{—— (1)}$$

For CF we have,

$$(D^2-2D+1)y = 0 \quad \text{— (2)}$$

Let $y = e^{mx}$ be a trial soln of (2)

$$Dy = me^{mx}, \quad D^2y = m^2e^{mx}$$

Putting the value in (2)

$$(m^2-2m+1)e^{mx} = 0$$

Since, $e^{mx} \neq 0$

A.E is $m^2-2m+1 = 0$

$$(m-1)^2 = 0$$

$$m = 1, 1$$

$\therefore y_c =$ ~~e^x~~ $(C_1+C_2x)e^x$

(1) ~~cos x~~

$$y_p = \frac{1}{D^2-2D+1} e^x \sin x$$

$$= \frac{1}{(D-1)^2} e^x \sin x$$

$$= e^x \frac{1}{(D+1-1)^2} \sin x$$

$$= e^x \frac{1}{D^2} \sin x$$

$$= e^x \frac{1}{-1^2} \sin x$$

$$= -e^x \sin x$$

General soln

$$y = y_c + y_p$$

$$=$$

SA 03/01/24

Solution of Homogeneous Linear Equation

An eqⁿ of the form

$$P_0 x^n \frac{d^n y}{dx^n} + P_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + P_2 x^{n-2} \frac{d^{n-2} y}{dx^{n-2}} + \cdots + P_n y = R(x)$$

is called linear homogeneous eqⁿ with variable co-efficients where $P_0, P_1, P_2 \cdots, P_n$ are constants and $R(x)$ is a constant or a function of x. This equation is also called Cauchy-Euler equation.

Solve $(x^3D^3 + 3x^2D^2 - 2xD + 2)y = 0$

Soln: Given, $(x^3D^3 + 3x^2D^2 - 2xD + 2)y = 0$ — (1)

Put $x = e^z$

$z = \ln x$

$\frac{dz}{dx} = \frac{1}{x}$ — (2)

$\frac{dy}{dx} = \frac{dz}{dx}\frac{dy}{dz}$

$\frac{dy}{dx} = \frac{1}{x}\frac{dy}{dz}$ — (3)

$x\frac{dy}{dx} = \frac{dy}{dz}$

$x\,Dy = Dy \quad [D = \frac{d}{dz}]$ — (4)

$[D = \frac{d}{dy}]$

$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$

$= \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dz}\right)$

$= \frac{1}{x}\frac{d}{dx}\left(\frac{dy}{dz}\right) - \frac{1}{x^2}\frac{dy}{dz}$

$= \frac{1}{x}\frac{d}{dz}\left(\frac{dy}{dz}\right)\frac{dz}{dx} - \frac{1}{x^2}\frac{dy}{dz}$

$$= \frac{1}{x}\frac{d^2y}{dz^2} + \frac{1}{x} - \frac{1}{x^2}\frac{dy}{dz}$$

$$x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dz^2} - \frac{dy}{dz}$$

$$x^2D^2y \left[D : \frac{d}{dx}\right] = D(D-1)y \left[D : \frac{d}{dz}\right] \quad \text{—— (5)}$$

Similarly,

$$x^3D^3y \left[D : \frac{d}{dx}\right] = D(D-1)(D-2)y \left[D : \frac{d}{dz}\right] \quad \text{—— (6)}$$

Putting the values in (1)

$$[D(D-1)(D-2) + 3D(D-1) - 2D + 2]y = 0$$

$$(D^3 - 3D + 2)y = 0 \quad \text{—— (7)}$$

Let $y = e^{mz}$ be the trial soln of (7)

$$Dy = me^{mz}, \quad D^2y = m^2e^{mz}$$

$$D^3e^{mz} = m^3e^{mz}$$

Putting the values in (7)

$$(m^3 - 3m + 2)e^{mz} = 0$$

Since $e^{mz} \neq 0$, A.E is

$$m^3 - 3m + 2 = 0$$

$$m = 1, 1, -2$$

| must derive

∴ The general solution

$$y = (C_1 + C_2 z)e^{z} + C_3 e^{-2z}$$

$$= (C_1 + C_2 \ln x)x + C_3 x^{-2}$$

Solve $(x^2D^2 - xD + 1)y = \ln x$

Soln: Given $(x^2D^2 - xD + 1)y = \ln x$ — (1)

Put $x = e^z$

$z = \ln x$

$\frac{dz}{dx} = \frac{1}{x}$ — (2)

$\frac{dy}{dx} = \frac{dz}{dx} \cdot \frac{dy}{dz}$

$\frac{dy}{dx} = \frac{1}{x}\frac{dy}{dz}$ — (3)

$x\frac{dy}{dx} = \frac{dy}{dz}$

$xDy = Dy \left[D = \frac{d}{dz}\right]$ — (4)

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$= \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dz}\right)$$

$$= \frac{1}{x}\frac{d}{dx}\left(\frac{dy}{dz}\right) - \frac{1}{x^2}\frac{dy}{dz}$$

$$= \frac{1}{x}\frac{d}{dz}\left(\frac{dy}{dz}\right)\frac{dz}{dx} - \frac{1}{x^2}\frac{dy}{dz}$$

$$= \frac{1}{x}\frac{d^2y}{dz^2}\cdot\frac{1}{x} - \frac{1}{x^2}\frac{dy}{dz}$$

$$x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dz^2} - \frac{dy}{dz}$$

$x^2D^2y \quad \left[D = \frac{d}{dx}\right] = D(D-1)y \quad \left[D = \frac{d}{dz}\right]$ —— (5)

Putting the values in (1)

$$[D(D-1) - D + 1]y = z$$

$$(D^2 - 2D + 1)y = z$$

For C.I $(D^2 - 2D + 1)y = 0$

Let $y = e^{mx}$ be the trial soln of (7)

$Dy = me^{mz}$, $D^2y = m^2e^{mz}$

Putting the values in (7)

$$(m^2 - 2m + 1)\, e^{mz} = 0$$

Since, $e^{mz} \neq 0$, A.E is

$$m^2 - 2m + 1 = 0$$

$$(m-1)^2 = 0$$

$$m = 1, 1$$

$$\therefore y_c = (c_1 + c_2 z)\, e^z$$

$$= (c_1 + c_2 \ln x)\, x$$

Now, $y_p = \dfrac{1}{D^2 - 2D + 1}\, z$

1. Def^n
2. Formation

} A

1. Seperation of var
2. Homogeneous
3. Eq^n Reducible to homogenous
4. Exact
5. Linear
6. Bernauli

} B

1. Solvable for p
2. " " y
3. " " x
4. Clairaults
5. Lagranges

} C

1. $R(x) = 0$
2. $R(x) = x$
3. $R(x) = e^{ax}$
4. $R(x) = \sin ax / \cos ax$
5. $R(x) = e^{ax} v$

} D

1. Homogeneous
2. Linear Eqⁿ } E

1. Series Solⁿ } F

1. Bessels function
2. Legendre Polynomial } G

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1. a.
   b. } A
   c. } B or C

2. a.
   b.
   c. } B or C

3. a.
   b.
   c. } D

4. a.
   b.
   c. } E, F