Lagrange's Equation

The equation of the form $y = f(p)x + g(p)$ known as Lagrange's Equation which is solvable for y and leads to a differential equation in p and x which is linear in $\frac{dx}{dp}$

* Method of finding the solution of the Lagrange's Equation.

The Lagrange's Equation is

$$y = f(p)x + g(p) \quad \text{——— (1)}$$

Differentiating (1) w.r to x, we get

$$\frac{dy}{dx} = f(p) + x f'(p)\frac{dp}{dx} + g'(p)\frac{dp}{dx}$$

$$\text{or} \quad p - f(p) = \left[x f'(p) + g'(p)\right]\frac{dp}{dx}$$

$$\text{or} \quad \left[p - f(p)\right]\frac{dx}{dp} = x f'(p) + g'(p)$$

$$\therefore \frac{dx}{dp} = x\frac{f'(p)}{p - f(p)} + \frac{g'(p)}{p - f(p)}$$

This is linear equation in x and p and can be solved in the usual way.

Solve: $(y+1)p - xp^2 + 2 = 0$ where $p = \frac{dy}{dx}$

Solution: The given equation is

$$(y+1)p - xp^2 + 2 = 0 \quad \text{—(1)}$$

$$\text{or } (y+1)p = xp^2 - 2$$

$$\text{or } y+1 = \frac{xp^2 - 2}{p}$$

$$\text{or } y+1 = xp - \frac{2}{p}$$

$$\text{or } y = xp + \left(-\frac{2}{p} - 1\right)$$

which is the Clairaut's form $y = px + f(p)$

Hence putting c for p, the general solution is

$$y = xc + \left(-\frac{2}{c} - 1\right)$$

$$= cx - \left(\frac{2}{c} + 1\right)$$

Solve: $y = (1+p)x + p^2$ where $p = \frac{dy}{dx}$

Solution: The given differential equation is

$$y = (1+p)x + p^2 \quad \text{——— (1)}$$

which is of the form $y = f(p)x + g(p)$

Now differentiating (1) w.r to x we get

$$\frac{dy}{dx} = (1+p) + x\frac{dp}{dx} + 2p\frac{dp}{dx}$$

or $$p = 1 + p + x\frac{dp}{dx} + 2p\frac{dp}{dx}$$

or $$1 + (x+2p)\frac{dp}{dx} = 0$$

or, $$(x+2p)\frac{dp}{dx} = -1$$

$$\therefore \frac{dx}{dp} + x = -2p \quad \text{——— (2)}$$

which is a linear equation in x.

To solve it $I.F = e^{\int 1.dp} = e^p$

$$e^p\frac{dx}{dp} + xe^p = -2pe^p$$

$$\frac{d}{dp}(xe^p) = -2pe^p$$

Integrating, $xe^p = -2\int pe^p\,dp$

or $$xe^p = -2(pe^p - e^p) + c$$

$$\therefore x = ce^{-p} - 2(p-1) \quad \text{——— (3)}$$

Substituting the values of x in (1) we have

$$y = (1+p)[ce^{-p} - 2(p-1)] + p^2 \quad \text{——— (4)}$$

Relation (3) and (4) give the required solution.

solve the following differential equations

1. $y = p^2x + p$, where $p = \frac{dy}{dx}$
2. $y = (1+p)x + 2p^2$, where $p = \frac{dy}{dx}$
3. $3y = 3(1+p)x + p^2$, where $p = \frac{dy}{dx}$