27/09/23

59%

59/90

Math 103

Differential equation & Matrix

ODE, PDE

ODE

SecA

PDE & Matrix

Sec B

Ordiniary Differential Equation

Partial Differential Equation

Gen Points

(Academics)

- Student
- Classroom
- Light, fan, Air con
- Seating Arrangement
- Visitors chair
- Dusters, Markers
- Rub the board
- Class info
- Discipline
- Lecture
- Assignment
- Turnout and Bearing
- Dress
- Norms → How to talk/behave
- How to talk with mobile
- On time
- Salutation

D.E.

MD Raisinghania

Q. Define the followings

1. Differential Equation (DE)
2. ordinary differential Equation (ODE)
3. Partial Differential Equation (PDE)
4. order of differential equation
5. Degree of "

Syllabus

1. Defn, order & Degree
2. formation of DE (06)
3. first Order and first degree (DE)
4. first " but higher degree - (09)
5. second and " order DE - 06
6. Homogenous linear DE
7. Series Soln, Frobenius Method

7. Bessel's function

8. Legendre polynomial

$x+3=5$ $\qquad$ $2x+y=2$

$x+y=1$

$Z = f(x, y)$

D / / \

D I I

$\frac{\delta z}{\delta x}$, $\frac{\delta z}{\delta y}$

1. Differential Eqn (DE)

An equation involving dependent variable, independent variable, and differential coefficient of dependent variable with respect to independent variable is called a differential equation (D.E)

The variables may or may not lie in d.e but the differential co-efficient must be ~~line~~ lie in d.e. ~~any~~ x is called

independent variable and y is called dependent variable. Derivation of y with respect to x is denoted by

$\frac{dy}{dx}$

Example:-

1. $\frac{dy}{dx} + (1-y^2)\tan x = 0$

2. $\frac{dy}{dx} = 1$

3. $x\frac{dz}{dx} + y\frac{dz}{dy} = xz$

there are two types of DE

1. ODE
2. PDE

$x \quad y \quad \frac{dy}{dx}$

ODE

An eqn involving one dependent ~~one~~ variable one independent variable and derivative of dependent variable with respect to a single independent variable is called ordinary differential Eqn (ODE)

Example:

$$1.\ \frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$$

PDE

An eqn involving one dependent variable ~~or~~ two or more independent variable and partial differential coefficient with respect to more than one independent variable is called partial differential eqn (PDE)

Example: $1.\ x\frac{dz}{dx} + y\frac{dz}{dy} = 3z$

Order of a DE

The order of the highest order differential coefficient involving in a de is called the order of the d.e.

Example:-

1. $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 0$

2. $\frac{d^4y}{dx^4} + 7\frac{d^2y}{dx^2} + 5y = \text{const}$

In above the eqn (1) is of second order and the eqn 2 is of fourth order.

5. Degree of a DE

the degree on power of the highest order derivative of a d.e is called degree of de. when the eqn has been made free from

fraction

Example :-

1. $\frac{d^2y}{dx^2} + 3\left(\frac{dy}{dx}\right)^4 + y = 0$

2. $\frac{d^2y}{dx^2} = 4\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{5/3}$

2. Since the eqn (1) the powers of the highest third order derivative is one so the degree of eqn is one.

If so powers of $\frac{dy}{dx}$ is 4 but $\frac{dy}{dx}$ is not the higest order differential coefficient. So the degree of eqn is not 4. Hence the degree of eqn is one.

If eleminating the fractional powers of eqn (2) is the eqn is of the form $\left(\frac{d^2y}{dx^2}\right)^3 = k^3\left[1 + \left(\frac{dy}{dx}\right)^2\right]^5$

In this case the power of the higest second order derivative, is 3

Hence the degree of eqn is 3

Distinguish between ODE and PDE

18/10/2023

Q. Find out the order and Degree of the following with justification.

1) $\left(\frac{d^3y}{dx}\right)^2 + \left(\frac{dy}{dx}\right)^4 + 3y = 0$

2) $\sqrt[3]{\left(\frac{dy}{dx}\right)^2 + \frac{dy}{dx} + 7} = \sqrt[5]{\left(\frac{d^4y}{dx^4}\right)^3 + 7y}$

(1) $\left(\frac{d^3y}{dx}\right)^2 + \left(\frac{dy}{dx}\right)^4 + 3y = 0$

In the above, the highest differential coefficient is 3. So order of the equation is 3. and the powers of the hfgest 3rd order differential coefficient is 2. So degree of the eqn is 2.

Hence, Order = 3

Degree = 2

2.3 $\sqrt{\left(\frac{d^2y}{dx^2}\right)^2 + \frac{dy}{dx} + 7} = \sqrt[5]{\left(\frac{d^4y}{dx^4}\right)^3 + 2}$

2) $\left[\left(\frac{d^2y}{dx^2}\right)^2 + \frac{dy}{dx} + 7\right]^5 = \left[\left(\frac{d^4y}{dx^4}\right)^3 + 2\right]^3$

⇒ order = 4

degree = 9

25/10/2023

formation of DE

1. formation of DE of

$f(x, y, z) = 0$

Soln Given eqn $f(x, y, a) = 0$ ——— (1)

Differentiating (1) wrt to x

$\phi(x, y, a, \frac{dy}{dx}) = 0$ ——— (2)

Eliminating a form (1) and (2) we get

$\phi(x, y, \frac{dy}{dx}) = 0$

this is the required d.e

2Q. find the de ~~form~~ from a straight line equation $y = mx$

form the d.e. of all, straight line passing through the origin.

x, y → Dependent Variable

$\frac{dy}{dx}$ → Differential coefficient

arbitary → अचरांक

Sol^n Given $y = mx$ ---(i)

Let $y = mx$ be the eqn of straight line passing through the origin.

Differentiate (i) with respect to x

$$\frac{dy}{dx} = m1 = m \quad \text{---(2)}$$

Putting the value of m in (1)

$$y = \frac{dy}{dx}\, x$$

$$y = x \frac{dy}{dx}$$

which is the required d.e.

2. formation of DE of $f(x, y, a, b)$

Soln Given eqn $f(x, y, a, b) = 0$ ... (1)

Differentiating (1) wr. to x.

$$\phi\left(x, y, a, b, \frac{dy}{dx}\right) = 0 \quad \text{---(2)}$$

~~Eliminating a b/w (1)~~.

Again Differentiating (2) wr. to x

~~$\phi(x, y, a, b)$~~

$$\phi\left(x, y, a, b, \frac{dy}{dx}, \frac{d^2y}{dx^2}\right) = 0 \quad \text{---(3)}$$

Eliminating a, b from (1), (2), (3)

$$\phi\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}\right) = 0$$

this is the required d.e.

Q. Find the d.e of the curves $y = e^x(A\cos x + B\sin x)$, where A and B are arbitrary constants

Soln Given $y = e^x(A\cos x + B\sin x)$ ——— (i)

Differentiating (1)

$$\frac{dy}{dx} = Ae^x \sin x + Be^x \cos x$$

$$= e^x(A\cos x + B\sin x) + e^x(-A\sin x + B\cos x)$$

$$\frac{dy}{dx} = y + e^x(-A\sin x + B\cos x) \quad \text{——— (ii)}$$

Again Differentiating (2) w.r.t. x

$$\frac{d^2y}{dx^2} = \frac{dy}{dx} + e^x(-A\sin x + B\cos x) - e^x(A\cos x + B\sin x)$$

$$\frac{d^2y}{dx^2} = \frac{dy}{dx} + e^x(-A\sin x + B\cos x) - y$$

$$\frac{d^2y}{dx^2} = \frac{dy}{dx} + \left(\frac{dy}{dx} - y\right) - y$$

$$\frac{d^2y}{dx^2} - \frac{2dy}{dx} + 2y = 0$$

which is the required d.e.

8) Form the d.e. corresponding $y = ae^{2x} + be^{-3x} + ce^x$, where a, b, c are arbitrary constants.

Soln: Given eqn: $y = ae^{2x} + be^{-3x} + ce^x$ ..... (1)

Differentiating (1) wr. to x

$$\frac{dy}{dx} = 2ae^{2x} - 3be^{-3x} + ce^{x} \quad \text{---(ii)}$$

Substracting from (2)

$$\frac{dy}{dx} - y = ae^{2x} - 4be^{-3x} \quad \text{---(iii)}$$

Again differentiating (3) wr. to x.

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} = 2ae^{2x} + 12be^{-3x} \quad \text{---(iv)}$$

Multiplying eqn (3) by 3 and adding with eqn (4)

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} + 3\left(\frac{dy}{dx} - y\right) = 5ae^{2x} \quad \text{---(5)}$$

$$\frac{d^2y}{dx^2} + \frac{2dy}{dx} - 3y = 5ae^{2x} \quad \text{---(5)}$$

Again differentiating =>

$$\frac{d^3y}{dx^3} + 2\frac{dy^2}{dx^2} - 3\frac{dy}{dx} = 2(5ae^{2x})$$

$$\frac{d^3y}{dx^3} + 2\frac{dy^2}{dx^2} - 3\frac{dy}{dx} = 2\left(\frac{dy^2}{dx^2} + 2\frac{dy}{dx} - 3y\right)$$

$$\Rightarrow \frac{d^3y}{dx^3} - 7\frac{dy}{dx} + 6y = 0$$

3. $y = Ae^{2x} + Be^{-2x}$

$$\frac{dy}{dx} = 2Ae^{2x} - 2Be^{-2x}$$

$$\frac{dy}{dx} = 2$$

Again, $\frac{d^2y}{dx^2} = 4Ae^{2x} + 4Be^{-2x}$

$$\frac{d^2y}{dx^2} = 4y$$

$$\frac{d^2y}{dx^2} - 4y = 0$$

08/11/2023

first ~~ent~~ Order and first degree differential equation,

A differential equation of the form $M + N\frac{dy}{dx} = 0$ or $M\,dx + N\,dy = 0$ is called 1st Order and 1st degree differential equation. where both M and N are functions of x and y or constants.

Types of differential equation of first order and 1st degree.

1. Seperation of variables
2. Homogenous equation
3. Equation reducti. reducible to homogenous.
4. Exact equation
5. Linear equation
6. Reducible to linear equation.

Separation of Variables

If the D.E $M\,dx + N\,dy = 0$ can be expressed in the form $f(x)\,dx + \phi(y)$ then it is called variables are seperated.

Rule 1) If the de is of the form $f(x)\,dx + f(y).\,dy = 0$ .ie. function of with dx and function of y with dy then only integrate we can find the solution.

Solve: $\frac{dy}{dx} + \frac{y^2+y+1}{x^2+x+1} = 0$

$$\frac{dy}{dx} = -\frac{y^2+y+1}{x^2+x+1}$$

$$\frac{dy}{y^2+y+1} = -\frac{dx}{x^2+x+1}$$

or $$\frac{dx}{x^2+x+1} + \frac{dy}{y^2+y+1} = 0$$

Integrating : $$\int \frac{dx}{x^2+x+1} + \int \frac{dy}{y^2+y+1} =$$

$$\Rightarrow \frac{dx}{(x+\frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2} + \frac{dy}{(y+\frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2}$$

$$\Rightarrow \frac{2}{\sqrt{3}} \tan^{-1} \frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}} + \frac{2}{\sqrt{3}} \tan^{-1} \frac{y+\frac{1}{2}}{\frac{\sqrt{3}}{2}}$$

Rule-2| If any de of the form $f(x)$ $\phi(y)\,dx + f(y)\,\phi(x)\,dy = 0$ then divide the eqn by $\phi(x)\,\phi(y)$ the eqn is of the form $\frac{f(x)}{\phi(x)}\,dx + \frac{f(y)}{\phi(y)}\,dy = 0$

i.e. function of x with dx and function of y with dy then only integrate we can find the soln.

Prob: $\sec^2 x \tan y \, dx + \sec^2 y \tan x \, dy = 0$ — (1)

Soln: divide eqn (1) by $\tan x \tan y$

$$\frac{\sec^2 x}{\tan x} dx + \frac{\sec^2 y}{\tan y} dy = 0$$

Integrate $\int \frac{\sec^2 x}{\tan x} dx + \int \frac{\sec^2 y}{\tan y} dy = 0$

$\Rightarrow \ln(\tan x) + \ln(\tan y) = \ln C$

$\Rightarrow \ln(\tan x \cdot \tan y) = \ln C$

$\Rightarrow \tan x \tan y = C$

Solve: $x\sqrt{1-y^2}\, dx + y\sqrt{1-x^2}\, dy = 0$ — (1)

Ans: divide eqn (1) by $\sqrt{1-x^2} \cdot \sqrt{1-y^2}$

$$\frac{x}{\sqrt{1-x^2}} dx + \frac{y}{\sqrt{1-y^2}} dy = 0$$

Integrate, $\int \frac{x}{\sqrt{1-x^2}} dx + \int \frac{y}{\sqrt{1-y^2}} dy = 0$

$$= -\frac{1}{2}\int \frac{-2x}{\sqrt{1-x^2}} dx - \frac{1}{2}\int \frac{-2y}{\sqrt{1-y^2}} = 0$$

$\frac{1}{\sqrt{t}}$ $\frac{t^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}$

$$\Rightarrow -\frac{1}{2}\,2\sqrt{1-x^2} - \frac{1}{2}\,2\sqrt{1-y^2} = -c$$

$$\Rightarrow -\left(\sqrt{1-x^2} + \sqrt{1-y^2}\right) = -c$$

$$\therefore \sqrt{1-x^2} + \sqrt{1-y^2} = c$$

Class-5 (20.11.23)

Separation of Variables:

Rule 3: If the d.e is of the form

$\frac{dy}{dx} = f(ax+by+c)$ ~~and simply~~ simplify finally

then put $ax+by+c=z$ and simplify finally integrating we will get the soln.

~~Ques: Solve~~ $\frac{dy}{dx} = (4x+y+1)^2$

Ques: Solve $\frac{dy}{dx} = (4x+y+1)^2$ —①

Let $4x+y+1=z$

diff ~~with~~ wrt to x,

$$4 + \frac{dy}{dx} + 0 = \frac{dz}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{dz}{dx} - 4$$

Putting the values in ①,

$$\frac{dz}{dx} - 4 = z^2 \Rightarrow \frac{dz}{dx} = 4+z^2$$

$$\Rightarrow \frac{dz}{4+z^2} = dx \Rightarrow dx = \frac{dz}{2^2+z^2}$$

Integrating,

$$\int \frac{dz}{2^2+z^2} = dx$$

$$\Rightarrow \frac{1}{2} \tan^{-1}\frac{z}{2} = x + c$$

$$\Rightarrow \frac{1}{2} \tan^{-1}\frac{2x+y+1}{2} = x + c$$

Homogenous D.E:

Homogenous Eqn: An eqn in which the sum of the powers of x and y in every terms are equal than the eqn is called homogenous eqn.

Example: $ax^2 + 2hxy + by^2$ is a homogenous eqn of 2nd degree in x & y

Homogenous D.E: An eqn of the form $\frac{dy}{dx} = \frac{f(x,y)}{\phi(x,y)}$ in which $f(x,y)$ and $\phi(x,y)$ are homogenous functions of x and y of the same degree, then the eqn is called homogenous D.E

Working rule for solving homogenous DE:

1) Rewrite the given homogenous D.E in the form $\frac{dy}{dx} = \frac{f(x,y)}{\phi(x,y)}$ —①

2) Substitute $y = vx$

3) Differentiate $y = vx$, then we get $\frac{dy}{dx} = v + x\frac{dv}{dx}$

4) Put the values in ①

$$v + x\frac{dv}{dx} = \frac{f(v)}{\phi(v)} \Rightarrow v + x\frac{dv}{dx} = F(v)$$

where $F(v) = \frac{f(v)}{\phi(v)}$

$\Rightarrow x\frac{dv}{dx} = F(v) - v$

$\Rightarrow \frac{dv}{F(v)-v} = \frac{dx}{x}$ where variables have been ~~separated~~ seperated.

5) Then the general soln will be obtained integration.

Ques:

Ques: Solve $(x^2+y^2)\,dx = 2xy\,dy$

$$\frac{dy}{dx} = \frac{x^2+y^2}{2xy} \quad \text{---} (1)$$

put $y = vx$

$$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$$

Putting the values in (1),

$$v + x\frac{dv}{dx} = \frac{x^2+v^2x^2}{2xvx}$$

$$\Rightarrow v + x\frac{dv}{dx} = \frac{1+v^2}{2v}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{1+v}{2v} - v \Rightarrow x\frac{dv}{dx} = \frac{1-v^2}{2v}$$

$$\Rightarrow x\frac{dv}{dx} = -\frac{v^2-1}{2v} \Rightarrow \frac{2v\,dv}{v^2-1} = -\frac{dx}{x}$$

Integrating both sides,

$$\int \frac{2v\,dv}{v^2-1} + \int \frac{dx}{x} = 0 \Rightarrow \ln(v^2-1) + \ln x = \ln c$$

$$\Rightarrow \ln(v^2-1)x = \ln c \Rightarrow (v^2-1)x = c \Rightarrow \left(\frac{y^2}{x^2} - 1\right)x = c$$

$$\Rightarrow \frac{y^2-x^2}{x} = c \Rightarrow y^2 - x^2 = cx$$

Ques: Solve $(x^2+y^2)\,dx + 2xy\,dy = 0$ —— (1)

$$\frac{dy}{dx} = -\frac{x^2+y^2}{2xy} \text{ —— (1)}$$

put $y = vx$ then $\frac{dy}{dx} = v + x\frac{dv}{dx}$ —— (2)

Now put the values of y and $\frac{dy}{dx}$ in (1) we get,

$$v + x\frac{dv}{dx} = -\frac{x^2+v^2x^2}{2x \cdot vx} = -\frac{x^2(1+v^2)}{2x^2v} = -\frac{1+v^2}{2v}$$

$$\Rightarrow x\frac{dv}{dx} = -\frac{1+v^2}{2v} - v = \frac{-1-v^2-2v^2}{2v} = \frac{-1-3v^2}{2v} = -\frac{1+3v^2}{2v}$$

$$\Rightarrow \frac{2v}{1+3v^2}\,dv = -\frac{dx}{x}$$

Integrating both sides,

$$\int \frac{2v}{1+3v^2}\,dv = -\int \frac{dx}{x}$$

$1+3v^2 = z$
$\Rightarrow 6v\,dv = dz$
$\Rightarrow 2v\,dv = \frac{1}{3}dz$

$$\Rightarrow \frac{1}{3}\int \frac{dz}{z} = -\int \frac{dx}{x} \Rightarrow \frac{1}{3}\log z = -\log x + \log c$$

$$\Rightarrow \frac{1}{3}\log(1+3v^2) = -\log x + \log c$$

$$\Rightarrow \frac{1}{3}\log(1+3v^2) + \log x = \log c$$

$$\Rightarrow \log(1+3v^2)^{\frac{1}{3}} + \log x = \log c$$

$$\Rightarrow x(1+3v^2)^{\frac{1}{3}} = c$$

$$\Rightarrow x\left(1+\frac{3y^2}{x^2}\right)^{\frac{1}{3}} = c$$

22/11/2023

Homogeneous DE

Equation Reducible to homogeneous

If the de is of the form

$$\frac{dy}{dx} = \frac{ax+by+c}{a_1x+b_1y+c_1} \quad \text{when } \frac{a}{a_1} \neq \frac{b}{b_1}$$

This type of DE can be reduced homogeneous eqn by putting

$x = \alpha + h$, $y = \beta + k$, where α, β are new variable and h, k are anbitary constants.

Therefore putting $\beta = v\alpha$ and with the help of seperation of variable and then integrate we can find the soln.

Solve: $\frac{dy}{dx} = \frac{2x+y-2}{3x+y-3}$ ——(1)

Here $\frac{2}{3} \neq \frac{1}{1}$

$2x+y-3y+1$

$\frac{2}{3} \neq \frac{1}{1}$

put $x = \alpha + h$, $y = \beta + k$

$dx = d\alpha + 0$, $dy = d\beta + 0$

$dx = d\alpha$ $\quad dy = d\beta$

$\therefore \frac{dy}{dx} = \frac{d\beta}{d\alpha}$ — (2)

Putting the values in (1)

$$\frac{dy}{dx} = \frac{d\beta}{d\alpha}$$

$$\Rightarrow \frac{d\beta}{d\alpha} = \frac{2(\alpha+h) + \beta + k - 2}{3(\alpha+h) + \beta + k - 3}$$

$$\Rightarrow \frac{d\beta}{d\alpha} = \frac{2\alpha + 2h - 2 + \beta + k}{3\alpha + 4h - 3 + \beta + k} \quad (3)$$

Now, choose $2h + k - 2 = 0$

and $3h + k - 3 = 0$

$$\frac{h}{-6+6} = \frac{k}{-6+6} = \frac{1}{2-3}$$

$$\frac{h}{-1} = \frac{k}{0} = \frac{1}{-1}$$

$h = 1$, $k = 0$

Now

$$\frac{d\beta}{d\alpha} = \frac{2\alpha+\beta+2-2}{3\alpha+\beta+3-3}$$

$$\frac{d\beta}{d\alpha} = \frac{2\alpha+\beta}{3\alpha+\beta} \quad \ldots\ldots (4)$$

Now, put $\beta = v\alpha$

$$\frac{d\beta}{d\alpha} = v + \alpha\frac{dv}{d\alpha} \quad \ldots\ldots (5)$$

from (4)

$$v+\alpha\frac{dv}{d\alpha} = \frac{2\alpha+v\alpha}{3\alpha+v\alpha}$$

$$v+\alpha\frac{dv}{d\alpha} = \frac{2+v}{3+v}$$

$$\alpha\frac{dv}{d\alpha} = \frac{2+v}{3+v} - v$$

$$\alpha\frac{dv}{d\alpha} = \frac{2+v-3v-v^2}{3+v}$$

$$\text{or } \frac{(v+3)\,dv}{v^2+2v-2} = -\frac{d\alpha}{\alpha}$$

$$\text{or, } \frac{\left\{\frac{1}{2}(2v+1)+\frac{7}{2}\right\}dv}{v^2+2v-2} + \frac{d\alpha}{\alpha} = 0$$

Integrating both sides,

$$\frac{1}{2}\int \frac{(2v+2)\,dv}{v^2+2v-2} + 2\int \frac{dv}{v^2+2v-2} + \int \frac{d\alpha}{\alpha} = 0$$

$$\Rightarrow \frac{1}{2}\int \frac{(2v+2)}{v^2+2v-2}\,dv + 2\int \frac{dv}{(v+1)^2-(\sqrt{3})^2} + \int \frac{d\alpha}{\alpha} = 0$$

$$\Rightarrow \frac{1}{2}\ln(v^2+2v-2) + \frac{2}{2\sqrt{3}}\ln\left(\frac{v+1-\sqrt{3}}{v+1+\sqrt{3}}\right) + \ln\alpha = \frac{1}{2}C$$

$$\Rightarrow \ln(v^2+2v-2) + \ln\alpha^2 + \frac{2}{\sqrt{3}}\ln\frac{v+1-\sqrt{3}}{v+1+\sqrt{3}} = C$$

$$\text{or, } \ln(v^2+2v-2) + \ln\alpha^2 + \frac{2}{\sqrt{3}}\ln\frac{v+1-\sqrt{3}}{v+1+\sqrt{3}} = C$$

$$\Rightarrow \ln\left(\frac{\beta^2}{\alpha^2} + \frac{2\beta}{\alpha} - 2\right)\alpha^2 + \frac{2}{\sqrt{3}}\ln\left\{\frac{\beta+(1-\sqrt{3})\alpha}{\beta+(1+\sqrt{3})\alpha}\right\} = C$$

since $x = \alpha + 1$ and $y = \beta + 0$ so putting $\alpha = x - 1$ and $\beta = y$ in the above equation we get,

$$\ln\{y^2 + 2y(x-1) - 2(x-1)^2\} + \frac{2}{\beta}\ln\left\{\frac{y + (1-\sqrt{3})(x-1)}{y + (1+\sqrt{3})(x-1)}\right\} = c$$

OT.

1. Definition
2. formation of DE
3. Seperation of variables.

Linear D.E

If P and Q are only function of x or constants when the DE of the form $\boxed{\frac{dy}{dx} + Py = Q}$ is called first order linear d.e.

To solve this equation multiply by

$\boxed{e^{\int P.dx}}$ → IF (Integrating factor)

then it becomes,

$$e^{\int P dx} + P y e^{\int P dx} = Q e^{\int P dx}$$

or $\frac{d}{dx}(y e^{\int P dx}) = Q e^{\int P dx}$

Integrating both sides wrt x we get $y e^{\int P dx} = \int [Q e^{\int P dx}] dx + C$

Sol

Solve: $x\frac{dy}{dx} - 4y = x^6 e^x$ ——— (1)

divide (1) by x

$$\frac{dy}{dx} + \left(\frac{-4}{x}\right)y = x^5 e^x$$

which is a Linear DE of the form

$\frac{dy}{dx} + Py = Q$, where $P = \left(\frac{-4}{x}\right)$

$$IF = e^{\int -\frac{4}{x}\,dx}$$

$$= e^{-4\int \frac{1}{x}\,dx}$$

$$= e^{-4\ln x}$$

$$= e^{\ln x^{-4}}$$

$$IF = x^{-4}$$

Multiply IF by (2)

$$x^{-4}\frac{dy}{dx} + \left(-\frac{4}{x}\right)x^{-4}y = x^5 e^x x^{-4}$$

$$\Rightarrow x^{-4}\frac{dy}{dx} + (-4)x^{-5}y = x e^x$$

$$\Rightarrow \frac{dy}{dx}(x^{-4}y) = x e^x$$

=

Integrating factor (IF)

An integrating factor is a function used to solve d.e. It is a function in which an ordinary d.e can be multiplied to make the function Integrate.

$$\frac{dy}{dx} + Py = Q$$

$$IF = e^{\int P\,dx}$$

Integrating both sides

$$x^{-4}y = \int x e^x\,dx$$

$$= x\int e^x\,dx - \int\left(\frac{d}{dx}(x)\int e^x\right)dx$$

$$= xe^x - \int e^x\,dx$$

$$= xe^x - e^x + c$$

2. $(x^2+9)\frac{dy}{dx}+xy=0$ Ans: $y=\frac{c}{\sqrt{x^2+9}}$

3. $(1+e^x)\frac{dy}{dx}+e^x y=0$ Ans: $y=\frac{c}{1+e^x}$

(2) $(x^2+9)\frac{dy}{dx}+xy=0$

$$\frac{dy}{dx}+\left(\frac{x}{x^2+9}\right)y=0 \quad \text{— (2)}$$

Which is a linear D.E of the form

$\frac{dx}{dy}+Py=Q$ where $P=\left(\frac{x}{x^2+9}\right)$

$$\text{If} = e^{\int \frac{x}{x^2+9}dx}$$

$$= e^{\frac{1}{2}\int \frac{2x}{x^2+9}dx}$$

$$= e^{\frac{1}{2}\ln(x^2+9)}$$

$$= e^{\ln\sqrt{x^2+9}}$$

$$= \sqrt{x^2+9}$$

Multiply If by (2)

$$\sqrt{x^2+9}\,\frac{dy}{dx}+\left(\frac{x}{x^2+9}\sqrt{x^2+9}\right)y=0$$

$$\Rightarrow \sqrt{x^2+9}\frac{dy}{dx}+\frac{x}{\sqrt{x^2+9}}y=0$$

$$\Rightarrow \frac{d}{dx}\left(y\sqrt{x^2+9}\right)=0$$

$\Rightarrow$ Integrating

$$y\sqrt{x^2+9}=\int 0$$

$$y\sqrt{x^2+9}=C$$

$$y=\frac{C}{\sqrt{x^2+9}}$$

Bernaulli eqn

If P and Q are only function of x then the de of the form $\frac{dy}{dx} + Py = Qy^n$ is called Bernaulli de.

Dividing the eqn by y^n it can be reduced to linear d.e.

Solve: $\frac{dy}{dx} + \frac{y}{x} = y^2$ ——— (1)

divide eqn (1) by y^2

$\frac{1}{y^2}\frac{dy}{dx} + \frac{1}{xy} = 1$ ——— (2)

put $v = -\frac{1}{y}$

$\frac{dv}{dx} = \frac{1}{y^2}\frac{dy}{dx}$

Putting the values in (2)

$\frac{dv}{dx} + \left(-\frac{1}{x}\right)v = 1$

which is a linear eqn.

$$IF = e^{\int \left(-\frac{1}{x}\right) dx}$$

$$= e^{-\ln x}$$

$$= e^{\ln x^{-1}}$$

$$= x^{-1}$$

Multiply (3) by x^{-1}

$$x^{-1}\frac{dv}{dx} + \left(-\frac{1}{x}\right)x^{-1}v = x^{-1}$$

$$\Rightarrow x^{-1}\frac{dv}{dx} + \left(-\frac{1}{x^2}\right)v = \frac{1}{x}$$

$$\Rightarrow \frac{d}{dx}(vx^{-1}) = \frac{1}{x}$$

By Integrating

$$\Rightarrow vx^{-1} = \ln x + c$$

$$\Rightarrow -\frac{1}{y}\frac{1}{x} = \ln x + c$$

$$\Rightarrow -1 = (c + \ln x)xy$$

Class-8 (29.11.23)

Exact DE: Let M and N both are functions of x, y

If $Mdx + Ndy = 0$ is obtained by differentiating $f(x,y) = 0$ then $Mdx + Ndy = 0$ is called exact differential equation.

Example: $xy = c$ ——— (1)

$$d(x,y) = 0$$

$$ydx + xdy = 0 \quad \text{——— (2)}$$

Since (2) is called exact D.E.

Rules from finding the soln of an exact D.E:

(1) Compairing the given de with $Mdx + Ndy = 0$ and find M, N

then find $\frac{\partial M}{\partial y}$ and $\frac{\partial N}{\partial x}$. If $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, we conclude that the given eqn is exact.

(2) The soln of an exact eqn is given by

$$\int_{(y \text{ is constant})} M\,dx + \int (\text{Term of } N \text{ not containing } x)\,dy = \text{constant}$$

~~y is cons~~

Ques: Solve $(4x+3y+1)dx + (3y+2y+1)dy = 0$ ——— (1)

which is of the form $Mdx + Ndy = 0$

where $M = 4x+3y+1$

$N = 3x+2y=1$

$\frac{\partial M}{\partial y} = 3 \qquad \frac{\partial N}{\partial x} = 3$

$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

So given de is exact.

The soln of the eqn is,

$$\int (4x+3y+1)\,dx + \int (2y+1)\,dy = c$$

(y is constant)

$$\Rightarrow \frac{4x^2}{2} + 3xy + x + \frac{2y^2}{2} + y = c$$

$$\Rightarrow 2x^2 + y^2 + 3xy + x + y = c$$

Ques: Solve $(x^3+3xy^2)\,dx+(y^3+3x^2y)\,dy = 0$ ——— (1)

which is of the form $M\,dx+N\,dy=0$

where $M = x^3+3xy^2$

$N = y^3+3x^2y$

$\frac{\partial M}{\partial x} = 3x^2+3y^2$ $\quad \frac{\partial N}{\partial x} = 3y^2+3x^2$

$\frac{\partial M}{\partial x} = \frac{\partial N}{\partial x}$

$\frac{\partial M}{\partial y} = 6xy$ $\quad \frac{\partial N}{\partial x} = 6xy$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

So given de is exact

The soln of the given eqn

$$\int (x^3 + 3xy^2)\,dx + \int y^3\,dy = c$$

$$\Rightarrow \frac{x^4}{4} + 3\frac{x^2}{2}y^2 + \frac{y^4}{4} = c$$

$$\Rightarrow x^4 + 6x^2y^2 + y^4 = c_1$$

Ques: Solve $(2x+3y-5)\,dy + (3x+2y-5)\,dx = 0$ ——— (1)

which is of the form

$Mdx + Ndy = 0$, where

$M = 3x+2y-5 \quad N = 2x+3y-5$

$$\frac{\partial M}{\partial y} = 2 \qquad \frac{\partial N}{\partial x} = 2$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

So given D.E is exact

The solution of the given eqn is

$$\int (3x+2y-5)\,dx + \int (3y-5)\,dy = 0$$

$$\Rightarrow \frac{3x^2}{2} + 2yx - 5x + \frac{3y^2}{2} - 5y = c$$

$$\Rightarrow 3x^2 + 4yx - 10x + 3y^2 - 10y = c$$

(ans)

04/12/2023***

First order but higher degree dE

A de involving first order but degree higher than one is called first order but higher degree de.

ie $p^n + A_1 p^{n-1} + A_2 p^{n-2} + \dots + A_n = 0$

is a first order and nth degree de where $p = \frac{dy}{dx}$ and $A_1, A_2 \dots A_n$ are functional of x, y

The eqiation is solved by five methods.

1. Solvable for p
2. solvable for y
3. solvable for x
4. clairauts equations (ক্লেরোর সমীকরণ)
5. Lagranges equation

$\frac{dy}{dx}$	p^2	p^3	p^n
	$\left(\frac{dy}{dx}\right)^2$,	$\left(\frac{dy}{dx}\right)^3$	$\left(\frac{dy}{dx}\right)^n$
$O = 1$	$O = 1$	$O = 1$	$O = 1$
$D = 1$	$D = 2$	$D = 3$	$D = n$
	$p = \frac{dy}{dx}$		

$$\frac{d^2y}{dx^2}, \quad \frac{d^3y}{dy^3} \ldots\ldots\ldots \frac{d^ny}{dx^n}$$

Solvable for p

If the equation $p^n + A_1 p^{n-1} + A_2 p^{n-2} + \ldots - A_n = 0$ can be expressed into n-factors of first order and first degree $(p - \alpha_1)(p - \alpha_n) = 0$ then equating to zero each factor n equation of first order and first degree are obtained. Then the solution are obtained by integration.

solve: $p^2 + 2xp - 3x^2 = 0$ where $p = \frac{dy}{dx}$

soln:

Given, $p^2 + 2xp - 3x^2 = 0$

$\Rightarrow p^2 + 2xp + x^2 - 4x^2 = 0$

$\Rightarrow (p+x)^2 - (2x)^2 = 0$

$\Rightarrow (p+3x)(p-x) = 0$

$p - x = 0$ — (2) | $p + 3x = 0$ — (3)

from, $p - x = 0$

$\frac{dy}{dx} - x = 0$

$dy - x\,dx = 0$

Integrating,

$y - \frac{x^2}{2} = 0 - \frac{c}{2} = 0$

$\Rightarrow 2y - x^2 - c = 0$

from 3 $p + 3x = 0$

$\Rightarrow \frac{dy}{dx} + 3x = 0$

$\Rightarrow dy + 3x\,dx = 0$

Integrating,

$$y + \frac{3x^2}{2} - \frac{c}{2} = 0$$

$$2y + 3x^2 - c = 0 \quad - - \quad (5)$$

General solution of (1) is

(4) × (5) [When it has two value then, the general solution is the multiple of two values.]

$$(2y - x^2 - c)(2y + 3x^2 - c) = 0$$

Solve

$p^3 - p(x^2+xy+y^2) + xy(x+y) = 0$

=) Given equation,

$p^3 - p(x^2+xy+y^2) + xy(x+y) = 0$

=) $p^3 - px^2 - pxy - py^2 + x^2y + xy^2 = 0$

=) $p(p^2-x^2) - xy(p-x) - y^2(p-x) = 0$

=) $(p-x)[p(p+x) - xy - y^2] = 0$

=) $(p-x)[p^2 - y^2 + px - xy] = 0$

=) $(p-x)[(p+y)(p-y) + x(p-y)] = 0$

=) $(p-x)(p-y)(p+y+x) = 0$

$p - x = 0 \quad \cdots (1)$

$p - y = 0 \quad \cdots (2)$

$p + y + x = 0 \quad \cdots (3)$

From (2)

$p - x = 0$

$\frac{dy}{dx} - x = 0$

$dy - x\,dx = 0$

$\Rightarrow y - \frac{x^2}{2} - \frac{c^2}{2} = 0$

$\Rightarrow 2y - x^2 - c = 0$

From (2)

$p - y = 0$

$\frac{dy}{dx} - y = 0$

$\Rightarrow \frac{dy}{y} = dx$

$\Rightarrow \ln y = x + a$

$\Rightarrow y = e^{x+a}$

$\Rightarrow y = e^a e^x$

$\Rightarrow y = c\, e^x$

From 3,

$p + y + x = 0$

$\frac{dy}{dx} + y + x = 0$

$\Rightarrow \frac{dy}{dx} + y = -x \quad \cdots (4)$

$IF = e^{\int dx}$

$= e^x$

multiplying (4) by e^x

$e^x \frac{dy}{dx} + e^x y = -xe^x$

$\Rightarrow \frac{d}{dx}(ye^x) = -xe^x$

$\Rightarrow ye^x = -\left[x\int e^x dx - \int\left(\frac{d}{dx}x \int e^x\right)dx\right]$

$\Rightarrow ye^x = -\left[xe^x - \int e^x dx\right]$

$\Rightarrow ye^x = -(xe^x - e^x) + c$

$\Rightarrow ye^x = -xe^x + e^x + c$

$\Rightarrow ye^x + xe^x - e^x - c = 0$

$\Rightarrow y + x - 1 - ce^{-x} = 0$

: General Solution :

$$(2y - x^2 - c)(y - c \cdot e^x)(y + x - 1 - ce^{-x}) = 0$$

CT - 3

Date : 17. Dec

Day : Sunday

Time : 8.00 am

Syllabus

1. homogenous eqn
2. Linear eqn
   Bernauli eqn
3. ~~Ca~~ Clairauts eqn

6 December

Equation solvable for y

In this case we are to first ~~differentia~~ differentiate the equation with respect to x and then form an equation with p and x.

If the equation is solvable for y, then y can be expressed in terms of x and p.

i.e $y = f(x, p)$ - - (B)

Differentiating (1) w.r.t x

$$\frac{dy}{dx} = \phi\left(x, p, \frac{dp}{dx}\right)$$

or $p = \phi\left(x, p, \frac{dp}{dx}\right)$ ---(2)

which is a function of x and p,

Let the ~~soit~~ sol^n of (2) be

$$\boxed{F(x, p, c) = 0} \quad \text{---} \quad (3)$$

when c is an arbitary constant

Now eliminating p from (1) and (3) The required solution is obtained.

Solve $y = px = p^2x^4$

Soln Given $y = px = p^2x^4$ ——— (1)

Differentiate w.r.t. x

$$\frac{dy}{dx} + p.1 = x\frac{dp}{dx} = 2p\frac{dp}{dx}x^4 + 4x^3p^2$$

$$p + p + x\frac{dp}{dx} - 2px^4\frac{dp}{dx} + 4x^3p^2 - 4x^3p = 0$$

$$2p - 4x^3p + x\frac{dp}{dx} - 2px^4\frac{dp}{dx} = 0$$

$$2p(1 - 2x^3p) + x\frac{dp}{dx}(1 - 2x^3p) = 0$$

$$(1 - 2x^3p)\left(2p + x\frac{dp}{dx}\right) = 0$$

Omitting the first factor which leads to Singular solution

$$\therefore 2p + x\frac{dp}{dx} = 0$$

$$\therefore 2p = -x\frac{dp}{dx}$$

$$\frac{dx}{x} = -\frac{dp}{2p}$$

$$-\frac{dp}{2p} + \frac{2dx}{x} = \frac{dp}{p} + \frac{2dx}{x} = 0$$

Integrate,

$$\int\frac{dp}{p} + 2\int\frac{dx}{x} = 0$$

$$\ln p + \ln x^2 = \ln c$$

$$\ln px^2 = \ln c$$

$$px^2 = c$$

$$p = \frac{c}{x^2}$$

Now, substituting the values of p in the given equation.

$$y + \frac{cx}{x^2} = e^2$$

$$y + \frac{c}{x} = e^2$$

which the required solution.

Equation solvable for x

In this case we are to first differentiate the equation with respect to y and then form an equation with p and y

If the equation is solvable for x, then x can be expressed in terms of y and p.

$$\text{i.e } x = f(y, p) \quad -- (1)$$

Differentiating (1) w.r.t y

$$\frac{dx}{dy} = \phi\left(y, p, \frac{dp}{dy}\right)$$

or $\frac{1}{p} = \varphi\left(y, p, \frac{dp}{dy}\right)$ ---- (2)

which is a function of x and p. Let the solution of (2) be

$F(y, p, c) = 0$ ---- (3)

when c is an arbitary constant

Now eliminating p from (1) and (3), the required solution is obtained.

$$\boxed{\frac{dy}{dx} = p}$$

$$\boxed{\frac{dx}{dy} = \frac{1}{p}}$$

* Solve: $y = 2px + p^2y$

$\Rightarrow y = 2px + p^2y$

$\Rightarrow 2x = -py + \frac{y}{p}$ — ①

Differentiating ① w.r.t 'y'

$$2\frac{dx}{dy} = -p - y\frac{dp}{dy} + \frac{1}{p} - \frac{y}{p^2}\frac{dp}{dy}$$

$$\Rightarrow 2\cdot\frac{-1}{p} = -p - y\frac{dp}{dy} + \frac{-1}{p} - \frac{y}{p^2}\frac{dp}{dx}$$

$$\Rightarrow \frac{2}{p} + p - \frac{1}{p} = -y\frac{dp}{dy}\left(1 + \frac{1}{p^2}\right)$$

$$\Rightarrow p + \frac{1}{p} = -y\frac{dp}{dy}\left(1 + \frac{1}{p^2}\right)$$

$$\Rightarrow p\left(1 + \frac{1}{p^2}\right) + y\frac{dp}{dy}\left(1 + \frac{1}{p^2}\right) = 0$$

$$\Rightarrow \left(1 + \frac{1}{p^2}\right)\left(p + y\frac{dp}{dx}\right) = 0$$

Omitting the first factor which leads to a singular solution we get,

$$p + y\frac{dp}{dy} = 0$$

$$\Rightarrow p = -y\frac{dp}{dy}$$

$$\Rightarrow \frac{dy}{y} = -\frac{dp}{p}$$

$\Rightarrow \frac{dp}{p} + \frac{dy}{y} = 0 \quad -(3)$

Integrating,

$\int \frac{dp}{p} + \int \frac{dy}{y} = 0$

$\Rightarrow \ln p + \ln y = \ln c$

$\Rightarrow \ln py = \ln c$

$\Rightarrow py = c$

putting the value of p in the given eq.

$y = 2 \cdot \frac{c}{y} \cdot x + \frac{c^2}{y^2} \cdot y$

$\Rightarrow y = \frac{2cx}{y} + \frac{c^2}{y}$

$\Rightarrow y^2 = 2cx + c^2$

$\Rightarrow 2cx - y^2 + c^2 = 0$

11/12/2023****

11/12/23

Clairaut's eqn:

Aneqn of the form $y = px + f(p)$ is called clairauts equation.

where $p = \frac{dy}{dx}$

Completed and singular solution (S.S)

the given equation is $y + f(p) \rightarrow (1)$

Differentiating (1) wr to x we get

$$\frac{dy}{dx} = p + f'(p)\frac{dp}{dx} + x\frac{dp}{dx}$$

$$p = p + x\frac{dp}{dx} + f'(p)$$

$$\left[x + f'(p)\right]\frac{dp}{dx} = 0$$

$$\frac{dp}{dx} = 0 \rightarrow (2) \qquad x + f'(p) = 0 \rightarrow (3)$$

Now integrating (2) wr to x,

$p = c$ where c is an arbitary constant.

Now putting the value of p in (1)
$y = (x + f(c))$ which is the required general solution of (1)
Again eliminating p from (1) and (2).
to obtain another solution involving no arbitary constant this solution is called the singular solution (S.S)

Solve: $y = px + \frac{ap^2}{p}$ and find the singular solution (if exists)

Soln: Given equation,
$y = px + \frac{ap^2}{p}$ ——— (1)
which is of the form ~~etc~~ clairaut's eqn.
Now differentiating (i) w.r.to x.
~~$y = \frac{dy}{dx}$~~ , $\frac{dy}{dx} = p + x\frac{dp}{dx} - \frac{ap^2}{p^2}\frac{dp}{dx}$

$$p = p + \left(x - \frac{a^2}{p^2}\right)\frac{dp}{dx}$$

$$x - \frac{a^2}{p^2} = 0$$

$$\frac{dp}{dx} = 0 \quad \text{--- (2)}$$

Now Integrating (2) w.r.to x

$p = c$ where c is an arbitrary constant

Now putting the value of in (1)

$y = cx + \frac{a^2}{c}$ which is the required general soln of (1)

Again from (3) we get

$$x - \frac{a^2}{p^2} = 0$$

$$x = \frac{a^2}{p^2}$$

$$p = \frac{a}{\sqrt{x}}$$

putting the value of p in (1)

$$y = \frac{a}{\sqrt{x}}x + \frac{a^2}{\frac{a}{\sqrt{x}}}$$

$$y = a\sqrt{x} + a\sqrt{x}$$

$$\Rightarrow y = 2a\sqrt{x}$$

$y^2 = 4ax$, which is one singular solution (S.S)

Solve: $(y+1)p - xp^2 + 2 = 0$

where $p = \frac{dy}{dx}$

The given equation is,

$(y+1)p - xp^2 + 2 = 0$

$(y+1)p = xp^2 - 2$

$(y+1) = xp - \frac{2}{p}$

$y = xp - \frac{2}{p} - 1$

which is the clairaut's form

$y = px + f(p)$

Hence putting c for p the general solution is

$y = cx + \left(\frac{2}{c} - 1\right)$

$= cx - \left(\frac{2}{c} + 1\right)$

13/12/2023

Solution of second and higher Order

1. If $e^{m_1 x}, e^{m_2 x}, e^{m_3 x} \ldots, e^{m_n x}$ be the n independent solutions of the d.e. $P_0 \frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + \ldots P_n y = 0$

the general solution is,

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \ldots c_n e^{m_n x}$$

where $c_1, c_2 \ldots c_n$ are arbitary constants.

2. If the d.e. $D^2 y + P_1 D y + P_2 y = 0$ has two independent solutions $y = e^{m_1 x}$ and $y = e^{m_2 x}$ then it is general solution is

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$$

where C_1, C_2 are arbitrary constants and $D = \frac{d}{dx}$

Auxiliary equation (A.E)

Let the DE be $Dy + P_1 Dy + P_2 y = 0$ (1)

let the solution of (1) be $y = e^{mx}$

so it will satisfy (1)

$D^2(e^{mx}) + P_1 D(e^{mx}) + P_2 e^{mx} = 0$

$\Rightarrow m^2 e^{mx} + P_1 m e^{mx} + P_2 e^{mx} = 0$

$\Rightarrow (m^2 + P_1 m + P_2) e^{mx} = 0$

Since $e^{mx} \neq 0$ so. $m^2 + P_1 m + P_2 = 0$

This equation called Auxiliary equation (A E)

following three cases should be remembers.

Case 1 : Let the roots are real and unequal. If $m = \alpha, \beta$, then its general solution is

$$y = C_1 e^{\alpha x} + C_2 e^{\beta x}$$

Case 2: Let the roots are real and equal. If $m = \alpha$ then its general solution is $y = (C_1 + C_2 x) e^{\alpha x}$

case 3: Let the roots are imaginary. If $m = \alpha \pm i\beta$ then its general solution is

$$y = e^{\alpha x}(C_1 \cos\beta x + C_2 \sin\beta x)$$

Solve: $(2D^2 - 3D + 1)y = 0$

Soln: $(2D^2 - 3D + 1)y = 0 \quad \ldots\ldots (i)$

Let, $y = e^{mx}$ be the trial solution (i)

$Dy = me^{mx}$, $D^2y = m^2e^{mx}$

Putting value in (i)

$$(2m^2 - 3m + 1)e^{mx} = 0$$

Since $e^{mx} \neq 0$ therefore

auxiliary equation,

$$2m^2 - 3m + 1 = 0$$

$$\Rightarrow 2m^2 - 2m - m + 1 = 0$$

$$\Rightarrow 2m(m-1) - 1(m-1) = 0$$

$$\Rightarrow (m-1)(2m-1) = 0$$

$$\Rightarrow m = 1, \quad m = \frac{1}{2}$$

∴ then general solution is

$$y = C_1 e^x + C_2 e^{\frac{1}{2}x}$$

Solve: $(D^2 - 2D + 1)y = 0$ ——— (i)

Let $y = e^{mx}$ be the trial solution of (i)

$Dy = me^{mx}$, $D^2y = m^2e^{mx}$

Putting the values in (i)

$(m^2 - 2m + 1)e^{mx} = 0$

$m^2 - 2m + 1 = 0$

$(m-1)^2 = 0$

$m = 1$

The general solution,

$y = (c_1 + c_2)x e^x$

Solve: $(5D^2 - 2D + 3)y = 0$

$y = e^{mx}$

$Dy = me^{mx}$ $\quad D^2y = m^2e^{mx}$

$\therefore (5m^2 - 2m + 3)e^{mx} = 0$

$5m^2 - 2m + 3 = 0$

$$m = \frac{+2 \pm \sqrt{4 - 4.5.3}}{2.5}$$

$$= \frac{2 \pm \sqrt{-56}}{10}$$

$$= \frac{2 \pm i2\sqrt{14}}{10}$$

$$= \frac{1}{5} \pm i\frac{\sqrt{14}}{5} \quad (\alpha \pm i\beta)$$

The general solution is,

$$y = e^{\frac{1}{5}x}\left(C_1 \cos\frac{\sqrt{14}}{5} + C_2 \sin\frac{\sqrt{14}}{5}\right)$$

~~Solve: $(D^2+D-6)y=0$~~

4. $(D^2+D-6)y=0$

5. $(5D^2-D+3)y=0$

6. $(D^3-3D+2)y=0$

7. $(D^2-4D+13)y=0$

8. $(D^4+9)y=0$

Soln: $(D^2-4D+13)y=0$

$$(m^2-4m+13)e^{mx}=0$$

$$m=\frac{+4\pm\sqrt{16-4.13}}{2}$$

$$=\frac{4\pm\sqrt{-36}}{2}$$

$$2\pm 3i$$

$$=\alpha\pm i\beta$$

e.w.

18/12/2023

18/12/2023

The linear equation with constant coefficient has the general form $(D^2 + P_1 D + P_2) y = X$

where P_1, P_2 are constants and X is a function of x only or constants.

Working Rule to find the complete solution of $f(D)y = X$

1. Firstly we have to find CF (Complementary function) i.e complete solution of the eqn

$f(D) y = 0$

2. Secondly we have to find PI (Particular Integral the eqn

$f(D) y = X$

$$PI = \frac{1}{f(D)} X$$

3 The complete solution of the eqn $f(D)\,y = X$ is

$$y = C.F. + P.I.$$
$$= y_c + y_p$$

Solve: $(2D^2 + 5D + 2)\,y = x^2$

Remember following expansions,

1. $(1-D)^{-1} = 1 + D + D^2 + \cdots$
2. $(1+D)^{-1} = 1 - D + D^2 - \cdots$
3. $(1-D)^{-2} = 1 + 2D + 3D^2 + \cdots$
4. $(1+D)^{-2} = 1 - 2D + 3D^2 - \cdots$
5. $(1-D)^{-3} = 1 + 3D + 6D^2 + \cdots$
6. $(1+D)^{-3} = 1 - 3D + 6D^2 - \cdots$

Solve: $(2D^2+5D+2)y = x^2$

Soln: Given $(2D^2+5D+2)y = x^2$

for complementary function C.F we have $(2D^2+5D+2)y = 0$

Let $y = e^{mx}$ be the trial solution

$\Rightarrow Dy = me^{mx}$, $D^2y = m^2e^{mx}$

Putting the value of in (2)

$(2m^2+5m+2)e^{mx} = 0$

Since $e^{mx} \neq 0$

$\Rightarrow$ A.E is $2m^2+5m+2 = 0$

$\Rightarrow 2m^2+4m+1m+2 = 0$

$\Rightarrow 2m(m+2)+1(m+2) = 0$

$\Rightarrow (m+2)(2m+1) = 0$

$m = -2, -\frac{1}{2}$

$\therefore y_c = c_1e^{-2x} + c_2e^{-\frac{1}{2}x}$

For PI $= y_p = \frac{1}{2D^2+5D+2} x^2$

$= \frac{1}{2\left(\frac{1}{2}+\frac{2D^2+5D}{2}\right)} x^2$

$= \frac{\frac{1}{2}}{1+\frac{2D^2+5D}{2}}$

$= \frac{1}{2}\left(1+\frac{2D^2+5D}{2}\right)^{-1} x^2$

$= \frac{1}{2}\left[1-\frac{2D^2+5D}{2}+\left(\frac{2D^2+5D}{2}\right)^2-\ldots\right] x^2$

$= \frac{1}{2}\left[x^2-\frac{2D^2(x^2)+5(D)(x^2)}{2}\right.$

$\left.+\frac{20D^3(x^2)}{8}+\frac{25D^2(x^2)}{4}\right]$

$= \frac{1}{2}\left(x^2-\frac{4+10x}{2}+\frac{50}{4}\right)$

$= \frac{4x^2}{\ldots}$

Complete soln: $y = y_1 + y_p$

2. $(D^2 - 4D + 4)\, y = x^3$

let $y = e^{mx}$

$Dy = me^{mx}$, $D^2y = m^2e^{mx}$

$\therefore (m^2 - 4m + 4)e^{mx} = 0$

$\Rightarrow (m-2)^2 = 0$

$\Rightarrow m = 2$

$\therefore y_1 = (C_1 + 2C_2)e^{2x}$

for PI $= \dfrac{1}{D^2 - 4D + 4} x^3$

$= \dfrac{1}{4\left(1 + \dfrac{D^2 - 4D}{4}\right)} x^3$

$= \dfrac{1}{4}\left(1 + \dfrac{D^2 - 4D}{4}\right)^{-1} x^3$

$= \dfrac{1}{4}\left[1 - \dfrac{D^2 - 4D}{4} + \left(\dfrac{D^2 - 4D}{4}\right)^2\right.$

$$= \frac{1}{4}\left[x^3 - \frac{D^2(x^3) - 4D(x^3)}{4} + \frac{D^4(x^3) - 8D^3(x^3) + 16D^2(x^3)}{16} - \frac{D^6 - 12D^5 + 48D^3}{64} - \cdots\right]$$

$$= \frac{1}{4}\left(x^3 - 6x - 12x^2 + \frac{6x - 24x^2 + 96x}{16} - 6\right)$$

$$= \frac{1}{4}$$

finding the PI of the d.e

$f(D)\,y = X$ where $X = e^{ax}$

and 'a' is constant

1. If $f(a) \neq 0$

$$PI = \frac{1}{f(D)} e^{ax}$$

$$= \frac{1}{f(a)} e^{ax}$$

2. If $f(a) = 0$

$$PI = \frac{1}{f(D)} e^{ax}$$

$$= e^{ax} \frac{1}{f(D+a)} \cdot 1$$

Solve: $(D^2+5D+6)\,y = e^{2x}$

Soln: Given $(D^2+5D+6)\,y = e^{2x}$ ——— (1)

complementary function "CF"

we have $(D^2+5D+6)\,y = 0$ ——— (2)

Let $y = e^{mx}$ be the trial soln of (2)

$Dy = me^{mx}$ $\quad D^2y = m^2 e^{mx}$

Putting the value in (2)

$(m^2+5m+6) = 0$

Since $e^{mx} \neq 0$

AE is $m^2 + 5m + 6 = 0$

$m^2+3m+2m+6 = 0$

$(m+3)(m+2) = 0$

$m = -3, -2$

$y_c = C_1 e^{-3x} + C_2 e^{-2x}$

For PI $= y_p = \frac{1}{D^2+5D+6} e^{2x}$

$f(D) = D^2+5D+6$

$f(a) = a^2+5a+6$

$\frac{1}{2^2+5\cdot 2+6} e^{2x}$

$= \frac{1}{20} e^{2x}$

$\therefore y = y_c + y_p$

Solve! $(D^2-2D+1)y = e^x$

$(m^2-2D+1)^2 = 0$

$(m-1)^2 = 0$

$m = 1$

$y_c = (C_1 + C_2)e^x$

For PI $= y_p = \frac{1}{D^2-2D+1} e^x$

$= \frac{1}{1-2+1} e^x$

$= e^x \frac{1}{(D+1)^2 - 2(D+1)+1}$

$= e^x \frac{1}{D^2} \cdot 1$

$= e^x \frac{1}{D} \frac{1}{D}(1)$

$= e^x \frac{1}{D}(x)$

$= e^x \frac{x^2}{2}$

20/12/2023

RHS = $\sin ax$ or $\cos ax$

Determine the method of PI

$y_p = \frac{1}{f(D^2)} \sin ax$ and

$y_p = \frac{1}{f(D^2)} \cos ax$:

1. If $f(-a^2) \neq 0$ then

$\frac{1}{f(D^2)} \sin ax = \frac{1}{f(-a^2)} \sin ax$

$\frac{1}{f(D^2)} \cos ax = \frac{1}{f(-a^2)} \cos ax$

2. If $f(-a^2) = 0$ and $f'(-a^2) \neq 0$

then $\frac{1}{f(D^2)} \sin ax = x \frac{1}{f'(D^2)} \sin ax$

$\frac{1}{f(D^2)} \cos ax = x \frac{1}{f'(D^2)} \cos ax$

3. If $f(-a^2) = f'(-a^2) = 0$ and $f''(-a^2) \neq$

then $\frac{1}{f(D^2)} \sin ax = x^2 \frac{1}{f''(D^2)} \sin ax$

$\frac{1}{f(D^2)} \cos ax = x^2 \frac{1}{f''(D^2)} \cos ax$

$D^2 \longrightarrow -a^2$

Solve: $(D^2-3D+2)y = \sin 3x$

Soln: Given $(D^2-3D+2)y = \sin 3x$ ... —(1)

for complementary function CF we have

$(D^2-3D+2)y = 0$ —— (2)

Let $y = e^{mx}$ be the trial soln of (2)

$Dy = me^{mx}$, $D^2y = m^2e^{mx}$

Putting the values of (1) in (2)

$m^2-3m+2 = 0$

$e^{mx} \neq 0$

$\Rightarrow$ A.E is $m^2-3m+2 = 0$

$\Rightarrow m^2-2m-m+2 = 0$

$\Rightarrow m(m-2)-1(m-2) = 0$

$\Rightarrow (m-2)(m-1) = 0$

$m = 1, 2$

$y_1 = c_1e^x + c_2e^{2x}$

For PI

$$y_p = \frac{1}{D^2 - 3D + 2} \sin 3x$$

$$= \frac{1}{-3^2 - 3D + 2} \sin 3x$$

$$= \frac{1}{-9 - 3D + 2} \sin 3x$$

$$= -\frac{1}{3D + 7} \sin 3x$$

$D^2 \to -a^2$

$$f(D^2) = \frac{-(3D - 7)}{(3D + 7)(3D - 7)} \sin 3x$$

$$= \frac{-(3D - 7)}{9D^2 - 49} \sin 3x$$

$$= \frac{-(3D - 7)}{9(-3)^2 - 49} \sin 3x$$

$$= \frac{1}{130}(3D - 7)\sin 3x$$

$$= \frac{1}{130}[3D(\sin 3x) - 7\sin 3x]$$

~~sin3x~~ ∴

General solution, $y = y_1 + y_2$

2. $(D^2 - 2D + 3)\, y = \cos x + x^2$

3. $(D^2 - 4)\, y = e^x - \sin x$

3. $(D^2 - 4)\, y = e^x - \sin x$

$y = e^{mx}$, $Dy = me^{mx}$ $D^2y = m^2 e^{mx}$

$(m^2 - 4)\, e^{mx} = 0$

$m^2 - 4 = 0$

$m = \pm 2$

$m = +2, -2$

$y_1 = c_1 e^{2x} + c_2 e^{-2x}$

For PI

$$y_p = \frac{1}{D^2-4}(e^x - \sin x)$$

$$= \frac{1}{D^2-4} e^x - \frac{1}{D^2-4} \sin x$$

$$= \frac{1}{1^2-4} e^x - \frac{1}{-1^2-4} \sin x$$

$$= \frac{1}{-3} e^x + \frac{1}{5} \sin x$$

$$y = y_c + y_p$$

$$= C_1 e^{2x} + C_2 e^{-2x} - \frac{1}{3} e^x + \frac{1}{5} \sin x$$

Class-15

R.H.S / $R(x) = e^{ax} \cdot V$ where $V = \sin ax$ or $\cos ax$ or x^m

Rule-1:

If $y_p = \frac{1}{f(D)} e^{ax} \cdot V$ then put e^{ax} left side of $\frac{1}{f(D)}$ and replace D by $D+a$.

$$y_p = \frac{1}{f(D)} e^{ax} \cdot V$$

$$= e^{ax} \cdot \frac{1}{f(D+a)} \cdot V$$

Rule-2:

If $y_p = \frac{1}{f(D)} x^n \cos ax$ or $y_p = \frac{1}{f(D)} x^n \sin ax$.

then put $x = \frac{1}{f(D)} x^n \cos ax$

$$Y = \frac{1}{f(D)} x^n \sin x$$

$$x + iY = \frac{1}{f(D)} x^n (\cos ax + i \sin ax)$$

$$= \frac{1}{f(D)} x^n e^{iax}$$

$$= e^{iax} \frac{1}{f(D+ia)} x^n$$

Rule-3:

$$y = \frac{1}{f(D)} xV$$

$$= x \frac{1}{f(D)} V - \frac{f'(D)}{\{f(D)\}^2} V$$

* Solve: $(D^2-9)y = 2xe^{3x}$

$\Rightarrow$ Given,

$$(D^2-9)y = 2xe^{3x} \quad \text{---} \quad (1)$$

For C.F we have,

$$(D^2-9)y = 0 \quad \text{---} \quad (11)$$

Let $y = e^{mx}$ be the trial soln of (11)

$$Dy = me^{mx}, \quad D^2y = m^2e^{mx}$$

Putting the values in (11)

$$(m^2-9)e^{mx} = 0$$

Since $e^{mx} \neq 0$, A.E is $m^2-9 = 0$

$$\Rightarrow (m-3)(m+3) = 0$$

$$m = 3, -3$$

$$\therefore y_c = C_1e^{3x} + C_2e^{-3x}$$

$$y_p = \frac{1}{D^2-9} 2xe^{3x}$$

$$= e^{3x} \frac{1}{(D+3)^2-9} 2x$$

$$= e^{3x} \frac{1}{D^2+6D} \cdot 2x$$

$$= e^{3x} \frac{1}{6D\left(1+\frac{D}{6}\right)} \cdot 2x$$

$$= \frac{2e^{3x}}{6} \frac{1}{D}\left(1+\frac{D}{6}\right)^{-1} x$$

$$= \frac{e^{3x}}{3} \frac{1}{D} \left(1 - \frac{D}{6} + \frac{D^2}{36} - \dots\right) x$$

$$= \frac{e^{3x}}{3} \frac{1}{D} \left[x - \frac{D(x)}{6} + \frac{D^2(x)}{36} - \dots\right]$$

$$= \frac{e^{3x}}{3} \frac{1}{D} \left(x - \frac{1}{6}\right)$$

$$= \frac{e^{3x}}{2} \left(\frac{x^2}{2} - \frac{x}{6}\right)$$

** Solve : $(D^2 - 2D + 1)y = e^x \sin x$

$\Rightarrow$ Given eqn.

$$(D^2 - 2D + 1)y = e^x \sin x \quad \text{—— (i)}$$

Let $y = e^{mx}$ be a trial sol^n of

$$(D^2 - 2D + 1)y = 0 \quad \text{—— (ii)}$$

then $\frac{dy}{dx} = me^{mx}$, $\frac{d^2y}{dx^2} = m^2 e^{mx}$

(ii) $\Rightarrow (m^2 - 2m + 1)e^{mx} = 0$

A.E is $m^2 - 2m + 1 = 0$ since $e^{mx} \neq 0$

or, $(m-1)^2 = 0$

or, $m = 1, 1$

$\therefore y_c = (c_1 + c_2 x)e^x$

$$\therefore y_p = \frac{1}{D^2 - 2D + 1} e^x \sin x$$

$$= \frac{1}{(D-1)^2} e^x \sin x$$

$$= e^x \frac{1}{(D+1-1)^2} \sin x$$

$$= e^x \frac{1}{D^2} \sin x$$

$$= e^x \frac{1}{-1^2} \sin x$$

$$= -e^x$$

Solutions of Homogeneous Linear Equations:

An eqn of the form $P_0 x^n \frac{d^n y}{dx^n} + P_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}}$

$$+ P_2 x^{n-2} \frac{d^{n-2} y}{dx^{n-2}} + \cdots + P_n y = R(x)$$

is called linear homogeneous eqn with variable co-efficients where $P_0, P_1, P_2 \cdots, P_n$ are constants and $R(x)$ is a constant or a funtion of x. This eqn is also called Cauchy-Euler eqn.

Solve: $(x^3 D^3 + 3x^2 D^2 - 2xD + 2) y = 0$

Given,

$$(x^3 D^3 + 3x^2 D^2 - 2xD + 2) y = 0$$

put $x = e^z$

$z = \ln x$

$\frac{dz}{dx} = \frac{1}{x} \cdots (1)$

$$\frac{dy}{dx} = \frac{dz}{dx} \cdot \frac{dy}{dz}$$

$$\frac{dy}{dx} = \frac{1}{x} \frac{dy}{dz} \quad \cdots (III)$$

$$x \frac{dy}{dx} = \frac{dy}{dz}$$

$$xDy = Dy \left[D = \frac{d}{x\,dz}\right] \cdots (IV)$$

$$\left[D = \frac{d}{dx}\right]$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$= \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dz}\right)$$

$$= \frac{1}{x}\frac{d}{dx}\left(\frac{dy}{dz}\right) - \frac{1}{x^2}\frac{dy}{dz}$$

$$= \frac{1}{x}\frac{d}{dz}\left(\frac{dy}{dz}\right)\frac{dz}{dx} - \frac{1}{x^2}\frac{dy}{dz}$$

$$= \frac{1}{x}\frac{dy}{dz^2} \cdot \frac{1}{x} - \frac{1}{x^2}\frac{dy}{dz}$$

$$\Rightarrow x^2 \frac{d^2y}{dx^2} = \frac{d^2y}{dz^2} - \frac{dy}{dz}$$

$$\Rightarrow x^2 D^2 y \left[D=\frac{d}{dx}\right] = D(D-1)y \left[D=\frac{d}{dz}\right] \cdots (5)$$

Similarly,

$$x^3 D^3 y \left[D=\frac{d}{dx}\right] = D(D-1)(D-2)y \left[D=\frac{d}{dz}\right] \cdots (6)$$

Putting the values in (1),

$$[D(D-1)(D-2) + 3D(D-1) - 2D + 2]y = 0$$

$$\Rightarrow (D^3 - 3D^2 + 2D + 3D^2 - 3D - 2D + 2)y = 0$$

$$(D^3 - 3D + 2)y = 0 \cdots (7)$$

Let,

$y = e^{mz}$ be the trial soln of (7)

$$Dy = me^{mz}, \quad D^2 y = m^2 e^{mz}$$

$$D^3 e^{mz} = m^3 e^{mz}$$

Putting the values in (7),

$(m^3 - 3m + 2)e^{mz} = 0$

Since $e^{mz} \neq 0$, A.E is

$m^3 - 3m + 2 = 0$

$m = 1, 1, -2$

The general solution,

$y = (c_1 + c_2 z)e^z + c_3 e^{-2z}$

$= (c_1 + c_2 \ln x)x + c_3 x^{-2}$

$= (c_1 + c_2 \ln x)x + \frac{c_3}{x^2}$.

Solve: $(x^2D^2 - xD + 1)y = \log x$

Soln: Given,

$$(x^2D^2 - xD + 1)y = \log x \quad \cdots (1)$$

Put $x = e^z$

$$z = \log x$$

$$\frac{dz}{dx} = \frac{1}{x} \quad \cdots (2)$$

$$\frac{dy}{dx} = \frac{dz}{dx}\frac{dy}{dz}$$

$$\frac{dy}{dx} = \frac{1}{x}\frac{dy}{dz} \quad \cdots (3)$$

$$x\frac{dy}{dx} = \frac{dy}{dz}$$

$$xDy = Dy \quad [D = \frac{d}{dz}] \quad \cdots (4)$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$= \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dz}\right)$$

$$= \frac{1}{x}\frac{d}{dx}\left(\frac{dy}{dz}\right) - \frac{1}{x^2}\left(\frac{dy}{dz}\right)$$

$$= \frac{1}{x}\frac{d}{dz}\left(\frac{dy}{dz}\right)\frac{dz}{dx} - \frac{1}{x^2}\frac{dy}{dz}$$

$$= \frac{1}{x}\frac{dy}{dz^2}\cdot\frac{1}{x} = \frac{1}{x^2}\frac{dy}{dz}$$

$$\Rightarrow x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dz^2} - \frac{dy}{dz}$$

$$\Rightarrow x^2 + x^2D^2y \left[D = \frac{d}{dx}\right] = D(D-1)y \left[D = \frac{d}{dz}\right] \quad \cdots (5)$$

Putting the values in (1)

$$[D(D-1) - D + 1]\, y = e^z$$

$$\Rightarrow (D^2 - 2D + 1)y = z$$

For C.I $(D^2 - 2D + 1)y = 0 \quad \cdots (6)$

Let, $y = e^{mz}$ be the trial soln of (6)

$$Dy = me^{mz}, \; D^2y = m^2e^{mz}$$

Putting the values in ⑥,

$(m^2-2m+1)e^{mz}=0$

Since $e^{mz}\neq 0$, AE is

$m^2-2m+1=0$

$\Rightarrow (m-1)^2=0$

$\Rightarrow m=1,1$

$y_c=(c_1+c_2z)e^z$

$=\{c_1+c_2(\ln x)\}x$

$=(c_1+c_2\ln x)x.$

(A)
1. Defⁿ
2. Formation

(B)
1. Separation of variables
2. Homogenous
3. Eqⁿ Reducible to Homogenous
4. Exact
5. Euler Linear
6. Bernaulli

(C)
1. Solvable for P
2. " " y
3. " " x
4. Clairauits
5. langranges

(D)
1. $R(x) = 0$
2. $R(x) = x$
3. $R(x) = e^{ax}$
4. $R(x) = \sin ax / \cos ax$
5. $R(x) = e^{ax} \cdot V$

(E)
1. Homogenous
2. Linear Eqn

(F)
1. Series Soln

(G)
1. Bessels Function
2. Legendre Polynomial

1. a), b) → A
   c) → B or C

2. a), b), c) → B or C

3. a), b), c) → D

4. a), b), c) → E, F