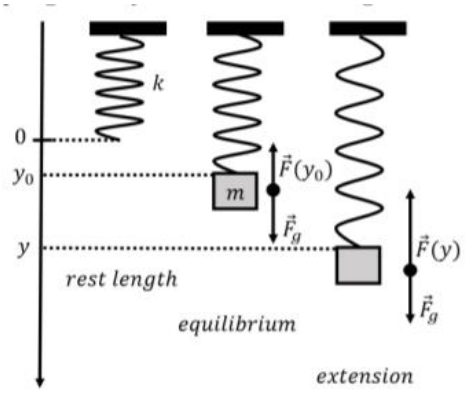
SHM with gravity: Pendulum

Influence of Gravity in Mass-Spring System

In mass-spring system, we can ignore gravity in both horizontal and vertical setting. The oscillation is independent of gravity. For horizontal setting, it is obvious. It is also true for vertical setup. Prove it by equilibrium condition.

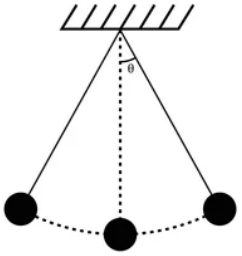


Simple Harmonic Oscillation is the motion of a particle or a body about equilibrium position. Addition of gravity is just going to change the equilibrium position. In the figure, without mass or gravity the equilibrium position is $y = 0$ but after adding the mass in the vertical setting, i.e. adding gravity, the equilibrium is shifted to $y = y_0$. Thus, in this system, the mass is going to oscillate about the equilibrium position y_0 instead of $y = 0$. Obviously, in the horizontal spring, this mass is going to oscillate about the position $y = 0$ (equilibrium).

In this section, we will study the pendulum system where instead of spring, we will use gravity to get the SHM.

Pendulum: Gravity instead of Spring

A body is suspended from a fixed point so that it can swing back and forth under the influence of gravity. This is also a SHM like mass-spring system. The main difference between these two systems is that in mass-spring system the spring was responsible for oscillation but here in pendulum, gravity is doing the job of the spring to generate the SHM.



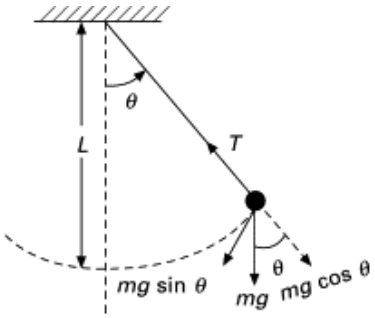
We can use any coordinate system to study this system but the Polar coordinate (r,θ) system is a 'good' choice as the mathematical structure is simpler. We can also use Cartesian coordinate system (x, y) as in mass-spring system. For simplicity, we choose the Polar coordinate system.

The equilibrium of a simple pendulum is the position where the pendulum hangs vertically downwards with no displacement from its original position. In this position, the gravitational force acting on the pendulum is balanced by the tension in the string or rod that supports it, resulting in a net force of zero. This position is also known as the "resting position" or "stable equilibrium position" of the pendulum.

When the pendulum is displaced from its equilibrium position, it experiences a restoring force that pulls it back towards the equilibrium position. This restoring force is proportional to the displacement and always acts towards the equilibrium position. As a result, the pendulum oscillates back and forth around its equilibrium position with a period that depends on the length of the string, the gravitational acceleration, and the amplitude of the oscillation.

Simple Pendulum

Definition: An ideal pendulum consisting of a point mass (called bob) suspended by an unstretchable massless thread or string and free to vibrate without friction is called Simple pendulum. In simple pendulum system, all the masses are concentrated in the bob (i.e. point mass, no mass in the thread).



Balance the forces in two independent directions, radial (r) and angular (θ), in the polar coordinate system:

$$F_r = mg\cos(\theta) - T = ma_r$$

$$F_\theta = mg\sin(\theta) = ma_\theta$$

where T is the tension force. As the motion, we want to study, is along the angular direction (θ), using the angular and linear acceleration relation ($\vec{a} = \vec{\alpha} \times \vec{r}$), the angular equation becomes,

$$mg\sin\theta = \mathrm{mL}\alpha$$

where L is the length of the massless thread or string.

$$\alpha = \frac{d^2\theta}{dt^2} = -\frac{g}{L}\, sin\theta$$

This is not the differential equation of SHM. But if we consider small angle approximation ($\theta \approx 0$), then $sin\theta \approx \theta$, we get,

$$\frac{d^2\theta}{dt^2} = -\frac{g}{L}\,\theta$$

This is the differential equation of SHM with angular velocity or frequency $\omega^2 = g/L$. Thus, the solution of the differential equation and the time period is given by,

$$\theta(t) = \theta_0 \sin(\omega t + \phi)$$
$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{L}{g}}$$

where θ_0 and ϕ depend on the initial conditions of the system. Taking derivatives of $\theta(t)$, we can get the expression for angular velocity and acceleration of the bob.

$$\Omega(t) = \frac{d\theta(t)}{dt} = \omega\theta_0 \cos(\omega t + \phi)$$
$$\alpha(t) = \frac{d\Omega(t)}{dt} = \frac{d^2\theta(t)}{dt^2} = -\omega^2\theta_0 \sin(\omega t + \phi) = -\omega^2\theta(t)$$

Similarly, like mass-spring system, we can determine the kinetic and potential energy of the system and show that total energy is conserved.

Compound Pendulum or Physical pendulum

Simple pendulum is an ideal pendulum system. But physical pendulums don't have massless thread and all the masses are not concentrated at bob. A compound pendulum has an extended mass and is free to oscillate about an axis.

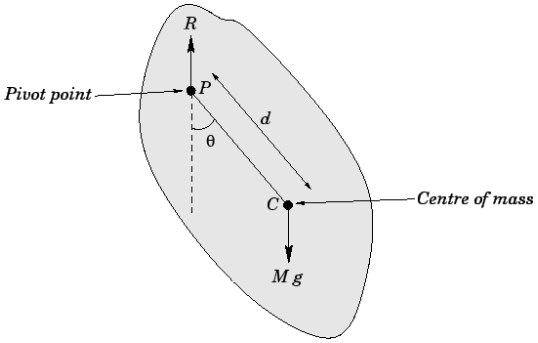
Definition: Any swinging rigid body free to rotate about a fixed horizontal axis is called a compound pendulum or physical pendulum.

Consider a rigid body shown in the figure. It is pivoted at point P. C is the center of mass of the body. Say, the distance between P and C is d. The torque of the body is given by,

$$\vec{\tau} = \vec{r} \times \vec{F}$$

The magnitude of the torque is,

$$\tau = mgd \sin\theta$$



From Newton's 2nd law of motion, $\sum\tau_{ext} = I\alpha$, we can write,

$$-mgd\, sin\theta = I\frac{d^2\theta}{dt^2}$$
$$\frac{d^2\theta}{dt^2} = -\frac{mgd}{I}\theta = -w^2\theta$$

where in the last line we consider the small angle approximation. Same differential equation! Thus, the form of the solution is same as in the case of simple pendulum, but with angular frequency,

$$w = \sqrt{\frac{mgd}{I}}$$

The Time period of a compound pendulum is then given by,

$$T = \frac{2\pi}{w} = 2\pi\sqrt{\frac{I}{mgd}}$$

Note: Here, 'I' is the moment of inertia of the body with respect to the axis passing through pivot point. We can get moment of inertia with respect to center of mass from ''moment of inertia table'' and then calculate the moment of inertia with respect to pivoted point using ''Parallel Axis Theorem''. If the distance between COM and pivot point is d, then moment of inertia with respect to axis passing through point P is given by.

$$I_P = I_{COM} + md^2$$

The radius of gyration is a measure of the distribution of mass around an object's center of mass. It is a distance that describes how much an object's mass is concentrated around its center of mass. The smaller the radius of gyration, the more compact the object's mass is and the more stable it will be when rotating. The radius of gyration is often used in engineering and physics to describe the dynamic properties of an object, such as its moment of inertia and angular velocity.

The radius of gyration (k) is given by the formula:

$$I_{COM} = mK^2$$

where, I is the moment of inertia of the object (a measure of an object's resistance to rotational motion) and m is the mass of the object. Thus the time period of the compound pendulum is given by,

$$T = 2\pi\sqrt{\frac{K^2 + d^2}{gd}}$$

Define effective length, then the time period equation looks similar to simple pendulum time period equation.

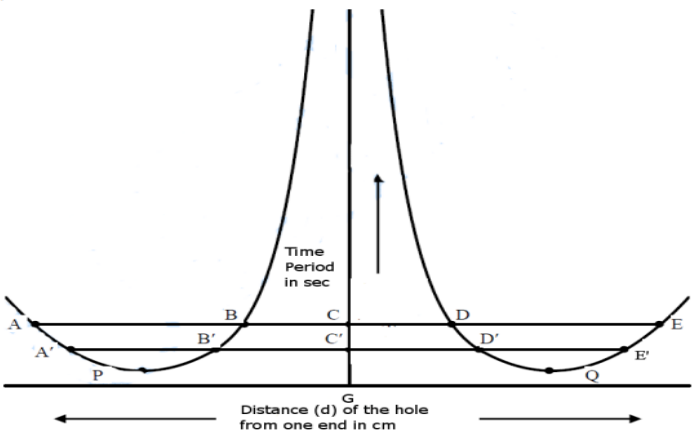
$$T = 2\pi\sqrt{\frac{L}{g}}$$

$$L = \frac{K^2 + d^2}{d}$$

But, notice, here L is not any physical length but an effective length. This is the length of the tread of an "equivalent" simple pendulum system. This is a quadratic equation in "d" and so there are two solutions in general. Consider the two solutions are $l_1 and\ l_2$. Then we can write,

$$L_1 + l_2 = L$$
$$l_1 * l_2 = K^2$$

Thus both l_1 and $l_2$are positive. This means that on one side of C.G there are two positions of the center of suspension about which the time periods are the same. Similarly, there will be a pair of positions of the center of suspension on the other side of the C.G about which the time periods will be the same. Thus there are four positions of the centers of suspension, two on either side of the C.G, about which the time periods of the pendulum would be the same. The distance between two such positions of the centers of suspension, asymmetrically located on either side of C.G, is the length L of the simple equivalent pendulum.

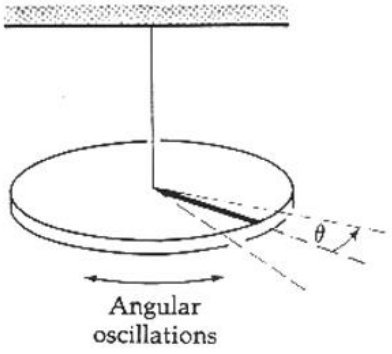


Torsional Pendulum

A disc is suspended by a wire. If we twist the disc from its rest position and release it, it will oscillate about that position in angular simple harmonic motion. Twisting the disc through an angle θ in either direction, introduces a restoring torque,

$$\tau = -\kappa\theta$$

where κ is a characteristic property of the wire, known as the torsional constant (restoring torque per unit angle of twist).



The time period of torsional pendulum is given by,

$$T = 2\pi\sqrt{\frac{I}{\kappa}}$$

Homework

Derive the differential equation of a torsional pendulum and hence find a general solution. Find an expression for angular frequency and time period of a torsional pendulum.