SA - SB

25/09/23

MATH - 103

ODE, PDE and Matrix

Matrix

$$A_{m \times n}^{\text{column}} = [a_{ij}]$$

↳ row

$$= \| \quad \|$$

$$= (\quad)$$

A matrix A_{RC} over a field R is a rectangular array of scalars usually presented in the following form:

$$A_{m \times n} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$A_{m \times 1}$ = column matrix

$A_{1 \times n}$ = row "

Elements

Equivalent matrix → row and col current same

Two matrices A and B are equal i.e. A = B if they have the same size and corresponding elements are equal.

—

row matrix, row vector

col n , col "

—

Real M $\begin{bmatrix} 3 & 4 & 5 \\ 1 & 6 & 8 \end{bmatrix}$

KER

Complex $\begin{pmatrix} 1-2i & i \\ 3i & -4 \end{pmatrix}$

KE ℂ

$$\begin{bmatrix} x+y & 2x+z \\ x-y & z-t \end{bmatrix} = \begin{bmatrix} 3 & 7 \\ 1 & 5 \end{bmatrix}$$

$x+y = 3,\ x-y = 1,\ 2x+z = 7,\ z-t = 5$

Matrix A, B, C (same size) $\quad k, k'$

i) $(A+B)+C = A+(B+C)$

ii) $A+0 = 0+A = A$

iii) $A+(-A) = 0 = (-A)+A$

iv) $A+B = B+A$

$k(A+B) = kA + kB$

$(k+k')A = kA + k'A$

$(kk')A = k(k'A)$

$1 \cdot A = A$

$(AB)C = A(BC)$

$A(B+C) = AB+AC$

$(B+C)A = BA + CA$

$k(AB) = (kA)B = A(kB)$

Transpose of a matrix

A^T

$$A = \begin{bmatrix} 5 & 1 & 3 \\ 4 & 8 & 9 \\ 11 & 13 & 2 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 5 & 4 & 11 \\ 1 & 8 & 13 \\ 3 & 9 & 2 \end{bmatrix}$$

—

$$(A+B)^T = A^T + B^T$$

$$(A^T)^T = A$$

$$(kA)^T = kA^T$$

$$(AB)^T = B^T A^T$$

Diagonal and Trace

Diagonal → a_{ij} ; $i = j$

Trace → summation of diagonal elements

For any square matrix A, the diagonal or main diagonal of A consists of the elements with the same subscripts that is $a_{11}, a_{22}, \cdots, a_{nn}$.

The trace of A, written $tr(A)$ is the sum of the diagonal elements, i.e. $tr(A) = a_{11} + a_{22} + \cdots + A_{nn}$.

$tr(AB) = tr(BA)$

$tr(A+B) = tr(A) + tr(B)$

$tr(kA) = k\,tr(A)$

$tr(A^T) = tr(A)$

Determinate

$D = \begin{vmatrix} & \\ & \end{vmatrix}$

$D = 0 \rightarrow$ non-~~invertible~~ invertiable matrix, singular matrix

$D \neq 0 \rightarrow$ invertiable " , non-singular matrix

$AB = BA = I$, $A^{-1} = B$ ~~or~~ $B^{-1} = A$ | $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Invert matrix is unique

Proof: $A \begin{cases} B_1 \\ B_2 \end{cases}$ $AB_1 = B_1A = I$

$AB_2 = B_2A = I$

$B_1 = B_1 I = B_1 (AB_2) = (B_1A) B_2 = I B_2$

$= B_2$

$(AB)^{-1} = B^{-1} A^{-1}$ not $A^{-1} B^{-1}$

Home work : inverse of 3x3, 4x4 matrix

HW/SB 02/10/23

Inverse of 3×3 matrix

Let,

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 0 \end{bmatrix}$$

Now,

$$\det(A) = 1(0-24) - 2(0-20) + 3(0-5)$$

$$= -24 + 40 - 15$$

$$= 1 \neq 0$$; matrix A has an inverse matrix

Now, Let,

$$a_{11} = \begin{vmatrix} 1 & 4 \\ 6 & 0 \end{vmatrix} = -24$$

$$a_{12} = \begin{vmatrix} 0 & 4 \\ 5 & 0 \end{vmatrix} = -20$$

$$a_{13} = \begin{vmatrix} 0 & 1 \\ 5 & 6 \end{vmatrix} = -5$$

$$a_{21} = \begin{vmatrix} 2 & 3 \\ 6 & 0 \end{vmatrix} = -18$$

$$a_{22} = \begin{vmatrix} 1 & 3 \\ 5 & 0 \end{vmatrix} = -15$$

$$a_{23} = \begin{vmatrix} 1 & 2 \\ 5 & 6 \end{vmatrix} = -4$$

$$a_{31} = \begin{vmatrix} 2 & 3 \\ 1 & 4 \end{vmatrix} = 5$$

$$a_{32} = \begin{vmatrix} 1 & 3 \\ 0 & 4 \end{vmatrix} = 4$$

$$a_{33} = \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} = 1$$

Hence the matrix of cofactors:

$$\begin{bmatrix} -24 & -20 & -5 \\ -18 & -15 & -4 \\ 5 & 4 & 1 \end{bmatrix} \times \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

$$= \begin{bmatrix} -24 & 20 & -5 \\ 18 & -15 & 4 \\ 5 & -4 & 1 \end{bmatrix}$$

And, $adj(A) = \begin{bmatrix} -24 & 18 & 5 \\ 20 & -15 & -4 \\ -5 & 4 & 1 \end{bmatrix}$

$$\therefore A^{-1} = \frac{1}{\det(A)} adj(A) = 1 \begin{bmatrix} -24 & 18 & 5 \\ 20 & -15 & -4 \\ -5 & 4 & 1 \end{bmatrix}$$

Inverse of 4×4 matrix

Let,

$$A = \begin{bmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \end{bmatrix}$$

$\therefore \det(A) = -16$ [using calculator]

Now the determinants of the submatrixes of A are:

$$a_{11} = \begin{vmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} = -4 \text{ [using calculator]}$$

Similarly,

$a_{12} = 4$ $a_{21} = 4$ $a_{31} = -4$ $a_{41} = -4$

$a_{13} = -4$ $a_{22} = -4$ $a_{32} = -4$ $a_{42} = -4$

$a_{14} = -4$ $a_{23} = -4$ $a_{33} = -4$ $a_{43} = 4$

$a_{24} = -4$ $a_{34} = 4$ $a_{44} = -4$

Hence the matrix of cofactors is,

$$\begin{bmatrix} +a_{11} & -a_{12} & +a_{13} & -a_{14} \\ -a_{21} & +a_{22} & -a_{23} & +a_{24} \\ +a_{31} & -a_{32} & +a_{33} & -a_{34} \\ -a_{41} & +a_{42} & -a_{43} & +a_{44} \end{bmatrix}$$

$$= \begin{bmatrix} -4 & -4 & -4 & 4 \\ -4 & -4 & 4 & -4 \\ -4 & 4 & -4 & -4 \\ 4 & -4 & -4 & -4 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A)$$

$$= -\frac{1}{16} \begin{bmatrix} -4 & -4 & -4 & 4 \\ -4 & -4 & 4 & -4 \\ -4 & 4 & -4 & -4 \\ 4 & -4 & -4 & -4 \end{bmatrix}$$

SB 02/10/23

Types of Matrix

Upper triangular matrix:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \quad a_{ij} = 0 \;;\; \forall\, i > j$$

Lower triangular matrix:

$$a_{ij} = 0; \quad i < j$$

Periodic matrix:

$A = [a_{ij}]_{n \times n}$ be a matrix

$A^{P+1} = A$

If for $P \in N$, $A^{P+1} = A$, then A is called periodic matrix and if p is the least +(ve) integer for which $A^{P+1} = A$ then A is called periodic matrix of period P.

$A^0 = I$

$A^2 = A \cdot A$

$A^P = A^{P-1} \cdot A$

$A^{P+1} = A^P \cdot A$

$= A^{P-1} \cdot A \cdot A$

$= A^{P-2} \cdot A \cdot A \cdot A$

$$A = \begin{bmatrix} 1 & -3 & 2 \\ -2 & 2 & 0 \\ -6 & 9 & -1 \end{bmatrix}$$

Check whether this matrix is periodic or not and find its period.

Procedure: Find A^2, A^3, A^4, A^5 (max)

In the above case, $A^3 = A$

Idempotent matrix

$A^2 = A$

e.g. $A = \begin{bmatrix} -1 & 1 & -1 \\ 3 & -3 & 3 \\ 5 & -5 & 5 \end{bmatrix}$

Nilpotent matrix

$A^n = 0$ but $A^{n-1} \neq 0$

$A_{n \times n}$, $A_{5 \times 5}$ } $A^5 = 0$, $A^4 \neq 0$

e.g., $$B = \begin{bmatrix} 5 & -3 & 2 \\ 15 & -9 & 6 \\ 10 & -6 & 4 \end{bmatrix}$$

Involuntary matrix:

$A^2 = I$

$A \cdot A = I$

If a square matrix is equal to its own inverse, then matrix is called involuntary.

Symmetric matrix:

$$A^T = A$$

$$\begin{pmatrix} a & h & g \\ h & b & f \\ g & f & c \end{pmatrix}$$

Skew matrix:

$$A^T = -A$$

$$\begin{pmatrix} 0 & h & g \\ -h & 0 & f \\ -g & -f & 0 \end{pmatrix}$$

Conjugate of a matrix (complex)

$$A = \begin{pmatrix} 1+i & 1-i \\ -i & 4i \end{pmatrix}$$

$$\bar{A} = \begin{pmatrix} 1-i & 1+i \\ i & -4i \end{pmatrix}$$

Hermitian matrix:

$$A^* = A$$

$$A^* = \bar{A}^T$$

$$\begin{pmatrix} 3 & 1-2i & 4+7i \\ 1+2i & -4 & -2i \\ 4-7i & 2i & 5 \end{pmatrix}$$

Skew Hermitian matrix:

$$A^* = -A$$

$$\begin{pmatrix} 1 & 1+i \\ -1+i & 0 \end{pmatrix}$$

Orthogonal matrix

$$A A^T = A^T A = I$$

$$A^T = A^{-1}$$

$$\begin{pmatrix} 1/3 & 2/3 & -2/3 \\ 2/3 & 1/3 & 2/3 \\ 2/3 & -2/3 & -1/3 \end{pmatrix}$$

Q. Given matrix is orthogonal, find its inverse.

Unitary matrix

$$A A^{*} = A^{*} A = I$$

$$A^{*} = A^{-1}$$

$$A = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$$

Normal matrix

$$A \cdot A^{*} = A^{*} A$$

$$A = \begin{pmatrix} 2-3i & -i \\ 1 & 1-2i \end{pmatrix}$$

Sub matrix

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \rightarrow \begin{bmatrix} 5 & 6 \\ 8 & 9 \end{bmatrix}$$

Minor of a matrix,

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \rightarrow M_{21} = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}$$

Co-factor of a matrix

$$A_{ij} = (-1)^{i+j} M_{ij}$$

Adjoint of a matrix

$$Adj(A) = [A_{ij}]^T$$

$$A^{-1} = \frac{1}{|A|} Adj\, A$$

H.W.

* Find the inverse of the matrix,

$$A = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 1 & 1 \\ -7 & 2 & 3 \end{bmatrix}$$

* Using Adjoint of a matrix, find the matrix A

if $(4A)^{-1} = \begin{pmatrix} 1 & -1 & 2 \\ 0 & 2 & 0 \\ 3 & -1 & 4 \end{pmatrix}$

* Solving the system by using cramers rule

$$x + 5y - z = 9$$

$$3x - 3y + 2z = 7$$

$$2x - 4y + 3z = 1$$

Find the value of x, y, z, t

$$2\begin{bmatrix} x & z \\ y & t \end{bmatrix} - 3\begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} = 3\begin{bmatrix} 3 & 5 \\ 4 & 6 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2x & 2z \\ 2y & 2t \end{bmatrix} = \begin{bmatrix} 9 & 15 \\ 12 & 18 \end{bmatrix} + \begin{bmatrix} 3 & -3 \\ 0 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 12 & 12 \\ 12 & 24 \end{bmatrix}$$

$\therefore x = 6;\quad z = 6;\quad y = 6;\quad t = 12$

Cramers Rule:

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

—

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$D = |A|$

$$D_{x_1} = \begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{23} & a_{33} \end{vmatrix}$$

$D_{x_2} = \ldots$

$D_{x_3} = \ldots$

$$x_1 = \frac{D_{x_1}}{D}$$

$$x_2 = \frac{D_{x_2}}{D}$$

$$x_3 = \frac{D_{x_3}}{D}$$

Linear eq.

$\sqrt{x} + 3y = \sin 2$ → non-linear

$4x + \sqrt{3}y = \ln 2$ → linear

SB

04/10/23

Block matrix

$$\left[\begin{array}{cc|cc} 1 & 2 & 4 & 7 \\ \hline 8 & 10 & 11 & 16 \\ 9 & 3 & 5 & 12 \end{array}\right]$$

—

$$ax_1 + by_1 = c$$

$$ex_1 + fy_1 = d$$

$$\begin{bmatrix} a & b \\ e & f \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} c \\ d \end{bmatrix}$$

A B

$$\left[\begin{array}{cc|c} a & b & c \\ e & f & d \end{array}\right]$$

Row/column operation

1. interchange any two row/col.
2. multiply each element of any row/col by a scalar
3. multiplied each element of any row/col by scalar then adding with another row or col..

Elementry row/col op.

The new matrix is equivalent of the prev one.

Row Echelon Form

Conditions

(i) zero ∅ row/col at last [if any]

(ii) below element of leading element of each row is zero. [leading el: first non-zero el from non-zero row]

(iii) lower rows leading el is right of upper row

$$\begin{bmatrix} 1 & 3 & 2 & 0 \\ 0 & 5 & 6 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

—

Convert to Row Echelon Form

$$\begin{bmatrix} 1 & 2 & -1 & 2 & 1 \\ 2 & 4 & 1 & -2 & 3 \\ 3 & 6 & 2 & -6 & 5 \end{bmatrix}$$

$$\begin{pmatrix} 1 & 2 & -1 & 2 & 1 \\ 0 & 0 & 3 & -6 & 1 \\ 0 & 0 & 15 & -12 & 2 \end{pmatrix}$$

$R_2' = R_2 - 2R_1$

$R_3' = R_3 - 3R_1$

$$\begin{bmatrix} 1 & 2 & -1 & 2 & 1 \\ 0 & 0 & 3 & -6 & 1 \\ 0 & 0 & 0 & -6 & 1 \end{bmatrix} \quad R_3' = 3R_3 - 5R_2$$

—

Next: Row Reduced Echelon Form

1.
2. } From Row Echelon Form
3.

4. Leading el must be 1

5. Both upper & lower el of leading el is zero

Almost converts to I

If I, then it is called:

SB 05/10/23

Row reduced Echelon Form

From prev class,

$$\begin{bmatrix} 1 & 2 & -1 & 2 & 1 \\ 0 & 0 & 3 & -6 & 1 \\ 0 & 0 & 0 & -6 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 2 & 1 \\ 0 & 0 & 1 & -2 & 1/3 \\ 0 & 0 & 0 & 1 & -1/6 \end{bmatrix} \quad R_2' = \frac{1}{3} R_2 \quad R_3' = -\frac{1}{6} R_3$$

$$\sim \begin{bmatrix} 1 & 2 & 0 & 0 & 4/3 \\ 0 & 0 & 1 & -2 & 1/3 \\ 0 & 0 & 0 & 1 & -1/6 \end{bmatrix} \quad R_1' = R_1 + R_2$$

$$\sim \begin{bmatrix} 1 & 2 & 0 & 0 & 4/3 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1/6 \end{bmatrix} \quad R_2' = R_2 + 2R_3$$

$\longrightarrow$ Row R. E. F.

H.W

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 6 & 0 & 1 & -2 \\ 8 & 1 & -2 & 1 \end{bmatrix}$$

Convert to R.E.F & R.R.E.F

System of Linear eq^{n}s.

$$\left.\begin{aligned} y + 2x &= 2 \\ x + \sqrt{5}\, y &= \sin x \end{aligned}\right\} \text{ linear} \qquad \left.\begin{aligned} \sqrt{y} + 2xy &= 2 \\ xy + 2y &= 2 \end{aligned}\right\} \text{ non-lin.}$$

~~leading variable~~

$ax + by = c$; $a, b \in \mathbb{R}$

leading const

leading var

$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = b$

$\sum_{i=1}^{n} a_i x_i = b$

$b = 0 \rightarrow$ homogeneous eq^n

$b \neq 0 \rightarrow$ non- " "

$\{s_1, s_2, \ldots, s_n\} \rightarrow$ sol^n set

#Find the soln set of $x_1 + 2x_2 = 4$

Solve for x_1

$x_1 = 4 - 2x_2$

↳ free variable

$$\begin{cases} x_2 = t \in \mathbb{R} \\ x_1 = 4 - 2t \end{cases}, \quad t = \text{parametric}$$

↳ soln set $\begin{cases} x_1 = 4 - 2t \\ x_2 = t \end{cases}$, $t \in \mathbb{R}$

Find the soln set of $x_1 - 2x_2 + x_3 = 5$

Solve for x_1,

$x_1 = 5 + 2x_2 - x_3$

$$\begin{cases} x_2 = t \\ x_3 = s \\ x_1 = 5 + 2t - s \end{cases} \qquad t, s \in \mathbb{R}$$

num of free var = num of var − num of eqn var

—

Find the soln set of

$$x + 2y - z = 2$$
$$y - z = 1$$

$y = 1 + z$

$z = t$

$y = 1 + t$

$x = 2 - 2y + z$

$= 2 - 2(1 + t) + t$

$= -t$

$\{x = -t,\ y = 1+t,\ z = t;\ t \in \mathbb{R}\}$ → General soln

$t = 2$

$x = -2,\ y = 3,\ z = 2$

$(-2, 3, 2)$ → Particular soln

A system two linear eq^{n}'s with two variable

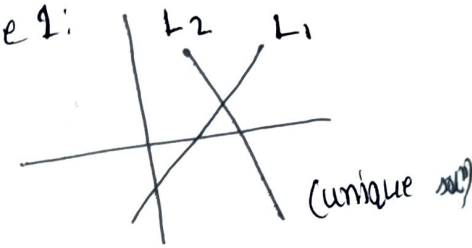
$$\left.\begin{aligned} L_1 &: ax + by = h \\ L_2 &: cx + dy = k \end{aligned}\right\} \rightarrow (1)$$

Case 1: System (1) has exactly one soln i.e, unique soln:

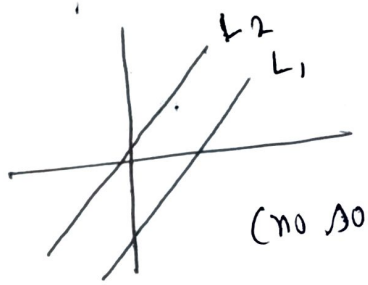
if L_1 and L_2 intersect at exactly one point i.e., when the lines have distinct slopes on co-efficient of x and y are not proportional.

Possible soln's

case 1:

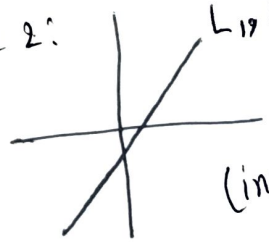


(unique soln)



(no soln)

Case 2:



(infinite soln)

Case2: System (1) has infinitely many soln's $L_1 \parallel L_2$ and coincident i.e, lines have the same slope and same y intercepts i.e,

$$\frac{a}{c} = \frac{b}{d} = \frac{h}{k}$$

Case 3: System ① have no soln [system known as 'inconsistent']

$L_1 \parallel L_2$ and distinct i.e., lines have same slope but distinct y intercept i.e.,

$$\frac{a}{c} = \frac{b}{d} \neq \frac{h}{k}$$

Case 1, 2 → consistent

case 3 → inconsistent

[one problem may come, verify all 3 cases.]

H.W Soln

Given matrix:

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 6 & 0 & 1 & -2 \\ 8 & 1 & -2 & 1 \end{bmatrix} = A$$

Conversion to Row Echelon Form:

$$A \sim \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 6 & 1 & -2 \\ 0 & 9 & -2 & 1 \end{bmatrix} \quad \begin{aligned} R_2' &= R_2 - R_1 \\ R_3' &= R_3 - 6R_1 \\ R_4' &= R_4 - 8R_1 \end{aligned}$$

$$\sim \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & -2 & 1 \end{bmatrix} \quad \begin{aligned} R_3' &= R_3 - 2R_2 \\ R_4' &= R_4 - 3R_2 \end{aligned}$$

$$\sim \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & -3 \end{bmatrix} \quad R_4' = R_4 + 2R_3$$

∴

It is the Row Echelon Form

Now, conversion to Row Reduced Echelon Form:

$$A \sim \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad R_2' = \frac{1}{3} R_2 \quad R_4' = -\frac{1}{3} R_4$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad R_1' = R_1 + R_2 \quad R_3' = R_3 + 2R_4$$

It is the Row Reduced Echelon Form

No sol^n

Inconsistent

$0 = a$, False

Consistent

$0 = 0$; True

A system of linear eq^n

$$\left.\begin{aligned} 2x - y &= 1 \\ 3x + 2y &= 12 \end{aligned}\right\} \quad \frac{2}{3} \neq \frac{1}{2}$$

$y = 2x - 1$

$3x + 4x - 2 = 12$

$\Rightarrow x = 2$

$y = 3$

#

$2x - y = 1$

$6x - 3y = 3$

$\frac{2}{6} = \frac{1}{3} = \frac{1}{3}$

Infinite sol^n

$y = 2x - 1$

$6x - 6x + 3 = 3$

$\Rightarrow 0 = 0 \rightarrow$ consistent

$$\begin{cases} x = t \; ; & t \in \mathbb{R} \\ y = 2t - 1 \end{cases}$$

$2x - y = 1$

$6x - 3y = 12$

$\frac{2}{6} = \frac{1}{3} \neq \frac{1}{12}$

No soln

$y = 2x - 1$

$6x - 6x + 3 = 12$

$\Rightarrow 0 = 9$

inconsistent system

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_m = b_m$$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

$$Ax = b$$

$[A|b]$ → Augmented matrix

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

Gaussian Elimination

Row Echelon Form

Consider 3×3 matrix

$$\left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{array}\right] \sim \left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1' \\ 0 & a_{22} & a_{23} & b_2' \\ 0 & 0 & a_{33} & b_3' \end{array}\right]$$

$$\sim \quad a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1'$$

$$\cancel{a_{21}x_1} + a_{22}x_2 + a_{23}x_3 = b_2'$$

$$a_{33}x_3 = b_3'$$

$$\therefore x_3 = \frac{b_3'}{a_{33}}$$

Gauss Jordan Elimination

Row Reduced Echelon Form

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & b_1' \\ 0 & 1 & 0 & b_2' \\ 0 & 0 & 1 & b_3' \end{array}\right] \rightarrow \begin{array}{l} x_1 = b_1' \\ x_2 = b_2' \\ x_3 = b_3' \end{array}$$

$3x - 2y + 8z = 9$

$-2x + 2y + z = 3$

$x + 2y - 3z = 8$

Sol^n:

The augmented matrix

$$\left[\begin{array}{ccc|c} 3 & -2 & 8 & 9 \\ -2 & 2 & 1 & 3 \\ 1 & 2 & -3 & 8 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 3 & -2 & 8 & 9 \\ 0 & 2 & 19 & 27 \\ 0 & 8 & -17 & 15 \end{array}\right] \quad \begin{array}{l} R_2' = 3R_2 + 2R_1 \\ R_3' = 3R_3 - R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 3 & -2 & 8 & 9 \\ 0 & 2 & 19 & 27 \\ 0 & 0 & -93 & -93 \end{array}\right] \quad R_3' = R_3 - 4R_2$$

$\approx A$

$3x - 2y + 8z = 9$

$2y + 19z = 27$

$-93z = -93$

$\therefore z = 1$

$2y + 19 = 27$

$\Rightarrow y = 4$

$3x - 8 + 8 = 9$

$\Rightarrow x = 3$

Using Gaussian Elimination

$$A \sim \left[\begin{array}{ccc|c} 1 & -2/3 & 8/3 & 3 \\ 0 & 1 & 19/2 & 27/2 \\ 0 & 0 & 1 & 1 \end{array}\right] \quad \begin{array}{l} R_1' = \frac{1}{3} R_1 \\ R_2' = \frac{1}{2} R_2 \\ R_3' = -\frac{1}{93} R_3 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 9 & 12 \\ 0 & 1 & 19/2 & 27/2 \\ 0 & 0 & 1 & 1 \end{array}\right] \quad R_1' = R_1 + \frac{2}{3} R_2$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$x = 3$

$y = 4$

$z = 1$

H.W

Solve:

$$2x - 3y + 5z = 1$$

$$3x + y - z = 2$$

$$x + 4y - 6z = 1$$

$$x - 4y + 5z = 8$$

$$3x + 7y - z = 3$$

$$x + 15y - 11z = -14$$

H.W soln / SB

Given equations

$$2x - 3y + 5z = 1$$
$$3x + y - z = 2$$
$$x + 4y - 6z = 1$$

The augmented matrix:

$$\left[\begin{array}{ccc|c} 2 & -3 & 5 & 1 \\ 3 & 1 & -1 & 2 \\ 1 & 4 & -6 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 2 & -3 & 5 & 1 \\ 0 & 11 & -17 & 1 \\ 0 & 11 & -17 & 1 \end{array}\right] \quad R_2' = 2R_2 - 3R_1, \quad R_3' = 3R_3 - R_2$$

$$\sim \left[\begin{array}{ccc|c} 2 & -3 & 5 & 1 \\ 0 & 11 & -17 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad R_3' = R_3 - R_2$$

$$\left[\begin{array}{ccc|c} 2 & -3 & 5 & 1 \\ 0 & 11 & -17 & 1 \\ 0 & 0 & 4 & 14 \end{array}\right] \quad R_3' = 3R_3 + 11R_1$$

Now, $2x - 3y + 5z = 1$

$$11y - 17z = 1$$

$$4z = 14$$

$$\therefore z = 7/2 = 3.5$$

$$11y - 17\ 7/2 = 1$$

$$\therefore y = 5.5$$

$$2x - 3 \times 5.5 + 5 \times 3.5 = 1$$

$$\therefore x = 0$$

Given equations:

$$x - 4y + 5z = 8$$
$$3x + 7y - z = 3$$
$$x + 15y - 11z = -14$$

The augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & -4 & 5 & 8 \\ 3 & 7 & -1 & 3 \\ 1 & 15 & -11 & -14 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & -4 & 5 & 8 \\ 0 & 19 & -16 & -21 \\ 0 & 38 & -32 & -45 \end{array}\right]$$

$R_2' = R_2 - 3R_1$

$R_3' = 3R_3 - R_2$

Now, $x - 4y + 5z = 8$

$19y - 16z = -21$

$38y - 32z = -45$

Here, $\frac{19}{38} = \frac{-16}{-32} \neq \frac{-21}{-45}$

Hence, no solution

Given equations

$$2x - 3y + 5z = 1$$

$$3x + y - z = 2$$

$$x + 4y - 6z = 1$$

The augmented matrix,

$$\left[\begin{array}{ccc|c} 2 & -3 & 5 & 1 \\ 3 & 1 & -1 & 2 \\ 1 & 4 & -6 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 2 & -3 & 5 & 1 \\ 0 & 11 & -17 & 1 \\ 0 & 11 & -17 & 1 \end{array}\right] \quad \begin{array}{l} R_2' = 2R_2 - 3R_1 \\ R_3' = 3R_3 - R_2 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 2 & -3 & 5 & 1 \\ 0 & 11 & -17 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad R_3' = R_3 - R_2$$

Now, $0x + 0y + 0z = 0$

Hence, infinite solution

SB 12/10/23

*** Find the values of τ and m such that the following system of linear eq^{n}s has ~~i) no~~

i) no soln ii) more than one soln

iii) a unique soln

$$\left.\begin{aligned} x+y+z &= 6 \\ x+2y+3z &= 10 \\ x+2y+\tau z &= m \end{aligned}\right\} \longrightarrow (1)$$

Augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & \tau & m \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & \tau-3 & m-10 \end{array}\right]$$

$R_2' = R_2 - R_1$

$R_3' = R_3 - R_2$

case 1:

$0 = a$

$\tau = 3, \; m \neq 10$

No soln

case 2:

$0 = 0$

$\tau = 3, \; m = 10$

more than 1 soln

case 3:

$\tau \neq 3$

unique soln

Elaborated

Case 1:

If $\tau = 3$ and $m \neq 10$, then 3rd eqn is of the form $0 = a$ where $a = m - 10 \neq 0$, which is not true. So the system is inconsistent.

Thus ① has no soln for $\tau = 3$ and $m \neq 10$

Case 2:

If $\tau = 3$ and $m = 10$ then 3rd eqn vanishes and having 2 eqns in 3 variables. So, it has $(3-2) = 1$ free variable.

Therefore ① has more than one soln for $\tau = 3$ and $m = 10$

Case 3:

For a unique soln the co-efficient of z in 3rd eqn must be non-zero i.e. $\tau \neq 3$ and m may have any value.

Hence, ① has unique soln for $\tau \neq 3$ and arbitrary values of m

Linear Combination:

Let, V be a vector space over a field, F, then $\vec{v} \in V(F)$ is said to be the linear combination of the vectors $\vec{v_1}, \vec{v_2}, \ldots, \vec{v_n} \in V(F)$ if there exists scalars $\alpha_1, \alpha_2, \ldots, \alpha_n \in F$ such that

$$\vec{v} = \alpha_1 \vec{v_1} + \alpha_2 \vec{v_2} + \cdots \alpha_n \vec{v_n}$$

$V \rightarrow \{\vec{v_1}, \vec{v_2}, \ldots \vec{v_n}\}$ ← vector space

+ identity → $a + 0 = a$

inverse → $a + (-a) = 0$

multiplicative identity

$a \cdot 1 = a$

inverse ... $a \cdot a^{-1} = 1$

$\{\vec{v_1}, \vec{v_2}, \ldots \vec{v_k}\} \rightarrow$ subspace

- V

$v_1, v_2, v_3 \cdots v_n$ $\in V$

$\alpha_1, \alpha_2 \cdots \alpha_n$

$v = \alpha_1 v_1 + \alpha_2 v_2 + \cdots \alpha_n v_n$

Show that $\vec{v} = (1, -2, 5)$ is a linear combination of

$\vec{v_1} = (1, 1, 1)$, $\vec{v_2} = (1, 2, 3)$ and $\vec{v_3} = (2, -1, 1)$

∃ → there exists

~~$v = \alpha_1 v_1 + \alpha$~~

Soln: In order to show that $\vec{v}$ is a linear combination of $\vec{v_1}, \vec{v_2}, \vec{v_3}$ ∃ $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ such that,

$$\vec{v} = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3$$

$$\Rightarrow (1, -2, 5) = \alpha_1 (1, 1, 1) + \alpha_2 (1, 2, 3) + \alpha_3 (2, -1, 1)$$

$$= (\alpha_1, \alpha_2, \alpha_1) + (\alpha_2, 2\alpha_2, 3\alpha_2) + (2\alpha_3, -\alpha_3, \alpha_3)$$

$$(1, -2, 5) = (\alpha_1 + \alpha_2 + 2\alpha_3,\ \alpha_1 + 2\alpha_2 - \alpha_3,\ \alpha_1 + 3\alpha_2 + \alpha_3)$$

Equating corresponding components from both sides and forms system of eqns

$$x_1 + x_2 + 2x_3 = 1$$

$$x_1 + 2x_2 - x_3 = -1$$

$$x_1 + 3x_2 + x_3 = 5$$

$$\sim \quad x_1 + x_2 + 2x_3 = 1$$

$$x_2 - 3x_3 = -3 \qquad R_2' = R_2' - R_1$$

$$2x_2 - x_3 = 4 \qquad R_3' = R_3 - R_1$$

$$x_1 + x_2 + 2x_3 = 1$$

$$\sim \quad x_2 - 3x_3 = -3$$

$$5x_3 = 10$$

By back substitution method from the above system

$$x_3 = 2, \quad x_2 = 3, \quad x_1 = -6$$

Thus, $\vec{v}$ is a linear com... of $\vec{v_1}, \vec{v_2}, \vec{v_3}$

For which value of γ

$\bar{v} = (1, \gamma, 5)$ is a linear combination of

$\bar{v}_1 = (1, -3, 2)$, $\bar{v}_2 = (2, -1, 1)$

$$\begin{cases} \alpha_1 + 2\alpha_2 = 1 \\ -3\alpha_1 - \alpha_2 = \gamma \\ 2\alpha_1 + \alpha_2 = 5 \end{cases}$$

Generator or span:

$\forall \rightarrow$ for all

Let, $S = \{\bar{v}_1 \cdots \vec{v_n}\}$ is a ~~de~~finite subset of a vector space $V(F)$. If

$$\bar{u} = \alpha_1 \bar{v}_1 + \cdots \alpha_n \bar{v}_n \quad \forall\ \bar{u} \in V$$

and $\alpha_1, \alpha_2 \cdots \alpha_n \in F$. Then S is called generator or span of $V(F)$

Linear combination

$$\alpha_1 \bar{v}_1 + \alpha_2 \bar{v}_2 + \cdots + \alpha_n \bar{v}_n = 0$$

$\alpha_1, \alpha_2, \cdots, \alpha_n = 0 \rightarrow$ linearly independent

if at least one is non-zero, linearly dependent

SB 16/10/23

Linear Dependence:

Let, $V(F)$ be a vector space over the field F. Then the vectors $\vec{v}_1 \dots \vec{v}_n \in V(F)$ is said to be linearly dependent or simply dependent if for any scalars $\alpha_1, \dots \alpha_n \in F$, such that $\alpha_1 \vec{v}_1 + \dots + \alpha_n \vec{v}_n = 0$

Here, at least one of the scalars of the set $\{\alpha_1, \alpha_n\}$ is non-zero.

Vice versa: Linearly independent:

Here, all scalars of the set $\{\alpha_1, \dots \alpha_n\}$ is zero.

Test whether the vectors are linearly independent or dependent

$(1, 2, 1), (2, 1, 0), (1, -1, 2)$

—

Soln: There exists scalars $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ such that

$$\alpha_1(1, 2, 1) + \alpha_2(2, 1, 0) + \alpha_3(1, -1, 2) = 0$$

$$(\alpha_1 + 2\alpha_2 + \alpha_3, 2\alpha_1 + \alpha_2 - \alpha_3, \alpha_1 + 0 + 2\alpha_3) = 0$$

Equating corresponding components and forming a system of eqn:

$$\alpha_1 + 2\alpha_2 + \alpha_3 = 0$$

$$2\alpha_1 + \alpha_2 - \alpha_3 = 0$$

$$\alpha_1 + 0 + 2\alpha_3 = 0$$

$$\left.\begin{aligned} \alpha_1 + 2\alpha_2 + \alpha_3 &= 0 \\ 0 + \alpha_2 - 3\alpha_3 &= 0 \\ 0 + -2\alpha_2 + \alpha_3 &= 0 \end{aligned}\right\} \quad \begin{aligned} R_2' &= R_2 - 2R_1 \\ R_3' &= R_3 - R_1 \end{aligned}$$

$$\left.\begin{aligned} x_1 + 2x_2 + x_3 &= 0 \\ x_2 - 3x_3 &= 0 \\ 0 + 2x_3 &= 0 \end{aligned}\right\} \quad R_3' = R_3 + R_1$$

By back substitution,

$$x_3 = 0, \quad x_2 = 0, \quad x_1 = 0$$

H.W

Test whether vectors are linearly independent or dependent

$(1, 1, 2), (1, 0, 1), (2, 1, 3)$

Basis:

Let V be a vector space over the field F and $\{\bar{v}_1, \bar{v}_2, \cdots, \bar{v}_n\}$ be a finite set of vectors in V then the set $\{\bar{v}_1, \cdots, \bar{v}_n\}$ is called basis of V if

i) $\{\bar{v}_1, \bar{v}_2, \cdots, \bar{v}_n\}$ is linearly independent

ii) $\{\bar{v}_1, \bar{v}_2, \cdots, \bar{v}_n\}$ span $\bar{V}$

Eigen value & Eigen vector

Let A be square matrix of ~~order~~ order n. A nonzero vector $\bar{x} \in \mathbb{R}^n$ is called an eigenvector of A if $A\bar{x}$ is a scalar ~~multiple~~ multiple of $\bar{x}$ i.e.,

$$A\bar{x} = \lambda\bar{x} \quad \text{for some scalar } \lambda.$$

The scalar λ is called an eigen value of A and $\bar{x}$ is called an eigenvector of A corresponding eigen value λ.

$$A\bar{x} = \lambda\bar{x}$$

$$A\bar{x} - \lambda\bar{x} = 0$$

$$(A-\lambda)\bar{x} = 0$$

$$(A - \lambda I)\bar{x} = 0$$

$A - \lambda I \rightarrow$ characteristics matrix

Characteristics matrix: $A - \lambda I$

" " : $|A - \lambda I|$

" " polynomial eqn = $|A - \lambda I| = 0$

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$$

$$\begin{vmatrix} 1-\lambda & 2 \\ 3 & 4-\lambda \end{vmatrix} = \lambda^2 - 5\lambda + 2 = 0$$

Cayley - Hamiltenthm:

Every square matrix satisfies it's ~~characte~~ ~~characteristic~~

Characteristic equation

A

$\lambda^2 - 6\lambda + 3 = 0$

$A^2 - 6A = -3$

HW

$$A = \begin{pmatrix} 3 & 2 & 8 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{pmatrix}$$

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 3-\lambda & 2 & 8 \\ 2 & -\lambda & 2 \\ 4 & 2 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow A^3 - 6A^2 - 15A - 8 = 0$$

$$\Rightarrow A^3 \cdot A^{-1} - 6A^2 A^{-1} - 15AA^{-1} - 8A^{-1} = 0$$

$$\Rightarrow 8A^{-1} = A^2 - 6A - 15$$

SB 19/10/23

verify Caley-Hamilton thm

for $A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Find A^{-1}, A^4

—

$|A - \lambda I| = 0$

$= \begin{vmatrix} 1-\lambda & 2 & 0 \\ 2 & -1-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0$

$= \lambda^3 - \lambda^2 - 5\lambda + 5 = 0$

$A^3 - A^2 - 5A + 5I = 0$

Multiply by A^{-1}

$A^2 - A - 5 + 5A^{-1} = 0$

$\therefore A^{-1} = \frac{1}{5}(\cdots)$

Multiply by A

$A^4 - A^3 - 5A^2 + 5A = 0$

$\therefore A^4 = \cdots$

(21) Find the eigen value and eigenvectors of A

$$A = \begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{pmatrix}$$

Soln: The characteristic eqn of A

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 3-\lambda & 2 & 4 \\ 2 & -\lambda & 2 \\ 4 & 2 & 3-\lambda \end{vmatrix} = 0$$

→ for 3X3 matrix

$$\Rightarrow \lambda^3 - \begin{bmatrix} \text{sum of} \\ \text{diagonal} \\ \text{elements} \end{bmatrix} \lambda^2 + \begin{bmatrix} \text{sum of} \\ \text{minor of} \\ \text{diagonal elements} \end{bmatrix} \lambda - |A| = 0$$

$$\Rightarrow \lambda^3 + 6\lambda^2 - 15\lambda - 8 = 0$$

$$\Rightarrow \lambda^3 + \lambda^2 - 7\lambda^2 - 7\lambda - 8\lambda - 8 = 0$$

$$\Rightarrow \lambda^2(\lambda+1) - 7\lambda(\lambda+1) - 8(\lambda+1) = 0$$

$$\Rightarrow (\lambda+1)(\lambda^2 - 7\lambda - 8) = 0$$

$\lambda + 1 = 0$ $\qquad$ $\lambda^2 - 7\lambda - 8 = 0$

$\Rightarrow (\lambda - 8)(\lambda + 1) = 0$

$\lambda = 8, -1, -1$

—

$(A - \lambda I)\vec{x} = 0$

$$\begin{bmatrix} 3-\lambda & 2 & 4 \\ 2 & -\lambda & 2 \\ 4 & 2 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

For $\lambda = -1$,

$$\begin{bmatrix} 4 & 2 & 4 \\ 2 & 1 & 2 \\ 4 & 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} 4 & 2 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$2x_1 + x_2 + 2x_3 = 0$

$x_1 = \frac{1}{2}(-x_2 - 2x_3)$

$x_2 = 2t$
$x_3 = s$ $\quad t, s \in \mathbb{R}$

$x_1 = \frac{1}{2}(-2t - 2s)$

$= -t - s$

$$\bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -t-s \\ 2t \\ s \end{bmatrix} = \begin{bmatrix} -t \\ 2t \\ 0 \end{bmatrix} + \begin{bmatrix} -s \\ 0 \\ s \end{bmatrix}$$

$$= t\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + s\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Now, for $\lambda = 8$

$$\begin{bmatrix} -5 & 2 & 4 \\ - & - & - \\ - & - & - \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\sim \begin{bmatrix} -5 & 2 & 4 \\ 0 & -36 & 18 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore$

$$\begin{cases} -5x_1 + 2x_2 + 4x_3 = 0 \\ -2x_2 + x_3 = 0 \end{cases}$$

$x_3 = 2t, \quad t \in \mathbb{R}$

$x_2 = x_3/2 = \frac{2t}{2} = t$

$-5x_1 = -2x_2 - 4x_3$

$= -2t - 8t$

$\Rightarrow x_1 = \frac{2}{5}t + \frac{8}{5}t = \frac{10}{5}t = 2t$

$$\bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2t \\ t \\ 2t \end{bmatrix}$$

$$= t \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$$

HW

Find the eigen value and eigenmatrix of A

$$A = \begin{bmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ 1 & 1 & 3 \end{bmatrix}$$

row space
col "
soln "
null " → nullity

} same dim → rank

Basis

cond 1. linearly independent

2. vector space span

Dimension:

If a vector space V has a basis consisting of n vectors then the number n is called the dimension of V, denoted by dim(V).

Row space:

The subspace of $\mathbb{R}^n$ generated on spanned by the row vectors of A is called the row space of A.

Column space:

The s...

column vectors of A is called the column space of A.

Solution space:

The soln of a homogeneous or non homogeneous system in n-unknowns is actually a subspace of $\mathbb{R}^n$, is called soln space.

Null space:

The soln space of the homogeneous system of eqns, which is a subspace of $\mathbb{R}^n$ is called the null space of A.

Rank:

The common dimension of the row space and column space of a matrix A is called rank of A, denoted by rank(A).

Nullity:

The dimension of null space of A is called nullity of A, denoted by nullity (A)

Rank + nullity = num of columns

Find the rank and nullity of A

$$A = \begin{pmatrix} -1 & 2 & 0 & 4 & 5 & -3 \\ 3 & -7 & 2 & 0 & 1 & 4 \\ 2 & -5 & 2 & 4 & 6 & 1 \\ 4 & -9 & 2 & -4 & -4 & 7 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & -4 & -28 & -37 & 13 \\ 0 & 1 & -2 & -12 & -6 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Basis of row space = Basis of column space

$$= 2$$

Rank(A) = 2

$$\begin{pmatrix} 1 & 0 & -4 & -28 & -37 & 13 \\ 0 & 1 & -2 & -12 & -6 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$x_1 - 4x_3 - 28x_4 - 37x_5 + 13x_6 = 0$$

$$x_2 - 2x_3 - 12x_4 - 6x_5 + 5x_6 = 0$$

F.V.

Solve for leading variable:

$$x_1 = 4x_3 + 28x_4 + 37x_5 - 13x_6$$

$$x_2 = 2x_3 + 12x_4 + 6x_5 - 5x_6$$

$x_3 = r$, $x_4 = s$, $x_5 = t$, $x_6 = u$

$$x_1 = 4r + 28s + 37t - 13u$$

$$x_2 = 2r + 12s + 6t - 5u$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} 4r + 28s + 37t - 13u \\ 2r + 12s + 6t - 5u \\ r \\ s \\ t \\ u \end{pmatrix}$$

$$= r\begin{pmatrix} 4 \\ 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + s\begin{pmatrix} 28 \\ 12 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} 37 \\ 6 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + u\begin{pmatrix} -13 \\ -5 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\text{nullity}(A) = 4$$

Rank(A) : The num of leading var in the general soln of $Ax = 0$

nullity(A) = The num of params/F.V in the general soln of $Ax = 0$

Procedure in exam:

determine row echelon form

write the later defn of rank & nullity

show rank and nullity

H.W

Find the rank and nullity of A

$$A = \begin{pmatrix} 1 & 2 & 0 & 1 & 1 \\ 2 & 4 & 1 & 3 & 0 \\ 3 & 6 & 2 & 5 & 1 \\ -4 & -8 & 1 & -3 & 1 \end{pmatrix}$$

Find the num of parameters in the general soln (nullity) of $Ax = 0$ if A is a 5×7 matrix of rank 3.

11.iv) Find rank and nullity of

$$x+2y-z+4t=0$$

$$2x-y+3z+3t=0$$

$$4x+y+3z+9t=0$$

$$y-z+t=0$$

$$2x+3y-z+7t=0$$

14.11

Solving linear system of eqn using ~~linear~~ inverse method

$$x + y + z = 6$$

$$3x + 3y + 4z = 20$$

$$2x + y + 3z = 13$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 3 & 3 & 4 \\ 2 & 1 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 20 \\ 13 \end{pmatrix}$$

$$A\,\bar{x} = \bar{b}$$

$$A^{-1} A\,\bar{x} = A^{-1}\,\bar{b}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 3 & 3 & 4 \\ 2 & 1 & 3 \end{pmatrix}^{-1} \begin{pmatrix} 6 \\ 20 \\ 13 \end{pmatrix}$$

$$= \begin{pmatrix} 5 & -2 & 1 \\ -1 & 1 & -1 \\ -3 & 1 & 0 \end{pmatrix} \begin{pmatrix} 6 \\ 20 \\ 13 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$$

ODE

$\frac{dy}{dx}$

PDE

$\frac{\partial y}{\partial x}$

independent variable

more than one

PDE: A diff^{t} eq^{n} involving derivatives of dependent variable with respect to (w r to) more than one independent variable is called partial differential eq^{n}

$$\frac{\partial v}{\partial s} + \frac{\partial v}{\partial t} = 5$$

Degree: The degree of PDE is the degree of the highest order derivative occuring in it after the diff^{t} eq^{n} has been made free from radicals and fractions.

Linear diff^{l} eq^{n}: Dependent variable and its differential co-efficients occur only in the first degree and are not multiplied together.

1st order, 2nd degree linear PDE: no such eq^{n} possible

$$\left(\frac{\partial^2 u}{\partial x}\right)^2 + \frac{\partial u}{\partial y} = 4$$

Non linear: D.V X D.V
D.D.V X D.V
DD.V X D.D.V

PDE:

$$\frac{\partial^2 y}{\partial x^2} + \frac{\partial^2 y}{\partial z^2} - \frac{\partial y}{\partial u} + y = (\ln x)^3 \quad \text{—— L}$$

$$\frac{\partial^2 y}{\partial x^2} + \frac{\partial y}{\partial t} + y^2 = \ln t \longrightarrow \text{NL}$$

$$y\frac{\partial y}{\partial x} + \frac{\partial y}{\partial t} = \sin t \longrightarrow \text{NL}$$

Homogeneous / non

Sol^n of PDE : relation between dep & indep

$z = cx + dt$ ⟶ general sol^n

$c, d \in \mathbb{R}$

$c = 2$
$d = 3$

$z = 2x + 3t$ ⟶ particular sol^n

$z = f(x, y)$

$p = \frac{\partial z}{\partial x}$, $q = \frac{\partial z}{\partial y}$, $r = \frac{\partial^2 z}{\partial x^2}$, $s = \frac{\partial^2 z}{\partial x \partial y}$, $t = \frac{\partial^2 z}{\partial y^2}$

$z(x_1, \cdots, x_n)$

$p_1 = \frac{\partial z}{\partial x_1}, \cdots, p_n = \frac{\partial z}{\partial x_n}$

$u(x, y)$

$u_x = \frac{\partial u}{\partial x}$ $\quad u_{xx} = \frac{\partial^2 u}{\partial x^2}$

$u_y = \frac{\partial u}{\partial y}$ $\quad u_{xy} = \frac{\partial^2 u}{\partial x \partial y}$

$u_{yy} = \frac{\partial^2 u}{\partial y^2}$

$z = ax + by + a^2 + b^2$ ——— ①

$a, b \rightarrow$ constant

Differentiating ① partially w.r. to x,

$$\frac{\partial z}{\partial x} = a$$

Diff... ① " " "

$$\frac{\partial z}{\partial y} = b$$

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = z$$

H.W

$z = ax + (1-a)y + b$

1 ques from today's class

Find P.D.E, $z = axe^y + \frac{1}{2}a^2e^{2y} + b$ —— (i)

Differentiating (i) partially w.r.to, x, y, respectively,

$$\frac{\partial z}{\partial x} = ae^y$$

$$\frac{\partial z}{\partial y} = axe^y + a^2e^{2y}$$

$$= x\,ae^y + (ae^y)^2$$

$$\frac{\partial z}{\partial y} = x\frac{\partial z}{\partial x} + \left(\frac{\partial z}{\partial x}\right)^2$$

which is the required PDE.

Find the PDE by eliminating A and p from

$$z = Ae^{pt} \sin px \quad \text{—— (1)}$$

Find the differential eqn of all spheres of radius λ, having centre in the xy-plane

Soln: From co-ordinate geometry of three dimensions the eqn of any ~~space~~ sphere of radius λ, having centre $(a, b, 0)$ in the xy-plane is given by

$$(x-a)^2 + (y-b)^2 + (z-0)^2 = \lambda^2 \quad \text{---} \quad (1)$$

a, b are arbitrary const.

Differentiating (1) w.r. to x and y we get

$$2(x-a) + 2z\frac{\partial z}{\partial x} = 0$$

$$\Rightarrow (x-a) + z\frac{\partial z}{\partial x} = 0$$

$$\Rightarrow x-a = z\frac{\partial z}{\partial x}$$

and, $(y-b) + z\frac{\partial z}{\partial y} = 0$

$$y-b = -z\frac{\partial z}{\partial y}$$

From ①

$$z^2\left(\frac{\partial z}{\partial x}\right)^2 + z^2\left(\frac{\partial z}{\partial y}\right)^2 + z^2 = \lambda^2$$

$$\Rightarrow z^2\left\{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1\right\} = \lambda^2$$

Find a partial D.E. by eliminating a, b, c from

$$u = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \quad \text{---} \quad (1)$$

Differentiating (1) w.r. to x, y ~~and z~~,

we get,

$$\frac{\partial u}{\partial x} \Rightarrow \frac{2x}{a^2} = 0$$

$$\frac{\partial u}{\partial y} \Rightarrow \frac{2y}{b^2} = 0$$

$$\frac{\partial u}{\partial z} =$$

$$\frac{2x}{a^2} + \frac{2z}{c^2}\frac{\partial z}{\partial x} = 0$$

$$\Rightarrow c^2 x + a^2 z \frac{\partial z}{\partial x} = 0 \quad \text{---} \quad (ii)$$

and, $c^2 y + z b^2 \frac{\partial z}{\partial y} = 0 \quad \text{---} \quad (iii)$

① $c^2 + a^2 z \frac{\partial^2 z}{\partial x^2} + a^2 \left(\frac{\partial z}{\partial x}\right)^2 = 0$

From ①

$$c^2 = -a^2 \frac{z}{x} \frac{\partial z}{\partial x}$$

$$-a^2 \frac{z}{x} \frac{\partial z}{\partial x} + a^2 z \frac{\partial^2 z}{\partial x^2} + a^2 \left(\frac{\partial z}{\partial x}\right)^2 = 0$$

$$\left(\frac{\partial z}{\partial x}\right)^2 + z \frac{\partial^2 z}{\partial x^2} - \frac{z}{x} \frac{\partial z}{\partial x} = 0$$

Find the differential eqn of the set of all right circular cones whose axis coincide with z-axis.

Soln: The general eqn of the set of all right circular cones whose axes coincide with z axis having semi-vertical angle α and vector at (a, b, c) is given by

$$x^2 + y^2 = (z-c)^2 \tan^2\alpha \quad \text{---} (1)$$

Diff (1) w.r.to x and y partially,

$$2x = 2(z-c) \frac{\partial z}{\partial x} \tan^2\alpha$$

$$x = (z-c) \frac{\partial z}{\partial x} \tan^2\alpha \quad \text{---} (2)$$

$$y = (z-c) \frac{\partial z}{\partial y} \tan^2\alpha \quad \text{---} (3)$$

y (1) & x (iii) $\Rightarrow y(z-c) \frac{\partial z}{\partial x} \tan^2\alpha = x(z-c) \frac{\partial z}{\partial y} \tan^2\alpha$

$$\Rightarrow y \frac{\partial z}{\partial x} = x \frac{\partial z}{\partial y}$$

Derivation of partial diff eqn by the elimination of arbitrary func of ϕ from the eqn $(u,v)=0$.
where u and v are func of x, y, z.

Or, show that $\phi(u,v)=0$ is the general soln of

$Pp + Qq = R$.

we have $\phi(u,v)=0$ — (1)

$z \rightarrow$ depen, $x, y \rightarrow$ ind

$$\frac{\partial z}{\partial x} = p, \quad \frac{\partial z}{\partial y} = q, \quad \frac{\partial x}{\partial y} = 0, \quad \frac{\partial y}{\partial x} = 0$$

Diff partially (1) w.r respect to x

$$\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial x}\right) +$$

$$\frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial v}{\partial z}\frac{\partial z}{\partial x}\right) = 0$$

$$\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial x}+p\frac{\partial x}{\partial z}\right)=-\frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial x}+p\frac{\partial v}{\partial z}\right)\longrightarrow (II)$$

Diff partially (I) w.r.to y

$$\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial x}\frac{\partial x}{\partial y}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial y}+\frac{\partial u}{\partial z}\frac{\partial z}{\partial y}\right)+$$

$$\frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial x}\frac{\partial x}{\partial y}+\frac{\partial v}{\partial y}\frac{\partial y}{\partial y}+\frac{\partial v}{\partial z}\frac{\partial z}{\partial y}\right)=0$$

$$=\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial y}+q\frac{\partial u}{\partial z}\right)=-\frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial y}+q\frac{\partial v}{\partial z}\right)\Longrightarrow (III)$$

$$\frac{\partial \phi}{\partial u}\Big/\frac{\partial \phi}{\partial v}=-\frac{\frac{\partial v}{\partial u}+q\frac{\partial v}{\partial z}}{\frac{\partial u}{\partial y}+q\frac{\partial u}{z}}\quad \text{from (III)}$$

$$\frac{\partial \phi}{\partial u}\Big/\frac{\partial \phi}{\partial v}=\frac{\frac{\partial u}{\partial x}+p\frac{\partial v}{\partial z}}{\frac{\partial u}{\partial x}+p\frac{\partial u}{\partial z}}\quad \text{from (II)}$$

$$\frac{\frac{\partial v}{\partial x} + p\frac{\partial v}{\partial z}}{\frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z}} = \frac{\frac{\partial v}{\partial y} + q\frac{\partial v}{\partial z}}{\frac{\partial u}{\partial y} + q\frac{\partial u}{\partial z}}$$

$$\Rightarrow \frac{\partial v}{\partial x}\frac{\partial u}{\partial y} + q\frac{\partial v}{\partial x}\frac{\partial u}{\partial z} + p\frac{\partial v}{\partial z}\frac{\partial u}{\partial y} + pq\frac{\partial v}{\partial z}\frac{\partial u}{\partial z} =$$

$$\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} + p\frac{\partial u}{\partial z}\frac{\partial v}{\partial y} + q\frac{\partial u}{\partial x}\frac{\partial v}{\partial z} + pq\frac{\partial v}{\partial z}\frac{\partial u}{\partial z}$$

P Q

$$\Rightarrow p\left(\frac{\partial u}{\partial y}\frac{\partial v}{\partial z} - \frac{\partial u}{\partial z}\frac{\partial v}{\partial y}\right) + q\left(\frac{\partial u}{\partial z}\frac{\partial v}{\partial x} - \frac{\partial u}{\partial x}\frac{\partial v}{\partial z}\right) =$$

$$\frac{\frac{\partial v}{\partial y}\frac{\partial u}{\partial x} - \frac{\partial v}{\partial x}\frac{\partial u}{\partial y}}{R}$$

$$\Rightarrow pP + qQ = R$$

Thus, we obtain a linear partial differential eqn of first order and first degree in p and q.

Show that $u(x, y, z) = a$ and $v(x, y, z) = b$ are two independent soln of

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

where, P, Q, R are fun^{c}s of x, y, and z

Soln:

Using total diff

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy + \frac{\partial u}{\partial z} dz = 0 \quad \text{—— (I)}$$

$$dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy + \frac{\partial v}{\partial z} dz = 0 \quad \text{—— (II)}$$

By cross multiplication method

$$\frac{dx}{\frac{\partial u}{\partial y}\frac{\partial v}{\partial z} - \frac{\partial u}{\partial z}\frac{\partial v}{\partial y}} = \frac{dy}{\frac{\partial u}{\partial z}\frac{\partial v}{\partial x} - \frac{\partial u}{\partial x}\frac{\partial v}{\partial z}} = \frac{dz}{\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}}$$

$$\Rightarrow \frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

where, $P = \cdots$

$Q = \cdots$

$R = \cdots$

SB

02/11/23

#Form a PDF by eliminating the func of from

$z = x^n f(y/x)$

Soln:

$z = x^n f(y/x)$ — ①

Differentiating ① w.r.to x & y partially

$$\frac{dz}{dx} = nx^{n-1} f(y/x) + x^n f'(y/x)(-y/x^2)$$

$$= nx^{n-1} f(y/x) - x^{n-2} f'(y/x)\, y \quad \text{— (ii)}$$

$$\frac{dz}{dy} = x^n f'(y/x)\frac{1}{x} = x^{n-1} f'(y/x) \quad \text{— (iii)}$$

multiplying (ii) by x and (iii) by y, then adding.

$$x\frac{dz}{dx} + y\frac{dz}{dy} = nx^n f(y/x) - x^{n-1} f'(y/x)\, y + x^{n-1} f'(y/x)\, y$$

$$= n x^n f(y/x) \rightarrow z = nz$$

$x\frac{dz}{dx} + y\frac{dz}{dy} = nz$ which is the required PDE

Form PDE by eliminating arbitrary func f and g

from $z = f(x^2-y) + g(x^2+y)$ — ①

Soln:

Differentiating ① partially w.r.to x and y

$$\frac{\partial z}{\partial x} = f'(x^2-y)\,2x + g(x^2+y)\,2x = 2x\{f'(x^2-y)+g'(x^2+y)\} - ②$$

$$\frac{\partial z}{\partial y} = -f'(x^2-y) + g'(x^2+y)$$

$$\frac{\partial^2 z}{\partial x^2} = 2x f''(x^2-y)\,2x + f'(x^2-y)\cdot 2 + g''(x^2+y)\,2x\,2x + g'(x^2+y)\cdot 2$$

$$= 4x^2\{f''(x^2-y) + g''(x^2+y)\} + 2\{f'(x^2-y) + g'(x^2+y)\}$$

$$\frac{\partial^2 z}{\partial y^2} = f''(x^2-y) + g''(x^2+y)$$

$$\frac{1}{2x}\frac{\partial z}{\partial x} = f'(x^2-y) + g'(x^2+y) \quad \text{from ②}$$

$$\frac{\partial^2 z}{\partial x^2} = 4x^2\frac{\partial^2 z}{\partial y^2} + \frac{1}{x}\frac{\partial z}{\partial x}$$

$$x\frac{\partial^2 z}{\partial x^2} = 4x^3\frac{\partial^2 z}{\partial y^2} + \frac{\partial z}{\partial x} \longrightarrow \text{required PDE}$$

H.W

Form a PDE by eliminating f and g. from

$$z = y f(x) + x g(y) \quad \text{---} \quad (1)$$

Hints

$$\frac{\partial z}{\partial x}, \quad \frac{\partial z}{\partial y}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)$$

Form a PDE by eliminating arbitrary func ϕ from

$$\phi(x^2+y^2+z^2,\ z^2-2xy) = 0 \quad \text{---} \quad (1)$$

Soln:

$$\phi(u,v) = 0 \longrightarrow (1)$$

Let, $u = x^2+y^2+z^2, \quad v = z^2-2xy$

Diff (1) partially w.r.to x & y,

$$\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z} p\right) + \frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z} q\right) = 0$$

$$\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} q\right) + \frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial y} + \frac{\partial v}{\partial z} q\right) = 0$$

$\frac{\partial u}{\partial x} = 2x, \quad \frac{\partial u}{\partial z} = 2z, \quad \frac{\partial v}{\partial x} = -2y, \quad \frac{\partial v}{\partial z} = 2z, \quad \frac{\partial u}{\partial y} = 2y, \quad \frac{\partial v}{\partial y} = -2x$

$$\frac{\partial \phi}{\partial u}(2x + 2zp) = -\frac{\partial \phi}{\partial v}(-2y + 2zp)$$

$$\frac{\partial \phi}{\partial u} \Big/ \frac{\partial \phi}{\partial v} = -\frac{2y + 2zq}{2x + 2zp} = \frac{y - zq}{x + zp}$$

$$= -\frac{\partial \phi}{\partial v}(-2x + 2zq)$$

$$\frac{\partial \phi}{\partial u} \Big/ \frac{\partial \phi}{\partial v} = -\frac{-2x + 2zq}{2y + 2zq} = \frac{x - zq}{y + zq}$$

$$\frac{y - zq}{x + zp} = \frac{x - zp}{y + zq}$$

$$\Rightarrow y^2 - zyp + yzq - z^2pq = x^2 + xzp - xzq - z^2pq$$

$$\Rightarrow p(x + y)z = (x + y)zq + x^2 - y^2 = 0$$

$$\Rightarrow z\frac{\partial z}{\partial x}(x + y) - (x + y)z\frac{\partial z}{\partial y} + (x - y)(x + y) = 0$$

$$z\frac{\partial z}{\partial x} - z\frac{\partial z}{\partial y} = y - x \Rightarrow \text{required PDE}$$

CT-1/ Syllabus: until linear combination (excl. L.C)

SB 02/11/2023

The eq^n of the type $Pp + Qq = R$

where P, Q, R are the func of x, y, z

$$p = \frac{\partial z}{\partial x}, \quad q = \frac{\partial z}{\partial y}$$

Lagrange's Linear PDE : $Pp + Qq = R$

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \rightarrow \text{auxiliary / subsidary eq}^n$$

$u = a, \quad v = b$

u, v func of x, y, z, $\phi(u, v) = 0$

Several methods to solve such eq^n

Grouping method

$$\frac{dx}{P} = \frac{dy}{Q} \qquad \frac{dy}{Q} = \frac{dz}{R}$$

$u = a$ $\qquad$ $v = b$

$$\phi(u, v) = 0$$

---

find the general sol^n of $\overset{P}{yz}p + \overset{Q}{zx}q = \overset{R}{xy}$

Sol^n: Subsidiary eq^n are

$$\frac{dx}{yz} = \frac{dy}{zx} = \frac{dz}{xy}$$

Grouping method:

$$\frac{dx}{y^2} = \frac{dy}{zx} \qquad \frac{dy}{zx} = \frac{dz}{xy}$$

$$\frac{dx}{y} = \frac{dy}{x} \Rightarrow x\,dx = y\,dy$$

$$\Rightarrow \frac{x^2}{2} = \frac{y^2}{2} + a'$$

$$\Rightarrow x^2 - y^2 = 2a' = a$$

$$\Rightarrow x^2 - y^2 = a$$

$$\frac{dy}{zy} = \frac{dz}{y} \Rightarrow y\,dy = z\,dz$$

$$\Rightarrow \frac{y^2}{2} = \frac{z^2}{2} + b'$$

$$\Rightarrow y^2 - z^2 = 2b' = b$$

Let $v = b \Rightarrow y^2 - z^2 = b$

$\Rightarrow \phi(x^2 - y^2, y^2 - z^2) = 0$

required general soln

Method of multipliers

Choose multipliers l, m, n such that

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{l\,dx + m\,dy + n\,dz}{pl + Qm + Rn}$$

$$\frac{dz}{R} = \frac{l\,dx + m\,dy + n\,dz}{0}$$

$\Rightarrow$ $R(l\,dx + m\,dy + n\,dz) = 0$ as $R \neq 0$

so, $l\,dx + m\,dy + n\,dz = 0$

multipliers problem

Find the general integral of $(mz-ny)p+(nx-lz)$

Soln: Subsidiary eqns are:

$$\frac{dx}{mz-ny} = \frac{dy}{nx-lz} = \frac{dz}{ly-mx}$$

Choosing x, y, z as multipliers

$$\frac{dx}{mz-ny} = \frac{dy}{nx-lz} = \frac{dz}{ly-mx} = \frac{xdx+ydy+zdz}{mxz-nxy+nyx-lyz+lzy-mz}$$

$$\frac{dz}{ly-mx} = \frac{xdx+ydy+zdx}{0}$$

$$(ly-mx)(xdx+ydz+zdx) = 0$$

$$\neq 0$$

$$xdx+ydy+zdz = 0$$

$$\Rightarrow \frac{x^2}{2}+\frac{y^2}{2}+\frac{z^2}{2} = c'$$

$$\Rightarrow x^2+y^2+z^2 = a \rightarrow u$$

2nd part: Choosing multipliers l, m, n

$$\frac{dx}{mz-ny} = \frac{dy}{nx-lz} = \frac{dz}{ly-mx} = \frac{ldx+mdy+ndz}{lmz-lny+mnx-mlz+nly-mnx}$$

$$ldx+mdy+ndz = 0$$

Integrating, $lx+my+nz = b \longrightarrow v$

$$\phi(x^2+y^2+z^2,\ lx+my+nz) = 0$$

Find the general integral of $(y^2+z^2-x^2)p - 2xyq + 2xz = 0$

Soln:

Sub... $\frac{dx}{y^2+z^2-x^2} = \frac{dy}{-2xy} = \frac{dz}{-2xz}$

1st part: $\frac{dy}{-2xy} = \frac{dz}{-2xz}$

$\frac{dy}{y} = \frac{dz}{z} \Rightarrow \ln y = \ln z + \ln a$

$\Rightarrow \ln y - \ln z = \ln a \Rightarrow \ln \frac{y}{z} = \ln a$

$\Rightarrow \frac{y}{z} = a$*

2nd part: Choosing x, y, z as multipliers

$$\frac{dx}{y^2+z^2-x^2} = \frac{dy}{-2xy} = \frac{dz}{-2xz} = \frac{x\,dx + y\,dy + z\,dz}{xy^2+xz^2-x^3-2xy^2-2xz^2}$$

$$\frac{dx}{-x^3-xy^2+xz^2} = \frac{dy}{-2yz} = \frac{dz}{-2z^2} = \frac{x\,dx+y\,dy+z\,dz}{-x^3-xy^2+xz^2}$$

$$\therefore \frac{dz}{-2z} = \frac{x\,dx+y\,dy+z\,dz}{(x^2+y^2+z^2)}$$

$$\frac{dz}{z} = \frac{2x\,dx+2y\,dy+2z\,dz}{x^2+y^2+z^2}$$

$$\ln z = \ln|x^2+y^2+z^2| + \ln b$$

$$\ln(x^2+y^2+z^2) - \ln z = -\ln b$$

$$\ln \frac{x^2+y^2+z^2}{z} = \ln \frac{1}{b}$$

$$\frac{x^2+y^2+z^2}{z} = c$$

$$\phi\left(y/z, \frac{x^2+y^2+z^2}{z}\right) = 0$$

Rough / ~~MT~~ CT - 2

Invert matrix unique:

Let, B_1 & B_2 are invert of matrix A.

Hence, $AB_1 = B_1A = I$ and

$AB_2 = B_2A = I$

Now, $B_1 = B_1I = B_1(B_2A) = B_2(B_1A)$

$= B_1(AB_2) = (B_1A)B_2 = IB_2 = B_2$

Hence, $B_1 = B_2$

Hermitian matrix: $A^* = A = \bar{A}^T$

Skew Hermitian matrix: $A^* = -A = \bar{A}^T$

Orthogonal matrix: $A^T = A^{-1}$ or, $AA^T = I = A^TA$

Row Echelon Form / Row Reduced...

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 6 & 0 & 1 & -2 \\ 8 & 1 & -2 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 6 & 1 & -2 \\ 0 & 9 & -2 & 1 \end{bmatrix} \quad \begin{aligned} R_2' &= R_2 - R_1 \\ R_3' &= R_3 - 6R_1 \\ R_4' &= R_4 - 8R_1 \end{aligned}$$

$$\sim \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & -2 & 1 \end{bmatrix} \quad \begin{aligned} R_3' &= R_3 - 2R_2 \\ R_4' &= R_4 - 3R_2 \end{aligned}$$

$$\sim \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & -3 \end{bmatrix} \quad R_4' = R_4 + 2R_3$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_1' = (3R_1 + R_2)\frac{1}{3}$

$R_2' = \frac{1}{3} R_2$

$R_3' = (3R_3 + 2R_2)\frac{1}{3}$

$R_4' = 2R_2 + 3R_3 - \frac{1}{3}$

System of eqns

Soln set of $x_1 - 2x_2 + x_3 = 5$

Solve for x_1

$$x_1 = 2x_2 - x_3 + 5$$

$$\left.\begin{aligned} x_2 &= t \\ x_3 &= s \\ x_1 &= 2t - s + 5 \end{aligned}\right\} \quad \begin{aligned} t &\in \mathbb{R} \\ s &\in \mathbb{R} \end{aligned}$$

$$x + 2y - z = 2$$
$$y - z = 1$$

$$y = 1 + z$$
$$z = t$$
$$y = 1 + t$$

$$x = -2y + z + 2$$
$$= -2(1+t) + t + 2$$
$$= -t$$

General soln: $\{x = -t;\ y = 1 + t,\ z = t;\ t \in \mathbb{R}\}$

SB 07/12/23

Homogeneous linear P.D.E with constant co-efficient:

$$\frac{\partial^n z}{\partial x^n} + k_1 \frac{\partial^n z}{\partial x^{n-1} \partial y} + k_2 \frac{\partial^n z}{\partial x^{n-2} \partial y^2} + \cdots + k_n \frac{\partial^n z}{\partial y^n} = F(x, y)$$

Complete soln = Complementary func + particular integral

To find complementary func

Symbolic form, put $\frac{\partial}{\partial x} = D$, $\frac{\partial}{\partial y} = D'$; $f(D, D') = F(x, y)$

Auxilary eqn, put $D = m$, $D' = 1$ put $f(m, 1) = 0$

Solve the auxilary eqn and find value of m

1. If roots are distinct $m_1, m_2, \cdots$

C.F. $= f_1(y + m_1 x) + f_2(y + m_2 x) + \cdots$

2. If roots are $m_1, m_1, m_2, \cdots$ two are repeated

C.F. $= f_1(y + m_1 x) + x f_2(y + m_1 x) + f_3(y + m_3 x) + \cdots$

3. Three are repeated,

C.F $= f_1(y + m_1 x) + x f_2(y + m_1 x) + x^2 f_3(y + m_1 x) + \cdots$

$$\frac{\partial^3 z}{\partial x^3} + 2\frac{\partial^3 z}{\partial x^2 \partial y} + 4\frac{\partial^3 z}{\partial y^3} = \sin(x+1)$$

$$D^3 + 3D^2 D$$

Rough/ midterm

Linear combination

vector space, $V \rightarrow \{\bar{v}_1, \bar{v}_2, \cdots, \bar{v}_n\}$

$\updownarrow \quad \updownarrow$ linearly combined

scalers, $F \rightarrow \{\alpha_1, \alpha_2, \cdots, \alpha_n\}$

$\bar{v} \in V(F)$ such that $\bar{v} = \alpha_1 \bar{v}_1 + \alpha_2 \bar{v}_2 + \cdots + \alpha_n \bar{v}_n$

Sol^n: $\bar{v}$ is a linear combination of $\bar{v}_1, \bar{v}_2, \bar{v}_3$ given there exists $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ such that,

$$\bar{v} = \alpha_1 \bar{v}_1 + \alpha_2 \bar{v}_2 + \alpha_3 \bar{v}_3$$

$$\Rightarrow (1, -2, 5) = \alpha_1 (1, 1, 1) + \alpha_2 (1, 2, 3) + \alpha_3 (2, -1, 1)$$

$$= (\alpha_1, \alpha_1, \alpha_1) + (\alpha_2, 2\alpha_2, 3\alpha_2) + (2\alpha_3, -\alpha_3, \alpha_3)$$

$$\Rightarrow (1, -2, 5) = (\alpha_1 + \alpha_2 + 2\alpha_3, \alpha_1 + 2\alpha_2 - \alpha_3, \alpha_1 + 3\alpha_2 + \alpha_3)$$

Now,

$$x_1 + x_2 + 2x_3 = 1$$

$$x_1 + 2x_2 - x_3 = -2$$

$$x_1 + 3x_2 + x_3 = 5$$

$$x_1 + x_2 + 2x_3 = 1$$

$$x_2 - 3x_3 = -3 \qquad R_2' = R_2 - R_1$$

$$x_2 + 2x_3 = 7 \qquad R_3' = R_3 - R_2$$

$$x_1 + x_2 + 2x_3 = 1$$

$$x_2 - 3x_3 = -3$$

$$5x_3 = 10$$

Using back substitution method from the above system,

$$x_3 = 2; \quad x_2 = 3, \quad x_1 = -6$$

Thus, $\bar{v}$ is a linear combination of comp. $\bar{v}_1, \bar{v}_2, \bar{v}_3$

Linear Dependence,

$$\alpha_1 \bar{v}_1 + \alpha_2 \bar{v}_2 + \dots + \alpha_n \bar{v}_n = 0$$

when, $\alpha_1 = \alpha_2 = \alpha_n = 0 \longrightarrow$ linearly independent

any of $\alpha_1, \alpha_2, \alpha_n \neq 0 \rightarrow$ " dependent

---

Basis: i) $\{\bar{v}_1, \bar{v}_2, \dots, \bar{v}_n\}$ is linearly independent

ii) " span $\bar{v}$

Eigen value & Eigen vector,

Let, A be a square matrix of order n

x " " non-zero vector $\in \mathbb{R}^n$

λ " " scaler

Then, $A\bar{x} = \lambda \bar{x}$ ⟶ eigen value

↳ eigen vector

$A\bar{x} = \lambda\bar{x}$

$\Rightarrow A\bar{x} - \lambda\bar{x} = 0$

$\Rightarrow (A - \lambda I)\bar{x} = 0$; $A - \lambda I \rightarrow$ char matrix

$0 = |A - \lambda I| \rightarrow$ polynomial eq^n

Sol^n: Using Caley-Hamilton th^m

$|A - \lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & 2 & 0 \\ 2 & -1-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0$$

$\lambda^3 - \lambda^2 + (-5\lambda) + 5 = 0$

$A^3 - A^2 - 5A + 5I = 0$

Now, multiply by A^{-1}

$A^2 - A - 5 + 5A^{-1} = 0$

$\Rightarrow A^{-1} = \frac{1}{5}(-A^2 + A + 5)$

$= \cdots$

multiply by A,

$A^4 = \cdots$

Soln. Using Caley - Hamilton thm,

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 3-\lambda & 2 & 4 \\ 2 & -\lambda & 2 \\ 4 & 2 & 3-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - 6\lambda^2 - 15\lambda - 8 = 0$$

$\lambda = 8, -1, -1$ [these are eigen values]

—

$$(A - \lambda I)\bar{x} = 0$$

$$\begin{pmatrix} 3-\lambda & 2 & 4 \\ 2 & -\lambda & 2 \\ 4 & 2 & 3-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

For $\lambda = -1$,

$$\begin{bmatrix} 4 & 2 & 4 \\ 2 & 1 & 2 \\ 4 & 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \quad \Bigg\} \quad \begin{bmatrix} 4 & 2 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

or, $2x_1 + x_2 + 2x_3 = 0$

or, $x_1 = \frac{1}{2}(-x_2 - 2x_3)$

$$\left.\begin{aligned} x_2 &= 2t \\ x_3 &= s \\ x_1 &= -t - s \end{aligned}\right\} \; t, s \in \mathbb{R}$$

$$\bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -t-s \\ 2t \\ s \end{bmatrix} = \begin{bmatrix} -t \\ 2t \\ 0 \end{bmatrix} + \begin{bmatrix} -s \\ 0 \\ s \end{bmatrix}$$

$$= t\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + s\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Now, for $\lambda = 8$,

$$\begin{bmatrix} -5 & 2 & 4 \\ 2 & -8 & 2 \\ 4 & 2 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{bmatrix} -5 & 2 & 4 \\ 0 & -18 & 9 \\ 0 & 18 & -9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

Rank + nullity = num. of cols

Rank(A): The num of leading variables in the gen soln of $Ax = 0$

nullity(A): The num of params in the gen soln of $Ax = 0$

---

rank = min(non-zero rows, non-zero cols)

nullity = var count - eqn count

or, var count - rank

PDE

$z = Ae^{pt} \sin px$ — (I)

Diff wr to x, t res...,

$\frac{\partial z}{\partial x} = pAe^{pt} \cos px$ — (II)

$\frac{\partial z}{\partial t} = Ape^{pt} \sin px$

$= pAe^{pt} \sin px$ — (III)

~~(III) ÷ (II) $\Rightarrow \frac{\partial z}{\partial t} \frac{\partial t}{\partial x} = \tan px$~~

(III) $\Rightarrow \frac{\partial z}{\partial t} = pz$

(II) $\Rightarrow \frac{\partial^2 z}{\partial x^2} = -p^2 Ae^{pt} \sin px$

$= -p \frac{\partial z}{\partial t}$

$\Rightarrow \frac{\partial^2 z}{\partial x^2} = -\frac{\partial z}{\partial t} \frac{1}{z} \frac{\partial z}{\partial t}$

$\Rightarrow z \frac{\partial^2 z}{\partial x^2} = -\left(\frac{\partial z}{\partial t}\right)^2$

$$z = axe^{y} + \frac{1}{2}a^2e^{2y} + b \quad \text{— (1)}$$

$$\frac{\partial z}{\partial x} = ae^{y}$$

$$\frac{\partial z}{\partial y} = axe^{y} + 2\frac{1}{2}a^2e^{2y}$$

$$= axe^{y} + a^2e^{2y}$$

$$\frac{\partial z}{\partial y} = x\frac{\partial z}{\partial x} + \left(\frac{\partial z}{\partial x}\right)^2$$

PDE of spheres of rad λ cent. in xy plane

soln: cent. $(a, b, 0)$

$$(x-a)^2 + (y-b)^2 + (z-0)^2 = \lambda^2 \quad \text{— (1)}$$

Diff. (1) wr to x, y,

$$2(x-a) + 2z\frac{\partial z}{\partial x} = 0$$

$$\Rightarrow x-a = -z\frac{\partial z}{\partial x}$$

Again,

$$2(y-b) + 2z\frac{\partial z}{\partial y} = 0$$

$$\Rightarrow y-b = -z\frac{\partial z}{\partial y}$$

$$(1) \Rightarrow z^2\left[\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1\right] = \lambda^2$$

$$u = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \quad \text{— (I)}$$

Diff. (I) by x, y,

$$\frac{2x}{a^2} + \frac{2z}{c^2}\frac{\partial z}{\partial x} = 0$$

$$\Rightarrow c^2 x + a^2 z \frac{\partial z}{\partial x} = 0 \Rightarrow c^2 = -a^2 \frac{z}{x}\frac{\partial z}{\partial x}$$

$$\Rightarrow c^2 + a^2 z \frac{\partial^2 z}{\partial x^2} + a^2 \left(\frac{\partial z}{\partial x}\right)^2 = 0 \quad \text{— (II)}$$

And,

$$\frac{2y}{b^2} + \frac{2z}{c^2}\frac{\partial z}{\partial y} = 0$$

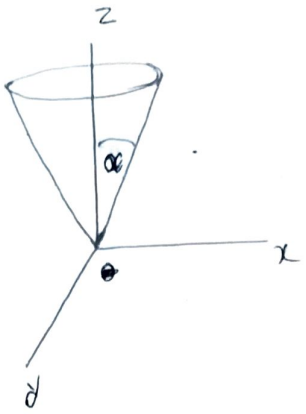
$$\Rightarrow c^2 y + b^2 z \frac{\partial z}{\partial y} = 0$$

$$\text{(II)} \Rightarrow -a^2 z \frac{\partial^2 z}{\partial x^2} + a^2 z \frac{\partial^2 z}{\partial x^2} + a^2 \left(\frac{\partial z}{\partial x}\right)^2 = 0$$

$$\Rightarrow z\frac{\partial^2 z}{\partial x^2} + \left(\frac{\partial z}{\partial x}\right)^2 - z\frac{\partial z}{\partial x} = 0$$

Soln: Gen eqn;

$$x^2+y^2 = (z-c)^2 \tan^2\alpha \quad -(I)$$



Diff (I) partially w.r.to x, y,

$$2x = 2(z-c)\tan^2\alpha \frac{\partial z}{\partial x}$$

$$\Rightarrow x = (z-c)\frac{\partial z}{\partial x}\tan^2\alpha \quad -(II)$$

$$y = (z-c)\frac{\partial z}{\partial y}\tan^2\alpha \quad -(III)$$

$y \times (II)$ & $x \times (III) \rightarrow y(z-c)\frac{\partial z}{\partial x}\tan^2\alpha = x(z-c)\frac{\partial z}{\partial y}\tan^2\alpha$

$$\therefore x\frac{\partial z}{\partial y} - y\frac{\partial z}{\partial x} = 0$$

SB

4/12/23

Rules for finding particular integral

$f(D, D')z = F(x, y)$

$D = \frac{\partial}{\partial x}$, $D' = \frac{\partial}{\partial y}$

$z = \frac{1}{f(D, D')} F(x, y)$

Types of $F(x, y)$	working rule
Zero	0
e^{ax+by}	Put $D = a$ and $D' = b$
$\sin(ax+by)$ / $\cos(ax+by)$	Put $D^2 = -a^2$, $D'^2 = -b^2$, $DD' = -ab$

$$\frac{1}{f(D^2, DD', D'^2)} \sin(ax+by)$$

$x^m y^n$ $\quad f(D,D')^{-1} F(x,y)$

Binomial expansion

$e^{ax+by} v(x,y) : e^{ax+by} \frac{1}{f(D+a, D'+b)} v(x,y)$

Put $D : D+a$

$D' = D'+b$

otherwise $\frac{1}{(Da-mD')} F(x,y)$

Factor

Solve the PDE:

$$\frac{\partial^3 z}{\partial x^3} - 3\frac{\partial^3 z}{\partial x^2 \partial y} + 4\frac{\partial^3 z}{\partial y^3} = e^{x+2y}$$

$$(D^3 - 3D^2D' + 4D'^2)z = e^{x+2y}$$

$$f(D, D') = D^3 - 3D^2D' + 4D'^3 \qquad D = m, \; D' = 1$$

$$f(m, 1) = m^3 - 3m^2 + 4 = 0$$

$$m^3 + m^2 - 4m^2 - 4m + 4m + 4 = 0$$

$$m^2(m+1) - 4m(m+1) + 4(m+1) = 0$$

$$(m+1)(m^2 - 2 \cdot 2m + 2^2)(m-2)^2 = 0$$

$$m = -1, 2, 2$$

$$C.F = f_1(y + 2x) + xf_2(y + 2x) + f_3(y - x)$$

Particular integral

$$z = \frac{1}{f(D,D')} F(x,y)$$

$$z = \frac{1}{D^3 - 3D^2D' + 4D'^3} e^{x+2y} \quad \because a=1,\ b=2$$

$$z = \frac{1}{1^3 - 3 \cdot 1^2 \cdot 2 + 4 \cdot 2^3} (e^{x+2y})$$

$$= \frac{1}{27} e^{x+2y}$$

Complete soln : $f_1() + xf_2() + f_3 + \frac{1}{27} e^{x+2y}$

Solve the PDE

$$\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial x \partial y} = \sin x \cos 2y$$

$$(D^2 - DD')z = \sin x \cos 2y = \frac{1}{2}[2\sin x \cos 2y] = \frac{1}{2}[\sin(x+2y) + \sin(x-2y)]$$

$$f(D, D') = D^2 - DD' \quad ; \quad D = m, \; D' = 1$$

$$f(m, 1) = m^2 - m = 0$$

$$\Rightarrow m(m-1) = 0$$

$$m = 0, 1$$

$$CF = f_1(y+x) + f_2(y)$$

$$z = \frac{1}{D^2 - DD'} \{\sin(x+2y) + \sin(x-2y)\} \frac{1}{2}$$

$$= \frac{1}{D^2 - DD'} \frac{\sin(x+2y)}{2} + \frac{1}{D^2 - DD'} \frac{\sin(x-2y)}{2}$$

$$= \frac{1}{-1^2 - (-1 \cdot 2)} \left\{ \frac{\sin(x+2y)}{2} \right\} + \frac{1}{-1^2 - (-1 \cdot -2)} \frac{\sin(x-2y)}{2}$$

$$= \frac{\sin(x+2y)}{2} - \frac{1}{6} \sin(x-2y)$$

HW

$$\frac{\partial^2 z}{\partial x^2} - 4\frac{\partial^2 z}{\partial x \partial y} + 4\frac{\partial^2 z}{\partial y^2} = e^{2x+y}$$

$$\frac{\partial^2 z}{\partial x^2} + 3\frac{\partial^2 z}{\partial x \partial y} + 2\frac{\partial^2 z}{\partial y^2} = 12xy$$

$$\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial x \partial y} - 2\frac{\partial^2 z}{\partial y^2} = (y-1)e^x$$

SB

21/12/23

Solve the PDE

$$(D^2+2DD'+D'^2)z = x^2+xy+y^2$$

$$m^2+2m+1=0 \;;\quad m=-1,-1$$

$$C.F. = f_1(y-x)+xf_2(y-x)$$

P.I:

$$z = \frac{1}{D^2+2DD'+D'^2}(x^2+xy+y^2)$$

$$= \frac{1}{(D+D')^2}(x^2+xy+y^2)$$

$$= \frac{1}{D^2\left(1+\frac{D'}{D}\right)^2}(x^2+xy+y^2)$$

$$= \frac{1}{D^2}\left(1+\frac{D'}{D}\right)^{-2}(x^2+xy+y^2)$$

$$= \frac{1}{D^2}\left\{1+(-2)\frac{D'}{D}+(-2)(-3)\frac{\left(\frac{D'}{D}\right)^2}{2!}\right\}(x^2+xy+y^2)$$

$$= \left(\frac{1}{D^2}-2\frac{D'}{D^3}+\frac{6}{2}\frac{D'^2}{D^4}\right)(x^2+xy+y^2)$$

$$= \frac{1}{D^2}(x^2+xy+y^2)-2\frac{D'}{D^3}(x^2+xy+y^2)+3\frac{D'^2}{D^4}(x^2+xy+y^2)$$

w.r.t to x → D; w.r.t to y → D' → differentiation

$\frac{1}{D}$, $\frac{1}{D'}$ → integrate (w.r.t to x; w.r.t to y)

$$= \frac{1}{D}\left(\frac{x^3}{3} + \frac{x^2}{2}y + y^2x\right) - 2\frac{1}{D^3}(x+2y) + 3\frac{D'}{D^4}(y x + 2y)$$

$$= \frac{x^4}{12} + \frac{x^3}{6} + \frac{y^2x^2}{2} - 2\frac{1}{D^2}\left(\frac{x^2}{2} + 2xy\right) + 3\frac{1}{D^4}(2)$$

$$= \frac{x^4}{12} + \frac{x^3}{6} + \frac{x^2y^2}{2} - 2\frac{1}{D}\left(\frac{x^3}{6} + x^2y\right) + 3\frac{1}{D^3}(2x)$$

$$= \frac{x^4}{12} + \frac{x^3y}{6} + \frac{x^2y^2}{2} - 2\left(\frac{x^4}{24} + \frac{yx^3}{3}\right) + 3\frac{1}{D^2}(x^2)$$

$$= \frac{x^3y}{6} - \frac{2x^3y}{3} + \frac{x^2y^2}{2} + 3\frac{x^4}{12}$$

$$= \frac{3x^3y}{6} + \frac{x^2y^2}{2} + \frac{x^4}{4}$$

$$= \frac{x^4}{4} + \frac{x^2y^2}{2} - \frac{x^3y}{2}$$

Solve the PDE

$$\frac{\partial z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} - 6\frac{\partial^2 z}{\partial y^2} = y \cos x$$

$$\Rightarrow (D^2 - D'^2) z = e^{x-y} \sin(x-y)$$

$e^{ax+by} V(x,y)$

$D = D + a$

$D' = D' + b$

$e^{x-y} \left(\frac{1}{D^2 - D'^2} V(x,y) \right)$

SB

28/12/23

Solve $(D^2 - D'^2)z = e^{x-y}\sin(x+2y)$ | $e^{ax+by} \cdot v(x,y)$

C.F. : HW

P.I : $z = \frac{1}{D^2 - D'^2} e^{x-y}\sin(x+2y)$

$= e^{x-y} \frac{1}{(D+1)^2 - (D'+1)^2} \sin(x+2y)$

$= C \frac{1}{D^2+2D-D'^2+2D'} \sin(x+2y)$

$D^2 = -a^2$ | $a = 1$
$D'^2 = -b^2$ | $b = 2$
$DD' = -ab$

$= C \frac{1 \times D}{D \times (2D+3+2D')} \sin(x+2y)$

$= C \frac{D}{2(-1)+3D+2(-12)} \sin(x+2y)$

$= C/3 \frac{D}{(D-2)} \sin(x+2y) \times (D+2) \frac{1}{D+2}$

$= \frac{C}{3} \frac{D^2+2D}{D^2-4} \sin(x+2y)$

$= \frac{C}{3} \frac{D^2+2D}{-5} \sin(x+2y)$

$= \frac{C}{15} D^2\{\sin(x+2y)\} + 2D\{\sin(x+2y)\}$

$$= + \frac{e^{x-y}}{15} \sin(x+2y) - \frac{2e^{x-y}}{15} \cos(x+2y)$$

wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \text{---(1)}$$



$l \rightarrow$ length

$u \rightarrow$ displacement

Boundary cond^n: $u(0, t) = u(l, t) = 0$

Initial cond^n: $u(x, 0) = f(x)$

$$\frac{\partial u}{\partial t} = 0, \quad \text{at } t = 0$$

Seperation of variable: $u(x, t) = X(x)\,T(t)$

Next page. - -

Soln: Let the complete soln of wave eqn is

$$u(x,t) = T(t)\,X(x) \quad \text{—— (2)}$$

$$\frac{\partial^2 u}{\partial x^2} = T(t)\,X''(x)$$

$$\frac{\partial^2 u}{\partial t^2} = T''(t)\,X(x)$$

$$\Rightarrow T''X = c^2 X'' T$$

$$\Rightarrow \frac{T''}{c^2 T} = \frac{X''}{X} = k$$

$$T'' - kc^2 T = 0 \quad \text{—(3)} \qquad X'' - kx = 0 \quad \text{— (4)}$$

Case - 1: when $k = 0$

$$T'' = 0$$

$$T' = c_1$$

$$T = c_1 t + c_2$$

$$X'' = 0$$

$$X' = c_3$$

$$X = c_3 x + c_4$$

Complete soln: $u(x,t) = (c_1 t + c_2)(c_3 x + c_4)$

Case-2: when k is positive. $k = p^2$ p is any num

From (3) and (4)

$$T'' - p^2c^2T = 0 \qquad X'' - p^2X = 0$$

$$(D^2 - p^2c^2)T = 0 \qquad (D'^2 - p^2)X = 0$$

$$D = \frac{d}{dt} \qquad D' = \frac{d}{dx}$$

$$m^2 - p^2c^2 = 0 \qquad m^2 - p^2 = 0$$

$$m = \pm pc \qquad m = \pm p$$

$$T(t) = c_1e^{pct} + c_2e^{-pct} \qquad X(x) = c_3e^{px} - c_4e^{-px}$$

Complete soln: $(c_1e^{pct} + c_2e^{-pct})(c_3e^{px} + c_4e^{-px})$

Case-3: when k is negative, $k = -p^2$

$T'' + p^2c^2T = 0$ $\qquad$ $X'' + p^2X = 0$

$m^2 + p^2c^2 = 0$ $\qquad$ $m^2 + p^2 = 0$

$m = \frac{-0 \pm \sqrt{0^2 - 4 \times 1 \times p^2c^2}}{2}$ $\qquad$ $m = \pm ip$

$= \frac{0 \pm i\sqrt{4p^2c^2}}{2}$

$= 0 \pm ipc$

$m_1 = 0 + ipc$

$m_2 = 0 - ipc$

$T(t) = e^0 (C_1 \cos pct + C_2 \sin pct)$ $\qquad$ $X(t) = C_3 \cos px + C_4 \sin px$

Complete soln: $(C_1 \cos pct + C_2 \sin pct)(C_3 \cos px + C_4 \sin px)$

wavy func $\longrightarrow$ sin/cos ; not algebric or expo

SB 04/01

$$A = \begin{bmatrix} 7 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{bmatrix} \longrightarrow \begin{aligned} \lambda_1 &= 7 \\ \lambda_2 &= 4 \\ \lambda_3 &= 3 \end{aligned}$$

$\det(A) = 0 \longrightarrow$ one of $\lambda = 0$

$$\begin{bmatrix} v_{xx} & v_{xy} & v_{xz} \\ v_{yx} & v_{yy} & v_{yz} \\ v_{zx} & v_{zy} & v_{zz} \end{bmatrix}$$

$\lambda = 0 \rightarrow$ parabola

$\lambda \rightarrow +, +, +$ / $-, -, -$ $\}$ → ecli...

$\lambda \rightarrow +, -, -$ / $-, +, +$ $\}$ - hy