Damped Harmonic Oscillator

The total energy of a harmonic oscillator remains constant for an undamped system. Once started, the oscillations continue forever with a constant amplitude (which is determined from the initial conditions) and a constant frequency (which is determined by the inertial and elastic properties of the system).

However, observation of the free oscillations of a real physical system reveals that the energy of the oscillator gradually decreases with time and the oscillator eventually comes to rest. For example, the amplitude of a pendulum oscillating in the air decreases with time, and it ultimately stops.

Definition: Damped Harmonic Oscillator

A damped harmonic oscillator is a physical system that undergoes a repetitive motion, such as back-and-forth oscillation or vibration, that is affected by a damping or frictional force. The damping force opposes the motion of the oscillator, causing it to lose energy and gradually come to a stop. The system can be modeled mathematically using an equation of motion that considers the mass of the oscillator, the damping coefficient, the spring constant, and the displacement of the oscillator from its equilibrium position.

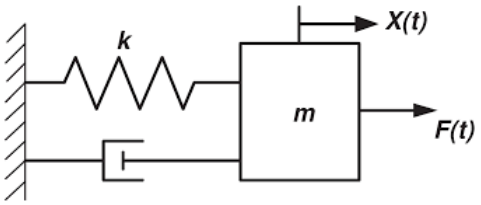
Damping Force

Damping forces are a special type of force that is used to slow down or stop a motion. The key point is that damping forces only work on vibratory motion or on systems that exhibit an oscillatory nature. It is a force that restricts the vibrations that can be of mechanical nature or electrical nature. The damping effect caused by damping forces is due to the dissipation of energy.

The damping force opposes the motion of the body. The magnitude of the damping force is directly proportional to the velocity of the body. The direction of the damping force is opposite to the velocity.

$$F_D = -bv$$

Here, v is the velocity of the body and b is the damping coefficient. This relation is true when the velocity of the oscillating system is small. If the velocity is not small, the damping force exerted on the system may be represented more closely by a force proportional to the square of the velocity. We shall not deal with such forces.



Differential Equation

The differential equation of a damped harmonic oscillator can be derived using Newton's second law of motion, which states that the net force acting on an object is equal to its mass times its acceleration.

Consider a mass m attached to a spring with spring constant k, and assume that the motion is damped by a frictional force proportional to the velocity of the mass, with damping coefficient b. Let $x(t)$ be the displacement of the mass from its equilibrium position at time t.

The force acting on the mass can be expressed as the sum of the spring force and the damping force, which are given by:

$$F_{Hook} = -kx \qquad\qquad F_{Drag} = -bv$$

Using Newton's second law ($\sum F = ma$), we have:

$$m\frac{d^2x}{dt^2} = -kx - b\frac{dx}{dt}$$

This is the differential equation of a damped harmonic oscillator. We can rearrange it into the standard form:

$$m\frac{d^2x(t)}{dt^2} + b\frac{dx(t)}{dt} + kx(t) = 0$$

$$\frac{d^2x(t)}{dt^2} + 2\beta\frac{dx(t)}{dt} + \omega_0^2 x(t) = 0$$

where, $\beta = \frac{b}{2m}$ and $\omega_0^2 = \frac{k}{m}$.

This equation describes the motion of the damped harmonic oscillator and can be used to solve for the displacement $x(t)$ of the mass as a function of time. The behavior of the oscillator depends on the values of the mass m, damping coefficient b, and spring constant k.

Solving Damped SHO for a General Solution

For solving the differential equation apply the "Guess the solution technique". Let the guess solution be:

$$x(t) = Ae^{\alpha t}$$

To find α, substitute this guess solution into the DE and simplify to get,

$$A\alpha^2 e^{\alpha t} + 2A\beta\alpha e^{\alpha t} + A\omega_0^2 e^{\alpha t} = 0$$

$$Ae^{\alpha t}(\alpha^2 + 2\beta\alpha + \omega_0^2) = 0$$

The term outside the parentheses ($Ae^{\alpha t}$) cannot be zero as it is the solution of the DE. $Ae^{\alpha t} = 0$ is called the trivial solution. For non-trivial solution:

$$\alpha^2 + 2\beta\alpha + \omega_0^2 = 0$$

This is a quadratic equation in α (i.e., two solutions, say α_1 and α_2). The solutions are,

$$\alpha_{1,2} = \frac{-2\beta \pm \sqrt{4\beta^2 - 4\omega_0^2}}{2}$$

Consider, two solutions are $\alpha_1 = -\beta + \sqrt{\beta^2 - \omega_0^2}$ and $\alpha_2 = -\beta - \sqrt{\beta^2 - \omega_0^2}$. Then, the general solution, using superposition principle, is given by,

$$x(t) = A\, e^{\alpha_1 t} + B\, e^{\alpha_2 t}$$

Substituting α_1 and α_2, we get the general solution of damped SHM as,

$$x(t) = A\, e^{(-\beta + \sqrt{\beta^2 - \omega_0^2})t} + B\, e^{(-\beta - \sqrt{\beta^2 - \omega_0^2})t}$$

Three Damped Oscillator Solutions

The solution to this differential equation is a function of time that describes the position of the oscillator as a function of time. Depending on the values of m, b, and k, the solution can take on different forms.

In the general solution there are three possible solutions depending on the value of β and ω_0:

- Case #1: $\beta < \omega_0$, called the Under Damped System.
- Case #2: $\beta \approx \omega_0$, called the Critically Damped System.
- Case #3: $\beta > \omega_0$, called the Over Damped System.

Case #1: Under Damped Solution ($\boldsymbol{\beta < \omega_0}$)

When the damping is weak and the oscillator is underdamped, the solution is a damped sinusoidal function, with a decreasing amplitude over time. As $\beta < \omega_0$, we can write:

$$\sqrt{\beta^2 - \omega_0^2} = i\sqrt{\omega_0^2 - \beta^2} = i\omega$$

where, $\omega^2 = \omega_0^2 - \beta^2$. Thus, the general solution can be written as:

$$x(t) = e^{-\beta t}\,[A\,e^{i\omega t} + Be^{-i\omega t}\,]$$

Using Euler's formula ($e^{i\theta} = \cos\theta + i\sin\theta$), we can simplify to get,

$$x(t) = e^{-\beta t}\,[(A+B)\cos\omega t + (A-B)\sin\omega t]$$

Now, as $x(t)$ is a real quantity, each term on RHS must be real. Thus, we can redefine,

$$(A+B) = x_0 \sin\phi$$

$$i(A-B) = x_0 \cos\phi$$

Then, the general solution becomes (applying trig. identities):

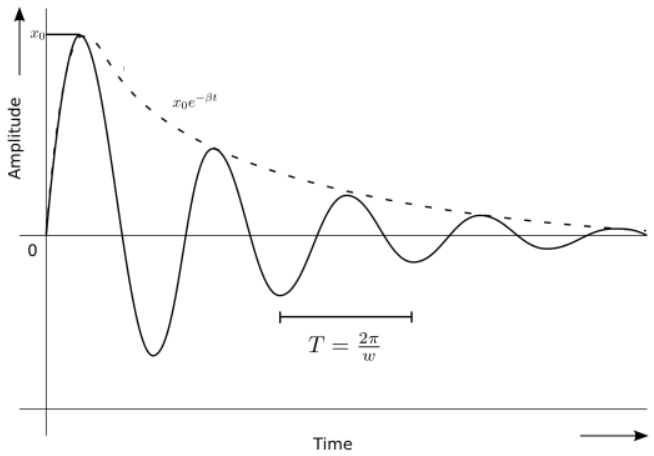
$$x(t) = x_0\, e^{-\beta t} \sin(\omega t + \phi)$$

where, $x_0\, e^{-\beta t}$ is the amplitude which is now depending on time (specifically it is decaying with time) and $\omega = \sqrt{\omega_0^2 - \beta^2}$ is the angular frequency of the system.

The solution implies that the body is executing SHM with frequency ω while the amplitude is decreasing with time exponentially. The effect of damping is two-fold:

(a) The amplitude of oscillation decreases exponentially with time.

(b) The angular frequency of the damped oscillator is less than the frequency of the undamped oscillation.



Under Damped Oscillation

Case #2: Critically Damped Solution ($\beta \approx \omega_0$)

In the case where the damping is critical, the oscillator exhibits a critically damped motion, which results in the quickest possible approach to equilibrium without oscillation.

Taylor expansion of exponential function is given by,

$$e^y = 1 + y + \frac{y^2}{2!} + \frac{y^3}{3!} + \cdots$$

If y is very small, then we can ignore the higher order terms. So, we can write:

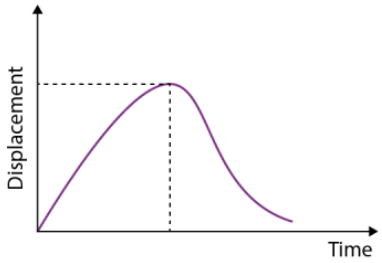
$$e^y \approx 1 + y$$

Critically damped is the case where $\beta \approx \omega_0$. Thus, we can Taylor expand and ignore the higher order temrs.

$$x(t) = e^{-\beta t}\left[A\left(1 + \sqrt{\beta^2 - \omega_0^2}\, t\right) + B\left(1 - \sqrt{\beta^2 - \omega_0^2}\, t\right)\right]$$

$$x(t) = e^{-\beta t}\left[(A + B) + (A - B)\sqrt{\beta^2 - \omega_0^2}\, t\right]$$

This oscillation does not execute SHM! Drag wins over oscillation.



Critically Damped Oscillation

Case #3: Over Damped Solution ($\beta > \omega_0$)

When the damping is strong and the oscillator is overdamped, the motion is characterized by a slower approach to equilibrium without oscillation. In this case, β > w0, the solution is given by,

$$x(t) = A\, e^{\left(-\beta + \sqrt{\beta^2 - \omega_0^2}\right)t} + B\, e^{\left(-\beta - \sqrt{\beta^2 - \omega_0^2}\right)t}$$

In this case, the terms under the square root are positive and so there is no imaginary factor. Thus, we cannot use Euler formula to convert exponential to oscillatory sine or cosine functions. The system has no oscillating frequency. It will not show SHM. Definition of hyperbolic function:

$$\sinh\theta = -i\sin(i\theta)$$

$$\cosh\theta = \cos(i\theta)$$

The Euler's formula for real argument:

$$e^{\pm\theta} = \cosh\theta \pm \sinh\theta$$

The solution for over damped situation is thus given by,

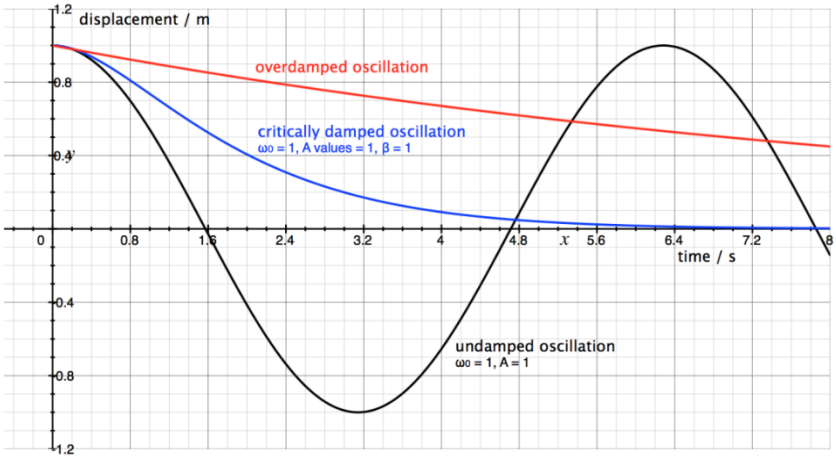
$$x(t) = e^{-\beta t}\left[\,(A + B)\cosh\left(\sqrt{\beta^2 - \omega_0^2}\, t\right) + \,(A - B)\sinh\left(\sqrt{\beta^2 - \omega_0^2}\, t\right)\right]$$

Similarly, set $A + B = x_0 \sinh\phi$ and $A - B = x_0 \cosh\phi$, the solution becomes,

$$x(t) = x_0 e^{-\beta t}\left[\sinh(\phi)\ \cosh\left(\sqrt{\beta^2 - \omega_0^2}\, t\right) + \cosh(\phi)\sinh\left(\sqrt{\beta^2 - \omega_0^2}\, t\right)\right]$$

$$x(t) = x_0\, e^{-\beta t}\sinh\left(\sqrt{\beta^2 - \omega_0^2}\, t + \phi\right)$$

This is the solution to over damped motion. This solution does not execute SHM.



Undamped(black), Critically(blue), Over(red) Damped Oscillation

Homework #01: LCR circuit. Derive the differential equation and find the solutions.

$$L\frac{d^2q(t)}{dt^2} + R\frac{dq(t)}{dt} + \frac{1}{C}\,q(t) = 0$$

Hins: $U_{capacitor} = \frac{1}{2}\frac{1}{C}q^2$ and $U_{inductor} = \frac{1}{2}Li^2$, where $i = \frac{dq}{dt}$.

- For LC circuit: $\frac{dU}{dt} = 0$ (EM energy conserved)
- For LCR circuit: $\frac{dU}{dt} = i^2R$ (EM energy changing into thermal energy)

Homework #02: Show that the average power dissipation (the rate of loss of energy) of a under damped SHM is given by,

$$P = 2\beta E_0 = ma_0^2\omega^2\beta e^{-2\beta t}$$

Hints: Write down the displacement equation for under damped SHO. Find the velocity expression. Then find the total energy (= kinetic energy + potential energy). Substitute displacement and velocity. Find the average total energy (E). Find the average power dissipation which is defined as:

$$P = -\frac{dE}{dt}$$

Mathematica Code

```
F[t_, b_, w_, A_, B_] := A Exp[(-b + Sqrt[b^2 - w^2]) t] + B Exp[(-b - Sqrt[b^2 - w^2]) t];

Plot1 = Plot[F[t, 1, 5, 5, 5], {t, 0, 1.75}, PlotStyle → {Green}]; (*Under_D*)
Plot2 = Plot[F[t, 5, 5, 5, 5], {t, 0, 1.75}, PlotStyle → {Blue, Dashed}] ; (*Critically_D*)
Plot3 = Plot[F[t, 10, 5, 5, 5], {t, 0, 1.75}, PlotStyle → {Red}]; (*Over_D*)
```

Dynamical Plot

```
Dynamic@Plot[F[x, b, w, 5, 5], {x, 0, 1.75}, PlotRange → All]
Slider[Dynamic@b, {0, 20}]
Slider[Dynamic@w, {0, 20}]
```

Plot 1 + Plot 2 + Plot 3

```
Legended[ Show[Plot1, Plot2, Plot3,
  PlotLabel → "Amplitude Vs time", AxesLabel → {"time", "Amplitude"}, PlotRange → All , GridLines → Automatic],
 SwatchLegend[{Green, Blue, Red}, {"Under Damped", "Critically Damped", "Over Damped"}]]
```