Undamped Simple Harmonic Motion

In simple harmonic motion (SHM), an object oscillates back and forth around an equilibrium position with a constant amplitude and frequency. The restoring force acting on the object always points towards the equilibrium position and is proportional to the displacement from that position.

Some examples of simple harmonic motion include the motion of a mass attached to a spring, the motion of a simple pendulum, and the motion of a vibrating guitar string. Simple harmonic motion occurs in many physical systems, including mechanical, electrical, and acoustic systems.

Equilibrium of SHM

Equilibrium is a state of balance or stability in a physical system where there is no net change or motion. In other words, when a system is in equilibrium, it is in a state of rest or constant motion with no acceleration. Equilibrium is achieved when the net force acting on an object is zero. This can occur when the object is at rest or moving with a constant velocity. In the case of rotational equilibrium, the net torque acting on an object is zero.

In the context of a simple harmonic oscillator, equilibrium refers to the position where the oscillator is at rest and the net force acting on it is zero. This position is also known as the "equilibrium position" or "resting position" of the oscillator.

For example, in the case of a mass attached to a spring (vertical setting), the equilibrium position is the position where the spring is neither stretched nor compressed and the mass is at rest. At this position, the force exerted by the spring is balanced by the force of gravity acting on the mass, resulting in a net force of zero.

$$\sum \vec{F}_{Net} = 0$$

The equilibrium of a simple pendulum is the position where the pendulum hangs vertically downwards with no displacement from its original position. In this position, the gravitational force acting on the pendulum is balanced by the tension in the string or rod that supports it, resulting in a net force of zero. This position is also known as the "stable equilibrium position" of the pendulum.

Differential Equation of Simple Harmonic Motion

Consider a mass m attached to a spring with spring constant k, and assume that the motion is damped by a frictional force proportional to the velocity of the mass, with damping coefficient c. Let x(t) be the displacement of the mass from its equilibrium position at time t.

The force acting on the mass can be expressed as the sum of the spring force and the damping force, which are given by:

Spring force, $F_H = -kx$ and Damping force, $F_D = -bv$. Using Newton's second law ($F_N = ma$), at equilibrium (and no external force applied to the system) we can write the equation of motion as:

$$ma = -kx - bv$$

$$m\frac{d^2x}{dt^2} = -kx - b\frac{dx}{dt}$$

This is the differential equation of a damped harmonic oscillator. We can rearrange it into the standard form:

$$m\frac{d^2x(t)}{dt^2} + b\frac{dx(t)}{dt} + kx(t) = 0$$

Here, m is the mass of the body attached to the spring, b is the damping coefficient and k is the spring constant. This equation describes the motion of the simple harmonic oscillator and can be used to solve for the displacement $x(t)$ of the mass as a function of time. The behavior of the oscillator depends on the values of the mass m, damping coefficient b, and spring constant k.

We need ODE (Ordinary Differential Equation) tools to solve these differential equations, i.e. solve for $x(t)$, then find velocity and acceleration. Then, we can study the kinetic and potential energy of the system. If we can write the conservation (of energy) laws, we can solve the system at any condition, more on this later. We will also study the system where two or more SHO are coupled with each other, called coupled SHO.

Undamped Simple Harmonic Motion

Undamped Simple Harmonic Motion is a type of periodic motion that occurs when an object oscillates back and forth around an equilibrium position without any external damping forces. In other words, it is a motion of an object under the influence of a restoring force that is directly proportional to the displacement from the equilibrium position and there is no energy loss due to any external force.

Thus, when external damping force is zero ($F_D = 0$), the differential equation of the system is given by,

$$m\frac{d^2x(t)}{dt^2} + kx(t) = 0$$

Define $\omega^2 = \frac{k}{m}$. Rearranging the differential equation, we get:

$$\frac{d^2x(t)}{dt^2} + \omega^2 x(t) = 0$$

Here, ω is called the angular frequency of the oscillator. The motion of the object in Undamped SHM can be described by an oscillatory function like sine or cosine. The amplitude of the oscillation remains constant, and the object moves back and forth across the equilibrium position with a constant frequency (ω). The frequency of oscillation depends on the properties of the system, such as the mass and spring constant in a mass-spring system or the length and gravity in a simple pendulum.

Solving Undamped SHM Differential Equation

So to solve for the trajectory of the block, we just have to ask ourselves what kind of function $x(t)$, when we take its derivative twice in a row, gives us back the same function times this negative number $-\omega^2$. The answer is sine or cosine! Recall that the derivatives,

$$\frac{\mathrm{d}}{\mathrm{d}t}\sin(\omega t) = \omega\cos(\Omega t)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\cos(\omega t) = -\omega\sin(\Omega t).$$

Those factors of ω on the right come from the chain rule: the derivative of $\sin(\omega t)$ with respect to its argument is $\cos(\omega t)$, and then we multiply by the derivative of ωt with respect to t, which contributes the factor of ω. Now taking the second derivative, we get

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\sin(\omega t) = -\omega^2\sin(\omega t)$$

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\cos(\omega t) = -\omega^2\cos(\omega t),$$

which is exactly what we wanted: the second derivative is $-\omega^2$ times the function that we started with. Then the general solution to the differential equation is:

$$x(t) = A\cos\omega t + B\sin\omega t$$

where A and B are constants determined by the initial condition of the system. If we redefine $A = x_0\cos\phi$ and $B = x_0\sin\phi$, then the general solution will be:

$$x(t) = x_0\cos(\omega t + \phi)$$

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These are the same solutions written with different constants. This solution can be written with exponential function using the Euler's identity ($e^{i\theta} = \cos\theta + isin\,\theta$).

$$x(t) = \alpha\, e^{i\omega t} + \beta\, e^{-i\omega t}$$

Our considered DE is a 2nd order differential equation. Thus, we need two constraints to solve the system. This is the reason we are getting two unknown constants still not fixed in the general formulas. We need either two boundary conditions (BC) or two initial conditions (IC) or one BC and one IC to fully specify the system.

Some Definitions

Displacement ($x(t)$): The distance of an object from its equilibrium position at a given time during its oscillation.

Amplitude (A): The maximum displacement of an oscillating object from its equilibrium position.

Phase ($\omega t + \phi$): A measure of the position of an object in its oscillation cycle at a particular point in time. State of oscillation at a certain time.

Initial Phase (ϕ): The phase of an object at its starting position, or the phase at t=0.

Time Period (T): The time taken by an oscillating object to complete one full cycle of motion.

Frequency (ν): The number of oscillations completed by an object in one second.

Time Period and Frequency

Consider the solution of the undamped SHM:

$$x(t) = A\cos(\omega t + \phi)$$

Choose the initial condition: $x(t=0) = A$ and $v(t=0) = 0$, we get the displacement:

$$x(t) = A\cos\omega t$$

If T is the time period (time for one complete oscillation) then,

$$x(t+T) = x(t)$$

Substitute the solution and get the time period (T) and frequency (ν) expression:

$$A\cos[\omega(t+T)] = A\cos\omega t$$

$$\omega T = 2\pi$$

$$\frac{1}{\nu} = T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}$$

Note! The angular velocity of a SHO is given by,

$$\omega^2 = \frac{k}{m}$$

where recall k and m must be constant in order to satisfy the 3rd restriction of SHM. Try to understand what is happening! Two constants of the system are specifying an angular velocity to the system and with this velocity the system is oscillating! We are considering a linear SHO (for example mass-spring system). Where does this angular velocity come from? Think about the correspondence!

In short: System = linear or angular motion BUT Description = angular!!

Velocity and Acceleration of the oscillator

Definition of velocity and acceleration:

$$v(t) = \frac{dx(t)}{dt} \qquad a(t) = \frac{dv(t)}{dt}$$

Differentiating the general solution $x(t) = A\cos(\omega t + \phi)$ and using trigonometric identities, we find the expression for velocity and acceleration as:

$$v(t) = -A\omega \sin(\omega t + \phi) = \omega\,(A^2 - \omega^2)^{\frac{1}{2}}$$

$$a(t) = -A\omega^2 \cos(\omega t + \phi) = -\omega^2 x(t)$$

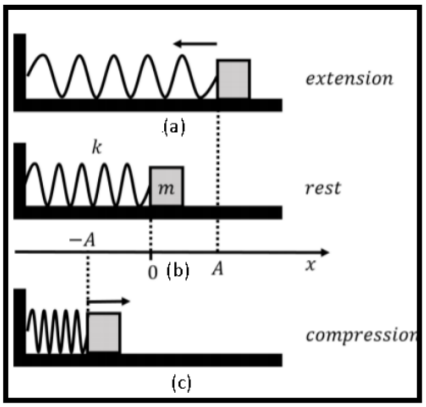
Maximum and Minimum of x, v, a in a SHM

Recall the expressions of x(t), v(t) and a(t). The trigonometric functions cosine and sine have the maximum value 1 and minimum value. So, the maximum and minimum of displacement, velocity, and acceleration of a SHM is given by,

$x_{max} = A$ $\quad v_{max} = -A\omega$ $\quad a_{max} = -A\omega^2$

$x_{min} = 0$ $\quad v_{min} = 0$ $\quad a_{min} = 0$

(a)	(b)	(c)
x_{max}	x_{min}	x_{max}
v_{min}	v_{max}	v_{min}
a_{max}	a_{min}	a_{max}



Mass-Spring system: (a) at max extension, (b) at equilibrium, (c) at max compression.

Energy: Kinetic, Potential and Total Energy

From Newtonian mechanics, we calculate the change in energy from the work done formula:

$$W = \int \vec{F}\,.\,d\vec{s} = \Delta E$$

Similarly, the impulse and change in momentum relation is given by the formula:

$$\vec{J} = \int \vec{F} dt = \Delta\vec{\mathrm{P}}$$

(**Homework**) Find the expression for kinetic and potential energy by integrating the forces. Kinetic and potential energy are derived from work done formula using the forces F_{Newton} and F_{Hook} expressions respectively. The expressions are given below:

$$T = \frac{1}{2}mv^2 \quad \& \quad V = \frac{1}{2}kx^2$$

Plug in the displacement and velocity expression of a SHO (and also use $\omega^2 = k/m$) to get the Kinetic and potential energy of a SHO in terms of amplitude, frequency, and phase:

$$T(t) = \frac{1}{2}m\omega^2 A^2 \sin^2(\omega t + \phi)$$

$$V(t) = \frac{1}{2}m\omega^2 A^2 \cos^2(\omega t + \phi)$$

Thus, the total energy $E = T(t) + V(t)$ is given by (use the trig. Identity $\sin^2\alpha + \cos^2\alpha = 1$),

$$E = \frac{1}{2}m\omega^2 A^2 = \frac{1}{2}kA^2 = \frac{1}{2}mv_{max}^2$$

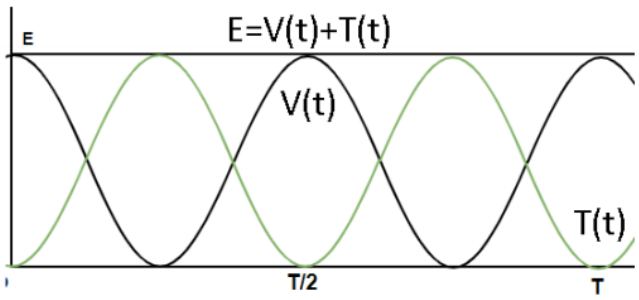
(**Homework**) Find the expression for average kinetic and potential energy. Hint: Average kinetic energy is defined as:

$$\langle T \rangle = \frac{\int_0^T T(t)\,dt}{\int_0^T dt}$$

Conservation of Energy

Conservation of energy: The total energy of an isolated system must remain constant over time. Consider two states of the system. $(x,v) = (x_1, v_1)$ at time $t = t_1$ and $(x, v) = (x_2, v_2)$ at time$t = t_2$. They are not independent but related by a constraint (energy conservation).

$$\frac{1}{2}mv_1^2 + \frac{1}{2}kx_1^2 = \frac{1}{2}mv_2^2 + \frac{1}{2}kx_2^2$$



Energy of a SHO

Proof

Total Energy of a harmonic oscillator is given by,

$$E = \frac{1}{2}mv^2(x) + \frac{1}{2}kx^2(x) = \frac{1}{2}kA^2$$

Clearly, the total energy is constant of a harmonic oscillator. Taking time derivative, we can show that,

$$\frac{dE}{dt} = 0$$

Thus, the total energy of an undamped SHM (with no external force applied) is constant.

Assignment: LC oscillatory Circuit.