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Linear partial differential eqns of second order:

Consider, the PDE of second order in two independent variable x and y as

$$A\frac{\partial^2 z}{\partial x^2} + B\frac{\partial^2 z}{\partial x \partial y} + C\frac{\partial^2 z}{\partial y^2} + f\left(x, y, z, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}\right) = 0 \quad \text{--- (1)}$$

Or, $Ar + Bs + Ct + f(x, y, z, p, q) = 0$

where, A, B, C are constants or continuous func of and y and $A > 0$, Now, eqn (1) is

(i) Elliptic if $B^2 - 4AC < 0$

(ii) parabolic if $B^2 - 4AC = 0$

(iii) Hyperbolic if $B^2 - 4AC > 0$

Classify the following

$$4\frac{\partial^2 u}{\partial x^2} + 4\frac{\partial^2 u}{\partial x \partial t} + \frac{\partial^2 u}{\partial t^2} = 0$$

Soln:

$A = 4, B = 4, C = 1$

$\therefore B^2 - 4AC = 4^2 - 4 \cdot 4 \cdot 1$

$= 0$

Hence, the P.D.E is parabolic

Classify the PDE.

$$\frac{\partial^2 u}{\partial t^2} + t\frac{\partial^2 u}{\partial x \partial t} + x\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} + 6u = 0$$

Soln:

Here, $A = 1, B = t, C = x$,

$B^2 - 4AC = t^2 - 4x$

1. The eqn is elliptic if

$t^2 - 4x < 0 \Rightarrow t^2 < 4x$

2. The eqn parabolic if

$$t^2 - 4x = 0 \Rightarrow t^2 = 4x$$

3. The eqn is Hyperbolic if

$$t^2 - 4x > 0 \Rightarrow t^2 > 4x$$

Classify the PDE.

$$xy\frac{\partial^2 z}{\partial x^2} - (x^2 - y^2)\frac{\partial^2 z}{\partial x \partial y} - xy\frac{\partial^2 z}{\partial y^2} + y\frac{\partial z}{\partial x} - x\frac{\partial z}{\partial y} = 2(x^2 - y^2$$

$$xy\frac{\partial^2 z}{\partial x^2} + (y^2 - x^2)\frac{\partial^2 z}{\partial x \partial y} + (-xy)\frac{\partial^2 z}{\partial y^2} + \cdots$$

$A = xy$, $B = y^2 - x^2$, $C = -xy$

$$B^2 - 4AC = (y^2 - x^2)^2 - 4xy(-xy)$$

$$= y^4 - 2x^2y^2 + x^4 + 4x^2y^2$$

$$= x^4 + y^4 - 2x^2y^2 + 4x^2y^2$$

$$= (x^2 + y^2)^2 + 2x^2y^2$$

$$= (x^2 + y^2)^2 > 0$$ Hyperbolic.

H.W এর

Hyperbolic

$x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial y^2} = 0$ $\rightarrow -4xy$

$\begin{cases} x>0, y<0 \\ x<0, y<0 \end{cases}$ Elliptic

$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x \partial y} - 2\cdot\frac{\partial^2 u}{\partial y^2} = 0$ $\rightarrow \begin{cases} x<0, y>0 \\ x>0, y<0. \end{cases}$

$y^2\frac{\partial^2 u}{\partial x^2} - 2xy\frac{\partial^2 u}{\partial x \partial y} + x^2\frac{\partial^2 u}{\partial y^2} = \frac{y^2}{x}\cdot\frac{\partial u}{\partial x} + \frac{x^2}{y}\frac{\partial u}{\partial y}$

(উপরে অধিক variable থাকলে)

More than two variables:

$\sum\sum a_{ij}\cdot\frac{\partial^2 u}{\partial x_i \partial x_j}$

$\sum_{j=1}^{3}\sum_{i=1}^{3} a_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{3} b_i\frac{\partial u}{\partial x_i} + cu = 0$

$u_x = \frac{\partial u}{\partial x}$

$u_{xy} = \frac{\partial^2 u}{\partial x \partial y}$

Consider, a matrix A = Coff of

$$\begin{matrix} u_{xx} & u_{xy} & u_{xz} \\ u_{xy} & u_{yy} & u_{yz} \\ u_{zx} & u_{zy} & u_{zz} \end{matrix}$$

$$\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2 \partial x_2} + \frac{\partial^2 u}{\partial x_3^2} + \frac{\partial^2 u}{\partial x_1 \partial x_2} + \frac{\partial^2 u}{\partial x_1 \partial x_2} +$$

$$\frac{\partial^2 u}{\partial x_1 \partial x_3} + \frac{\partial^2 u}{\partial x_2 \partial x_3} + \text{---}$$

$$u_{x_1 x_1} + u_{x_2 x_2} + u_{x_3 x_3} + u_{x_1 x_2} +$$

$$u_{x_1 x_3} + u_{x_2 x_3} + \text{---}$$

Consider, a matrix A = coff of

$$\begin{bmatrix} u_{xx} & u_{xy} & u_{xz} \\ u_{xy} & u_{yy} & u_{yz} \\ u_{zx} & u_{zy} & u_{zz} \end{bmatrix}$$

then if

i) λ of A are non-zero and all have same sign elliptic.

ii) If any λ is zero, then parabolic

iii) λ are non-zero and have same

sign except one then hyperbolic.

$u_{xx} + 2u_{xy} + u_{zz} + u_{yz} + u_{zy} = 0$

The matrix A of given eqn

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Det = 0 ∴ parabolic.

$1(1-1) = 0 \Rightarrow$ parabolic.

Classify $u_{xx} + u_{yy} = u_{zz}$

Classify $u_{xx} + u_{yy} + 2u_{zz} = 0$

H.W.

Soln:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Det $= 1(1+1) - 0$

$= 2 \rightarrow$ hyperbolic