Superposition Principle

If two waves are traveling through the same medium and they intersect, the displacement of the medium at any given point is the sum of the displacements of each wave at that point.

1. Superposition Principle for Linear Homogeneous Equation: Consider a differential equation.

$$A_n \frac{d^n x(t)}{dt^n} + A_{n-1} \frac{d^{n-1} x(t)}{dt^{n-1}} + \cdots + A_1 \frac{dx(t)}{dt} + A_0 x(t) = 0$$

Say, x_1 and x_2 are two solutions of the differential equation. Then, $x = x_1 + x_2$ is also a solution.

2. Superposition Principle for Linear Inhomogeneous Equation: Consider ''Driven SHO'' type equation,

$$A_n \frac{d^n x(t)}{dt^n} + A_{n-1} \frac{d^{n-1} x(t)}{dt^{n-1}} + \cdots + A_1 \frac{dx(t)}{dt} + A_0 x(t) = F(t)$$

where $F(t)$ is the driven force independent of x. If $x_1(t)$ is the solution of DE with driven force $F_1(t)$ and x_2 is the solution of DE with driven force $F_2(t)$, then system with total driven force$F_1(t) + F_2(t)$, the corresponding solution is given by,

$$x(t) = x_1(t) + x_2(t)$$

3. Superposition of SHM along a straight line: Consider, N numbers of SHMs along x-axis.

$$x_i = a_i \sin(\omega_i t + \phi_i)$$

Where $i = 1,2, \ldots, N$. Then, the resultant motion is given by,

$$x = \sum x_i = \sum a_i \sin(\omega_i t + \phi_i)$$

4. Superposition of two SHM at right angles: Consider two SHM at right angles. The angle between x-direction and y-direction is 90 degrees.

$$x = a \sin(\omega_1 t + \phi_1)$$

$$y = b \sin(\omega_2 t + \phi_2)$$

Then, the position of the particle in the X − Y plane is given by,

$$\vec{r} = x\hat{\imath} + y\hat{\jmath}$$

where, $\hat{\imath}$ and $\hat{\jmath}$ are the unit vectors along x and y coordinate.

Lissajous figure

Definition

A Lissajous figure is a visual pattern that results from the intersection of two sinusoidal waveforms that have different frequencies and phases. When these two waveforms are plotted against each other in a two-dimensional coordinate system, the resulting pattern traced out over time is called a Lissajous figure.

The shape of the Lissajous figure depends on the relative frequencies and phases of the two waveforms. When the frequencies are the same, a Lissajous figure is a straight line. As the frequencies and phases of the two waveforms are changed, the Lissajous figure can take on a variety of shapes, including ellipses, circles, and complex curves.

Construction

To create a Lissajous figure, two sinusoidal waves are plotted against each other on a coordinate plane. One wave is plotted on the x-axis, and the other wave is plotted on the y-axis. The amplitude, frequency, and phase of each wave can be adjusted to create different Lissajous figures. The resulting pattern is a closed curve, which can have a variety of shapes depending on the ratio of the frequencies of the two waves.

Mathematical description

The mathematical description of Lissajous figures involves plotting two sinusoidal waves on a coordinate plane and observing the resulting pattern. The equation for a Lissajous figure can be written as:

$$x = A\sin(at + \delta)$$

$$y = B\sin(bt)$$

where A and B are the amplitudes of the waves, a and b are the frequencies of the waves, t is time, and δ is the phase difference between the waves. The phase difference can be adjusted to create different Lissajous figures.

Examples

Example 1: When $a{:}\,b = 1{:}\,1$, we get a Lissajous figure that is a perfect circle. This can be seen by setting $a = b = 1$ and $\delta = 0$ in the equations.

$$x = A\sin(t)$$

$$y = B\sin(t)$$

If we plot this on a coordinate plane with x on the horizontal axis and y on the vertical axis, we get a perfect circle.

Example 2: When $a{:}\,b = 2{:}\,1$, we get a Lissajous figure with two lobes along the x-axis and one lobe along the y-axis. This can be seen by setting $a = 2$ and $b = 1$, and $\delta = 0$ in the equations.

$$x = A\sin(2t)$$

$$y = B\sin(t)$$

If we plot this on a coordinate plane with x on the horizontal axis and y on the vertical axis, we get a pattern that looks like two lobes connected by a line.

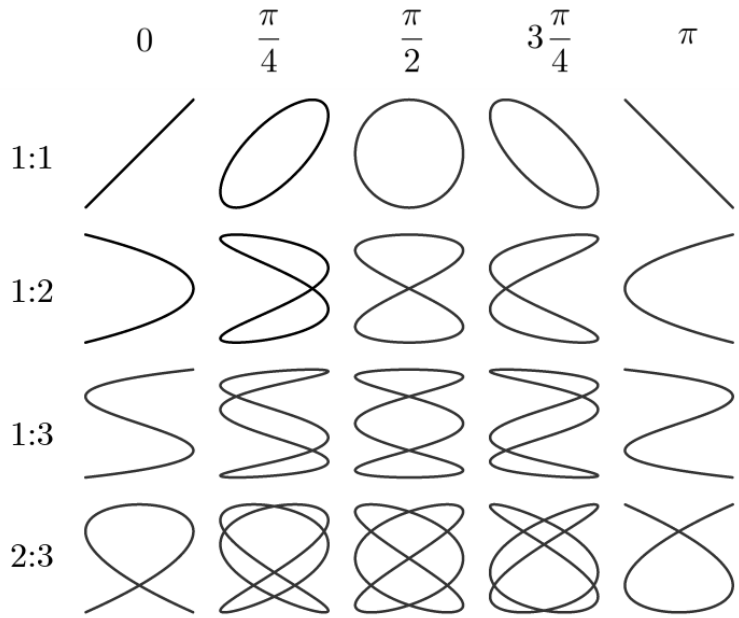
Example 3: When $a: b = 3: 2$, we get a Lissajous figure with three lobes along the x-axis and two lobes along the y-axis. This can be seen by setting $a = 3$ and $b = 2$, and $\delta = 0$ in the equations.

$$x = A\sin(3t)$$

$$y = B\sin(2t)$$

If we plot this on a coordinate plane with x on the horizontal axis and y on the vertical axis, we get a pattern that looks like a figure-eight shape.

Some examples of Lissajous Figures are given below with different frequency ratio and phase differences:



Application: Measurements using Lissajous Figure

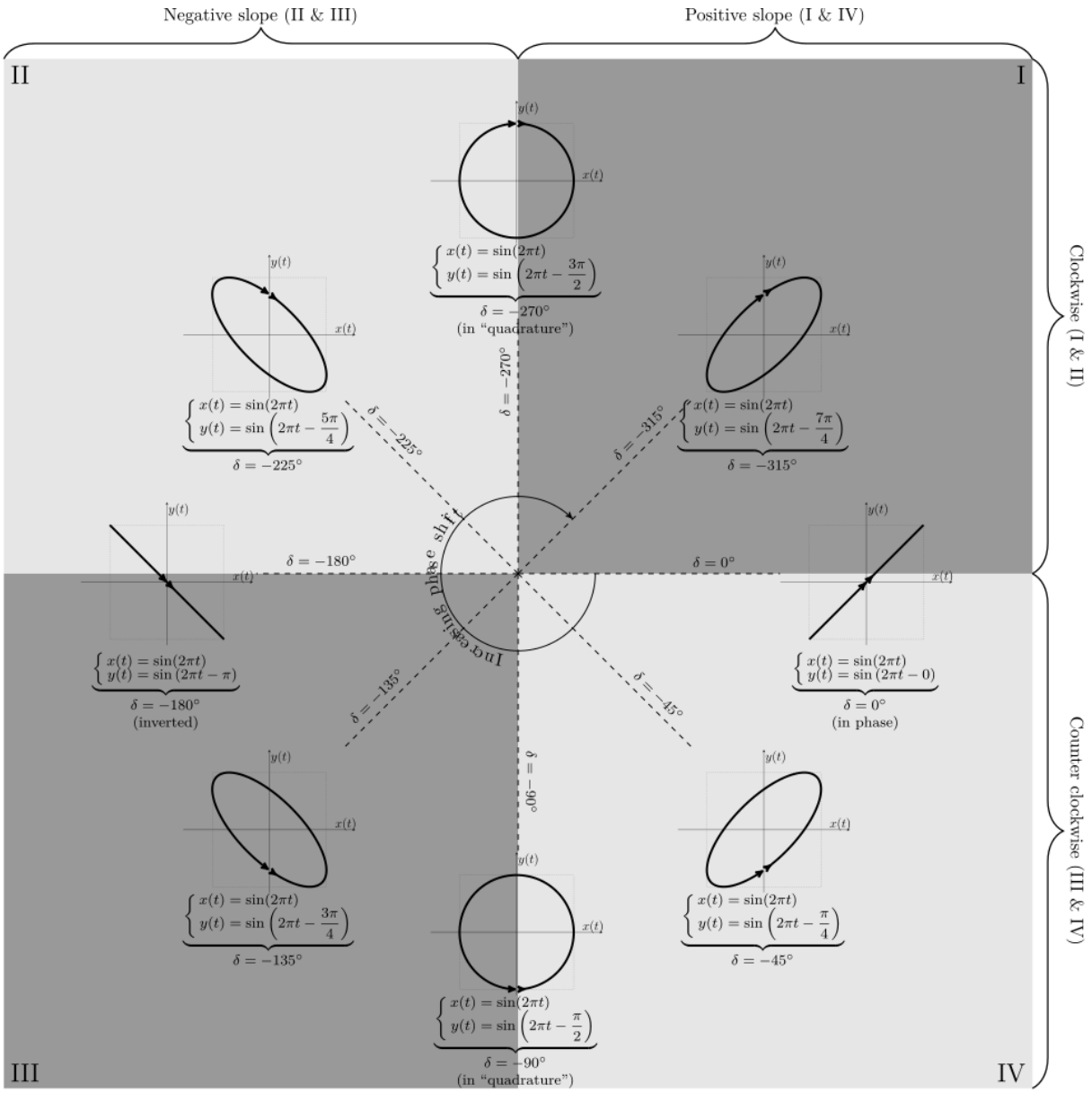
Lissajous figures can be used to make several types of measurements, including the measurement of frequency, phase, and amplitude. Here are some examples of how Lissajous figures can be used for measurements:

Frequency Measurement: As explained earlier, Lissajous figures can be used to measure the frequency of an unknown signal by comparing it with a known reference signal. By counting the number of lobes in the resulting Lissajous figure, we can determine the ratio of the frequencies of the two signals and thus calculate the frequency of the unknown signal.

Phase Measurement: Lissajous figures can also be used to measure the phase difference between two signals. The phase difference can be determined by observing the shape of the Lissajous figure. If the figure is elongated along the X-axis, it means that the phase difference between the two signals is zero or a multiple of 2π. If the figure is elongated along the Y-axis, it means that the phase difference is π/2 or a multiple of π/2. By observing the shape of the Lissajous figure, we can thus determine the phase difference between the two signals.

Amplitude Measurement: Lissajous figures can also be used to measure the amplitude of a signal. This is done by adjusting the amplitude of one of the signals until the Lissajous figure becomes a perfect circle. At this point, the amplitude of the signal is equal to the radius of the circle. By comparing the radius of the circle with the amplitude of the other signal, we can determine the ratio of the amplitudes of the two signals and thus calculate the amplitude of the unknown signal.

Harmonic Analysis: Lissajous figures can be used for harmonic analysis of a periodic signal. By comparing the shape of the Lissajous figure with the known shape of Lissajous figures for different ratios of frequencies, we can determine the harmonic content of the signal. This can be useful in analyzing the quality of audio signals or in detecting harmonics in power systems.



Example: Lissajous Figure of Two SHO with Same Frequency (Frequency Ratio 1:1)

The shape of the curve depends on the time period, phase difference and amplitude of the two SHM. Consider two SHM:

$$x(t) = a\,sin(wt + \varphi)$$

$$y(t) = b\,sin(wt)$$

with phase difference φ. For simplicity, we choose the same frequency for both SHO. Now, we can write,

$$sin(wt) = y/b \qquad \text{and} \qquad cos(wt) = \sqrt{1 - \frac{y^2}{b^2}}$$

Use trigonometric formula,

$$sin(a + b) = sin(a)\,cos(b) + cos(a)sin(b)$$

and substitute $sin(wt)$ and $cos(wt)$ from above equations to get,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 2xy\,ab\,cos(\varphi) = \sin^2\phi$$

This is the general equation for the path of the particle in the Lissajous Figure (2 SHO with same frequency).

Case 01: No phase difference $\varphi = 0$, then we get,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 2xy\, ab = 0$$

$$\left(\frac{x}{a} - \frac{y}{b}\right)^2 = 0$$

$$y = \pm\frac{b}{a}x$$

This is an equation of straight line passing through the origin with slope $\pm\, b/a$ in the Lissajous Figure.

Case 02: Phase difference $\varphi = \pi/2$ and $a \neq b$, then we get,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

This is a general equation of Ellipse.

Case 03: Phase difference $\varphi = \pi/2$ and $a = b$, then we get,

$$x^2 + y^2 = a^2$$

This is a general equation of Circle with radius a.

Case 04: Phase difference $\varphi = \pi/4$, then we get,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \sqrt{2}xy\, ab = \frac{1}{2}$$

This is a general equation of Oblique Ellipse.

Case 05: Phase difference $\varphi = 3\pi/4$, then we get,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \sqrt{2}xy\, ab = \frac{1}{2}$$

This is a general equation of Oblique Ellipse.

Case 06: No phase difference $\varphi = \pi$, then we get,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 2xy\, ab = 0$$

$$\left(\frac{x}{a} + \frac{y}{b}\right)^2 = 0$$

$$y = \pm\frac{b}{a}x$$

This is an equation of straight line passing through the origin with slope $\pm\, b/a$ in the Lissajous Figure.

Homework: Lissajous Figure of Two SHO with Frequency Ratio 1:2, 1:3, 2:3 for phase difference $\phi = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi$. You can use Mathematica to plot the figures.

Mathematica Code: Lissajous Figure

The equation for a Lissajous figure can be written as:

$x = A \sin(a \times t + \delta)$

$y = B \sin(b \times t)$

where A and B are the amplitudes of the waves, a and b are the frequencies of the waves, t is time, and δ is the phase difference between the waves.

Manipulate

```
Manipulate[ParametricPlot[{Sin[a t + delta], Sin[b t]}, {t, 0, 2 Pi}],
 {a, 0, 5}, {b, 0, 5}, {delta, 0, 2 π}]
```

Plot

```
In[ ]:= a = 5; b = 15; delta = π / 2;
ParametricPlot[{Sin[a t + delta], Sin[b t]}, {t, 0, 2 Pi}]
```

Frequency Ratio

```
In[ ]:= delta = 0; abratio = 3; b = 5; a = b * abratio;
ParametricPlot[{Sin[a t + delta], Sin[b t]}, {t, 0, 2 Pi}]
```