Coupled Oscillator

Coupled oscillators have applications in various fields, including physics, engineering, biology, and social sciences. They can be used to model physical systems such as pendulums, electrical circuits, and mechanical systems. They can also be used to model biological systems such as heartbeats or the synchronization of firefly flashing patterns.

Definition

Coupled oscillators refer to a system of two or more oscillators that interact with each other. The oscillators can be mechanical, electrical, or electromagnetic, and the coupling can be through various physical interactions, such as springs, electrical circuits, or magnetic fields. The behavior of each oscillator is affected by the other oscillator(s) in the system, leading to complex and interesting dynamics. The ends of the oscillator can be fixed end or open end.

Example

One example of a two-body coupled oscillator with a fixed end is a system consisting of two identical masses attached to a horizontal spring, with one end of the spring attached to a rigid wall. As two ends are fixed, the boundary condition is fixed.

Coupling-Interaction Between Oscillators

The coupling between the two oscillators arises from the fact that the motion of each oscillator affects the motion of the other. As one oscillator moves, it exerts a force on the other oscillator, causing it to move as well. The phase difference between the two oscillators can lead to interesting behavior, such as synchronization or anti-synchronization.

Phase Locked: Synchronization vs Anti-Synchronization

Synchronization refers to the phenomenon where the two oscillators settle into a synchronized motion, such that their phases are locked together. This can occur for various reasons, such as the presence of a common driving force or a coupling mechanism that favors synchronization.

Anti-synchronization, on the other hand, refers to the phenomenon where the two oscillators settle into an anti-synchronized motion, such that their phases are locked in an anti-phase relationship. This can occur in systems with non-linear coupling mechanisms, such as electrical circuits or chemical reactions.

Coupled Oscillator and Wave

To get to waves from oscillators, we have to start coupling them together. For large numbers of coupled oscillators, we find solutions like waves. Certain features of waves such as resonance and normal modes can be understood with finite numbers of oscillators.

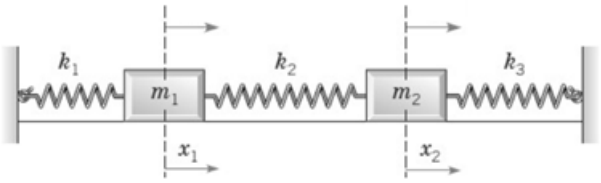
Modes in Coupled Oscillator

Normal modes: These are the modes in which the system vibrates with a natural frequency. The normal modes are characterized by their frequencies and shapes, which depend on the mass, stiffness, and damping of the system. In a two-body coupled oscillator system, there are two normal modes, which correspond to the motion of the masses in phase (symmetric mode) or out of phase (anti-symmetric mode).

Collective modes: These are the modes in which the motion of the system is a combination of normal modes. Collective modes can arise due to the interaction between different oscillators in the system. In a two-body coupled oscillator system, collective modes can arise due to the coupling between the masses through the spring.

Two Body Coupled Oscillator: A Simple Case

Consider a system with two masses with springs shown in the figure. Consider the simple case: both masses are same ($m_1 = m_2 = m$) and the spring constants of the springs connecting masses with the wall are same ($k_1 = k_3 = k$) but a different spring constant of the spring connecting the two masses ($k_2 = \kappa$).



Construction of the Differential Equation

Then the force acting on body 1 from moving body 1 is,

$$F_{on1,from1} = -kx_1 - \kappa x_2$$

and the force acting on body 1 from moving body 2 is,

$$F_{on1,from2} = +\kappa x_2$$

Thus, the differential equation for body 1 is given by,

$$m\ddot{x_1} = -(k + \kappa)x_1 + \kappa x_2$$

Similarly for body 2, we can write the differential equation as,

$$m\ddot{x_2} = -(k + \kappa)x_2 + \kappa x_1$$

Solution of the Differential Equation

These are coupled DE as the solutions x_1 and x_2 are mixed in the DEs (entangled). To solve the DE, we must first disentangle them. We get two solutions with two frequencies as disentanglement must not affect the physical system. Two solutions (x_1, x_2) → two solutions $(x_1 + x_2, x_1 - x_2)$.

Adding the differential equations and redefining $y = x_1 + x_2$, we get the DE and solution:

$$m\ddot{y} + ky = 0$$

$$y = x_1 + x_2 = A_s \cos(\omega_s t + \phi_s)$$

with $\omega_s^2 = \frac{k}{m}$. And subtracting the differential equations and redefining $z = x_1 - x_2$, we get the DE and the solution:

$$m\ddot{z} + (k + s\kappa)z = 0$$

$$z = x_1 - x_2 = A_f \cos(\omega_f t + \phi_f)$$

with $\omega_f^2 = \frac{k+2\kappa}{m}$. The subscript '$f$' and '$s$' refer to fast and slow respectively. (As $\omega_f > \omega_s$ from the DE solution.)

Thus, we have found two solutions, each of which oscillates with a fixed frequency. These are the normal modes of the system. A general solution is a linear combination of these two solutions (by superposition principle).

$$x_1 = \frac{1}{2}[(x_1 + x_2) + (x_1 - x_2)] = \frac{1}{2}[A_s \cos(\omega_s t + \phi_s) + A_f \cos(\omega_f t + \phi_f)]$$

$$x_2 = \frac{1}{2}[(x_1 + x_2) - (x_1 - x_2)] = \frac{1}{2}[A_s \cos(\omega_s t + \phi_s) - A_f \cos(\omega_f t + \phi_f)]$$

Now we will determine the conditions for generating different modes. We are constructing the theory with longitudinal displacement. But this is also true for transverse (vibration direction is perpendicular to the energy transfer direction) displacement.

Symmetric oscillation mode: Pull both masses to the right by same amount so that,

$$x_1(t=0) = x_2(t=0) \rightarrow A_f = 0$$

Then the solution is $x_1 = x_2$ and both oscillates with frequency ω_s and amplitudeA_s.

Anti-symmetric oscillation mode: If we excite the masses such that,

$$x_1(t=0) = -x_2(t=0) \rightarrow A_s = 0$$

Then the solution is $x_1 = -x_2$ and both oscillating with frequency ω_f and amplitude A_f.

Thus, two body coupled oscillator system can oscillate in two modes mentioned above. Can you guess the situation for 3 body coupled system?

Anti-symmetric Oscillation Mode vs Anti-Synchronization of Oscillators

An anti-symmetric oscillation mode is not the same as anti-synchronization of oscillators.

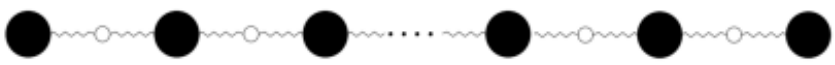
An anti-symmetric oscillation mode refers to a specific mode of oscillation where two objects or systems oscillate with the same frequency but with opposite phase, such that when one object is at its maximum displacement, the other is at its minimum displacement.

On the other hand, anti-synchronization of oscillators refers to a phenomenon where two or more oscillators are coupled such that they oscillate with opposite phase, but not necessarily with the same frequency. In other words, when one oscillator is at its maximum displacement, the other is at its minimum displacement, but the period of oscillation may not be the same for both oscillators.

Homework: Show that Beat is an example of coupled oscillator and Beats occur when two normal mode frequencies get close.

Generalizing to n Coupled Oscillator

As we have already done two coupled oscillator system in longitudinal vibrations, but this is true for both longitudinal and transverse oscillations, in this section we will derive the general equation for n coupled oscillator in the transverse oscillation. The physics is same both.



Coupled Oscillation (Generalized to n oscillators)

Now let's generalize the system to n coupled oscillator (equally spaced). Consider a mass inside the system and label it p. Thus, p runs from 1 to n as there are n numbers of coupled masses. Suppose there is a tension (T) on the p^{th} mass from both sides (i.e. $(p+1)^{th}$and $(p-1)^{th}$ mass, due to coupling between them).

The differential equation for the p^{th} particle can be written as.

$$m\ddot{y_p} = -T\sin(A_{p-1}) + T\sin(A_p)$$

where we can write the sine functions as,

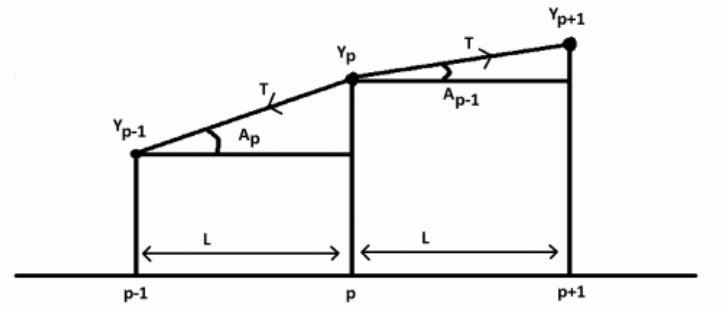
$$\sin A_{p-1} = \frac{y_p - y_{p-1}}{L}$$

$$\sin A_p = \frac{y_{p+1} - y_p}{L}$$

Define $\omega_0^2 = \frac{T}{mL}$ and simplifying, we get the differential equation,

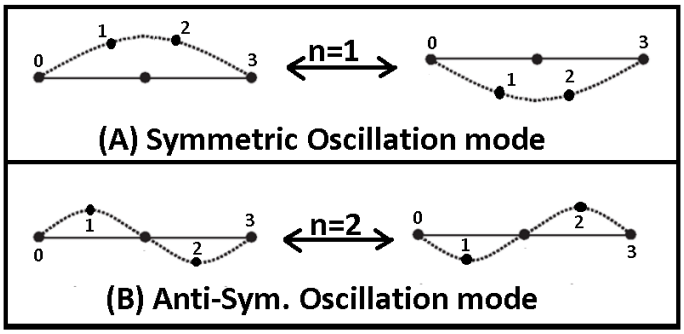
$$\ddot{y_p} + 2\omega_0^2 y_p - \omega_0^2(y_{p+1} + y_{p-1}) = 0$$

There are n differential equations, one for each mass. Thus, there are n modes. We need to fix the boundary condition $y_0 = 0$ and$y_{n+1} = 0$, as the end points of the system are not moving (they are fixed points).



$p, p+1$ and $p-1$ particle displacement from equilibrium

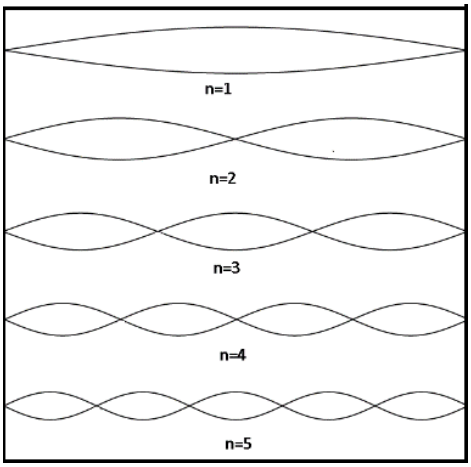
We will not solve this DE in this course. If we set $n = 2$, then we will get back our known two body coupled oscillator system. Then there will be two differential equations, one for $p = 1$ and another for$p = 2$, where $p_0 = p_3 = 0$ for the boundary condition. The two oscillation modes are given in the figure below.



Now, if we consider $n = 5$, we will get 5 modes. Here,

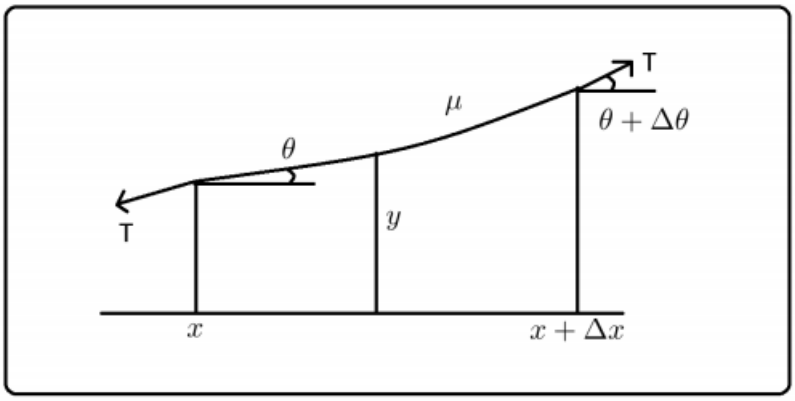
$$\omega_{n=1} < \omega_{n=2} < \omega_{n=3} < \omega_{n=4} < \omega_{n=5}$$

are the frequencies of the five modes. Thus, we can solve any systems with any numbers of n easily with diagrams once we know the trick. Always the number of modes is equal to the number of masses coupled in the system and number of modes is equal to the number of nodes plus one. Check the initial condition: if the number of particles is zero i.e. $\sum p = 0$, then no energy propagation.



Infinite Coupled Oscillator

As the number of coupled oscillators is now infinite, there is no notion of point mass. So we have to define mass per length or density of the medium of oscillation (no more discrete, now the game is on continuous level). Consider a small segment of the string oscillating with mass density $\mu = \frac{m}{l}$. The tension force, say T acting on both sides.



To approximate the tension to be same (which is the physical case), both θ and $\Delta\theta$ must be very small. Now apply the Newton's second law of motion (y-direction) to get,

$$F_y = -T\sin\theta + T\sin(\theta + \Delta\theta) = (dm)\ddot{y}$$

where, $dm = \mu\, dx$ is the mass of the segment. As the angle is very small, with small angle approximation ($\sin\theta \approx \theta$), the force along y-direction is,

$$F_y = T\, d\theta = (dm)\ddot{y}$$

Now the slope of the curve is $\tan\theta = \frac{\partial y}{\partial x}$, Differentiating with respect to x to get,

$$\frac{1}{\cos^2\theta}\frac{d\theta}{dx} = \frac{\partial^2 y}{\partial x^2}$$

Now apply small angle approximation ($\cos\theta \approx 1$) and substitute $dm = \mu\, dx$ in the above equation and get,

$$\frac{\mu}{T}\frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 y}{\partial x^2}$$

This is the differential equation of the system. To solve this we will apply the Guess technique. The form of the guess function is$f(x \pm ct)$, where c is a constant. This will solve the differential equation if,

$$c = \sqrt{\frac{T}{\mu}}$$

If we look at the argument of the guess function, from dimensional consistency, we see that the dimension of c is the dimension of velocity. We denote this with v and thus the differential equation becomes,

$$\frac{1}{v^2}\frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 y}{\partial x^2}$$

This is the differential equation of 'wave' or also called wave equation.

A Powerful Technique to Solve Coupled Oscillator Differential Equation

Consider the two body system. The equations of motion are linear, we expect them to be solved by exponentials $x_1 = c_1 e^{i\omega t}$ and $x_2 = c_2 e^{i\omega t}$ for some ω, c_1 and c_2. We are using complex solutions to make the math simpler, then we can always take the real part at the end. Plugging in these guesses, the differential equation of the system,

$$-m_1\omega^2 c_1 = -(k+\kappa)c_1 + \kappa c_2$$

$$-m_2\omega^2 c_2 = -(k+\kappa)c_2 + \kappa c_1$$

Next, we will write these equations in matrix form. To do so, we define a vector $\vec{c}$ as,

$$\vec{c} = (c_1 \quad c_2)^T$$

Then the equations of motion become,

$$M.\vec{c} = -\omega^2\ \vec{c}$$

where, the matrix $M = \begin{bmatrix} \frac{-k-\kappa}{m_1} & \frac{\kappa}{m_1} \\ \frac{\kappa}{m_2} & \frac{-k-\kappa}{m_2} \end{bmatrix}$.

Homework Using Eigenvalue-Eigenvector concept of a matrix, solve for the eigen values of the matrix M. Show that the eigen values are (setting $m_1 = m_2 = m$),

$$\omega_s = \sqrt{\frac{k}{m}} \quad and \quad \omega_f = \sqrt{\frac{k+2\kappa}{m}}$$

These are the normal modes of the system. We can generalize to n coupled oscillator system. We just need to increase the dimension of matrix and solve for eigen value and eigen vector.