Forced Simple Harmonic Oscillation

Forced oscillation refers to a type of oscillation or vibration that occurs when an external force or input is applied to a system that is capable of oscillating. The external force causes the system to vibrate with a frequency that is equal to the frequency of the applied force, or a multiple of it, depending on the properties of the system.

For example, consider a simple pendulum. If the pendulum is set into motion and left to swing freely, it will oscillate at its natural frequency. However, if an external force is applied to the pendulum, such as a periodic push, the pendulum will oscillate with a frequency that is equal to the frequency of the applied force.

Resonance

The behavior of a system under forced SHM can be described using the concept of resonance. Resonance occurs when the frequency of the driving force matches one of the natural frequencies of the system. When this happens, the amplitude of the oscillations can increase dramatically, leading to a phenomenon known as resonance amplification.

Resonance amplification can be observed in many real-world situations, such as the collapse of the Tacoma Narrows Bridge in 1940. In this case, strong winds caused the bridge to vibrate at its natural frequency, leading to an increase in amplitude that eventually caused the bridge to collapse.

Example: Forced SHO

Think about a simple test. In a viscous fluid, attach a mass m to a spring. The spring's free end is attached to a disk that is powered by a motor with variable speed. With an angular driving frequency of w, the motor rotates. The rotating disk provides energy to the system by the work done by the driving force $F_d = F_0 \sin(w_0 t)$.

Differential Equation

Using Newton's second law $\vec{F}_{net} = m\vec{a}$, we can analyze the motion of the mass. The resulting equation is similar to the force equation for the damped harmonic oscillator, with the addition of the driving force:

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = F_0 \sin(\omega_0 t)$$

$$\frac{d^2x}{dt^2} + 2\beta\frac{dx}{dt} + \omega^2 x = f_0 \sin(\omega_0 t)$$

where, $2\beta = \frac{b}{m}$, $\omega^2 = \frac{k}{m}$ and $f_0 = \frac{F_0}{m}$.

General Solution

When an oscillator is forced with a periodic driving force, the motion may seem chaotic. The motion of the oscillator is known as transients. After the transients die out, the oscillator reaches a steady state, where the motion is periodic. After some time, the steady state solution to this differential equation is,

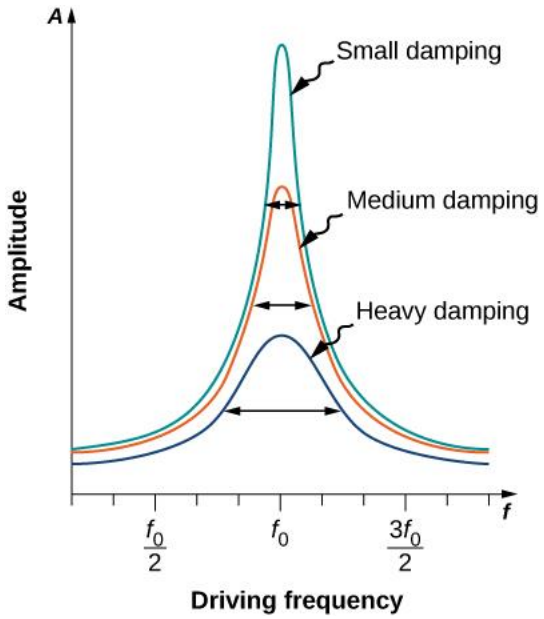
$$x(t) = A \sin(w_0 t - \phi)$$

Once again, it is left as an exercise to prove that this equation is a solution. Taking the first and second time derivative of *x*(*t*) and substituting them into the force equation shows that $x(t) = A \sin(w_0 t - \phi)$ is a solution as long as the amplitude and phase are given by (**Homework**)

$$A = \frac{F_0/m}{\sqrt{(w^2 - w_0^2)^2 + 4\beta^2 w_0^2}}$$

$$\phi = \tan^{-1}\left(\frac{2\beta\omega_0}{(\omega^2 - \omega_0^2)}\right)$$

where $\omega = \sqrt{\frac{k}{m}}$ is the natural angular frequency of the system of the mass and spring. Recall that the angular frequency and therefore the frequency of the motor can be adjusted. Looking at the denominator of the equation for the amplitude, when the driving frequency is much smaller, or much larger, than the natural frequency, the square of the difference of the two angular frequencies $(w^2 - w_0^2)^2$ is positive and large, making the denominator large, and the result is a small amplitude for the oscillations of the mass. As the frequency of the driving force approaches the natural frequency of the system, the denominator becomes small, and the amplitude of the oscillations becomes large. The maximum amplitude results when the frequency of the driving force equals the natural frequency of the system $A_{max} = \frac{F_0}{bw}$ (Resonance).



Amplitude vs Driving Frequency Graph

Figure "Amplitude Vs Driving Frequency" shows a graph of the amplitude of a damped harmonic oscillator as a function of the frequency of the periodic force driving it. Each of the three curves on the graph represents a different amount of damping. All three curves peak at the point where the frequency of the driving force equals the natural frequency of the harmonic oscillator. The highest peak, or greatest response, is for the least amount of damping, because less energy is removed by the damping force. Note that since the amplitude grows as the damping decreases, taking this to the limit where there is no damping $b = 0$, the amplitude becomes infinite.

Note that a small-amplitude driving force can produce a large-amplitude response. This phenomenon is known as resonance. A common example of resonance is a parent pushing a small child on a swing. When the child wants to go higher, the parent does not move back and then, getting a running start, slam into the child, applying a great force in a short interval. Instead, the parent applies small pushes to the child at just the right frequency, and the amplitude of the child's swings increases. It is interesting to note that the widths of the resonance curves depend on damping: the less the damping, the narrower the resonance. The consequence is that if you want a driven oscillator to resonate at a very specific frequency, you need as little damping as possible.

Phase of the Driven Oscillator

The displacement of a SHO vibrating in a damping medium and subjected to an external force is given by,

$$x(t) = A \sin(w_0 t - \phi)$$

The equation suggests that the phase of the driven oscillator lags behind the driven force by an angle given by,

$$\phi = \tan^{-1}\left(\frac{2\beta\omega_0}{(\omega^2 - \omega_0^2)}\right)$$

The phase angle ϕ depends upon damping and the relative values of ω and ω_0. The following three cases arise:

(i) When $\omega_0 < \omega$, i.e. driving frequency is less than the natural undamped frequency of the oscillator:

$\tan\phi$ will be positive and $0 < \phi < \pi/2$ for all values of β.

(ii) When $\omega_0 = \omega$, i.e. driving frequency is equal to the natural undamped frequency of the oscillator:

$\tan\phi = \infty$ and $\phi = \pi/2$ for all values of β.

(iii) When $\omega_0 > \omega$, i.e. driving frequency is greater than the natural undamped frequency of the oscillator:

$\tan\phi$ will be negative and $\pi/2 < \phi < \pi$ for all values of β.

Homework: Derive the differential equation for driven LCR circuit with external emf $E = E_0 \sin(\omega_0 t - \phi)$. Find the expression for solution, amplitude, and phase.

Mathematica Code

```
F[w_, F0_, b_, w0_, m_, A_] := A F0 / Sqrt[m (w^2 - w0^2)^2 + b^2 w^2];

Plot1 = Plot[F[w, 10, 2, 10, 2, 8], {w, 0, 20},
  PlotLabel → "Amplitude Vs Driving Frequency", PlotStyle → {Green}]
```

Dynamical Plot

```
Dynamic@Plot[F[w, 10, d, a, 0.5, 8], {w, 0, 10},
  AxesLabel → {"Driving Frequency", "Amplitude"},
  PlotLabel → "Amplitude Vs Driving Frequency", PlotRange → All]
Slider[Dynamic@a, {2, 10}] (*Natural Frequency*)
Slider[Dynamic@d, {0, 2}] (*damping*)
(*Amplitude Vs Driving frequency*)
```