Wave- Progressive wave & Standing wave

Waves are essential to many areas of science and technology, including communication, medicine, and engineering. For example, radio waves are used to transmit information over long distances, sound waves are used in medical imaging and therapy, and water waves are used in oceanography and engineering.

Definition of Wave

A wave is a disturbance that propagates through space and time, transferring energy from one point to another without transferring matter. Waves can take many forms, such as sound waves, light waves, water waves, and electromagnetic waves, among others. In general, waves can be characterized by their wavelength, frequency, amplitude, and velocity.

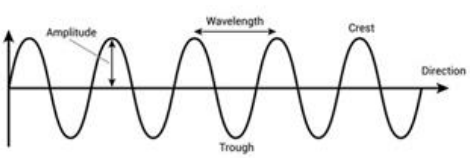
Transverse wave and Longitudinal wave

A transverse wave is a type of wave that propagates perpendicular to the direction of particle motion. In a transverse wave, the disturbance or oscillation is perpendicular to the direction of the wave propagation. This means that the particles of the medium through which the wave travels move in a direction perpendicular to the direction of the wave.

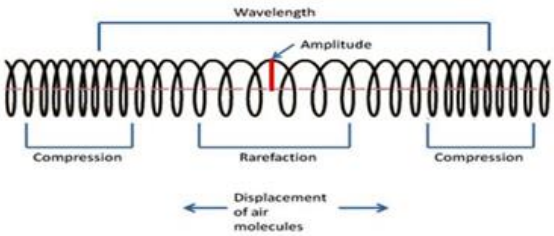
An example of a transverse wave is a water wave. When a stone is thrown into a still pond, ripples are created that propagate outwards from the point of impact. As the wave moves away from the point of impact, the particles of water move up and down in a direction perpendicular to the direction of the wave propagation.

longitudinal wave is a type of wave that propagates parallel to the direction of particle motion. In a longitudinal wave, the disturbance or oscillation is in the same direction as the wave propagation. This means that the particles of the medium through which the wave travels move in a direction parallel to the direction of the wave.

An example of a longitudinal wave is a sound wave. When a sound wave is generated, the particles of the medium (such as air, water, or solid material) oscillate back and forth parallel to the direction of wave propagation. The sound wave is created by the compression and rarefaction of the medium, which causes the particles to move closer together and then farther apart.



Transverse wave



Longitudinal wave

Mechanical wave vs Electromagnetic wave

Mechanical waves are waves that require a physical medium to travel through, such as air, water, or a solid material. Examples of mechanical waves include sound waves, water waves, and seismic waves. These waves transfer energy by causing particles in the medium to vibrate in a back-and-forth motion. Mechanical waves can be transverse or longitudinal, depending on the direction of the particle motion relative to the direction of wave propagation.

Electromagnetic waves are waves that do not require a physical medium to travel through. They can propagate through a vacuum, such as in outer space. Examples of electromagnetic waves include radio waves, microwaves, infrared radiation, visible light, ultraviolet radiation, X-rays, and gamma rays. Electromagnetic waves are transverse waves that consist of oscillating electric and magnetic fields that are perpendicular to each other and to the direction of wave propagation. They transfer energy through the interaction of the electric and magnetic fields.

Some Definitions Connected with Wave Motion

Wavelength (λ): The distance between two adjacent points on a wave that are in phase. The wavelength is usually represented by the Greek letter lambda (λ) and is measured in meters (m). Mathematically, wavelength is expressed as:

$$\lambda = v/f$$

where v is the velocity of the wave and f is the frequency.

Frequency (f): The number of cycles of the wave that pass a given point per unit time. Frequency is usually measured in Hertz (Hz), which represents the number of cycles per second. Mathematically, frequency is expressed as:

$$f = 1/T$$

where T is the period of the wave, which is the time it takes for one complete cycle.

Amplitude (A): The maximum displacement of a point on the wave from its equilibrium position. Amplitude is usually measured in meters (m). Mathematically, amplitude is expressed as:

$$A = (y_max - y_min) / 2$$

where y_max is the maximum displacement of the wave and y_min is the minimum displacement.

Velocity (v): The speed at which the wave propagates through a medium. Velocity is usually measured in meters per second (m/s). Mathematically, velocity is expressed as:

$$v = \lambda * f$$

where λ is the wavelength and f is the frequency.

Time Period (T): The time it takes for one complete cycle of the wave to pass a given point. Period is usually measured in seconds (s). Mathematically, period is expressed as:

$$T = 1/f$$

where f is the frequency.

Angular frequency: The angular frequency (ω) of a wave is the rate at which the wave oscillates in radians per second. It is related to the regular frequency (f) of the wave by the equation:

$$\omega = 2\pi f$$

where π (pi) is the mathematical constant representing the ratio of a circle's circumference to its diameter. The unit of angular frequency is radians per second (rad/s).

Angular wave number: The angular wave number (k) of a wave is the rate of change of phase with respect to distance, measured in radians per meter (rad/m). It is related to the regular wave number (k') by the equation:

$$k = 2\pi/\lambda$$

where λ is the wavelength of the wave. The unit of angular wave number is radians per meter (rad/m).

Wavefront: A wavefront is a surface of constant phase in a wave, that connects points of equal phase within the wave. It can be thought of as the leading edge of a wave, or the boundary between regions of the wave that are in phase and regions that are out of phase.

The shape of the wavefront depends on the geometry of the source of the wave and the properties of the medium through which it is propagating. For example, in the case of a point source in a homogeneous medium, the wavefronts are spherical, while in the case of a line source in a homogeneous medium, the wavefronts are cylindrical.

Plane Progressive Wave

A plane progressive wave is a type of wave that propagates uniformly in a straight line, with a constant amplitude and wavelength. It is also known as a plane wave or a traveling wave.

In a plane progressive wave, the wavefronts are parallel planes that move in the direction of propagation of the wave. The wave is said to be "progressive" because the wavefronts are constantly moving forward, while the shape of the wave remains the same. The mathematical equation for a plane progressive wave can be expressed as:

$$y(x,t) = A\sin(kx - \omega t + \phi)$$

where $y(x,t)$ is the displacement of the wave at position x and time t, A is the amplitude of the wave, k is the wave number (related to the wavelength λ by the equation $\lambda = 2\pi/k$), ω is the angular frequency (related to the frequency f by the equation$\omega = 2\pi f$), and ϕ is the phase constant.

Phase Velocity

To derive an expression for phase velocity, we can start with the general equation for a wave:

$$y(x,t) = A\sin(kx - \omega t + \phi)$$

where y(x,t) is the displacement of the wave at position x and time t, A is the amplitude of the wave, k is the wave number, ω is the angular frequency, and ϕ is the phase constant.

To find the phase velocity, we need to determine how the phase angle changes with time. The phase angle is defined as the argument of the sine function, which is given by:

$$\varphi = kx - \omega t$$

Since (kx – ωt) is a constant quantity, we have $\frac{d}{dt}(kx - \omega t) = 0$. The phase velocity of the wave should be given by $\frac{dx}{dt}$. Thus, we get the phase velocity,

$$\omega - k\frac{dx}{dt} = 0$$

$$\frac{dx}{dt} = \frac{\omega}{k}$$

But

$$\frac{\omega}{k} = \frac{2\pi n}{2\pi/\lambda} = \frac{\lambda}{T}$$

Which is wave velocity. Thus, for a single wave, in a given medium, phase velocity is the wave velocity.

Group Velocity

Group velocity is a property of a wave packet or a group of waves that describes the velocity at which the envelope of the wave packet propagates through space. In other words, it is the velocity at which the information or energy of the wave is transmitted. The group velocity is given by the expression:

$$v_g = \frac{d\omega}{dk}$$

where ω is the angular frequency of the wave and k is the wave number. The group velocity is the rate of change of the angular frequency with respect to the wave number.

To understand the physical meaning of the group velocity, let's consider a wave packet composed of several individual waves with different frequencies and wavelengths. The envelope of the wave packet is the overall shape of the wave, which varies more slowly than the individual waves.

As the wave packet propagates through space, the individual waves may interact and interfere with each other, causing the envelope of the wave packet to change shape. The group velocity describes the velocity at which this overall shape or envelope moves through space.

Energy Density of a Plane Progressive Wave

In a plane progressive wave motion, the energy is passed from particle to particle. So there is a regular transmission of energy across every section of the medium. In order to obtain an expression for energy density, let us start with the equation of a plane progressive wave, which is,

$$y = a\sin\frac{2\pi}{\lambda}(vt - x)$$

Then the velocity of the particle is given by,

$$u = \frac{dy}{dt} = \frac{2\pi v}{\lambda}\ a\cos\frac{2\pi}{\lambda}(vt - x)$$

And the acceleration of the particle is given by,

$$\frac{du}{dt} = \frac{d^2y}{dt^2} = -\frac{4\pi^2v^2}{\lambda^2}a\sin\frac{2\pi}{\lambda}(vt - x) = -\frac{4\pi^2v^2}{\lambda^2}y$$

Kinetic energy per unit volume:

Let us consider unit volume of the medium in the form of an extremely thin element of the medium parallel to the wavefront. The density of the medium is the mass of the element. Since the layer is thin, the velocity of all the particles in the layer may be considered to be the same. Thus, the average kinetic energy,

$$K_{av} = \frac{1}{2}\rho u^2 = \frac{2\pi^2a^2v^2}{\lambda^2}\rho\cos^2[\frac{2\pi}{\lambda}(vt - x)]$$

Potential energy per unit volume:

The work done per unit volume for a small displacement dy of the layer is = $force \times displacement(dy)$. Here,

$$force = mass\ density \times acceleration$$

Thus, the total work done when the layer is displaced from 0 to y, the potential energy,

$$P_{av} = \int_0^y \frac{4\pi^2v^2\rho}{\lambda^2}y\,dy = \frac{2\pi^2v^2\rho}{\lambda^2}a^2\sin^2[\frac{2\pi}{\lambda}(vt - x)]$$

Total energy density:

Thus, the total energy per unit volume of the medium $E = K + P$, is given by,

$$E = \frac{2\pi^2a^2v^2\rho}{\lambda^2}\left\{\cos^2\left[\frac{2\pi}{\lambda}(vt - x)\right] + \sin^2\left[\frac{2\pi}{\lambda}(vt - x)\right]\right\} = 2\pi^2f^2\rho a^2$$

It is interesting to note that although both kinetic and potential energy densities depend upon the values of x and t, its total energy density is independent of either.

Wave Differential Equation

The wave differential equation is a mathematical equation that describes the behavior of waves in various physical systems. It is a partial differential equation that relates the second derivative of the wave function with respect to time to the second derivative of the wave function with respect to space. The wave equation is one of the fundamental equations in physics and is used to describe a wide range of wave phenomena, including light, sound, and electromagnetic waves.

The one-dimensional wave equation can be expressed as:

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$$

where $y(x,t)$ is the wave function, c is the wave speed, and x and t are the space and time coordinates, respectively. This equation states that the acceleration of the wave at any point is proportional to the curvature of the wave at that point, with the constant of proportionality being the square of the wave speed.

The wave equation is a linear equation, which means that the principle of superposition applies. This principle states that the sum of two solutions to the wave equation is also a solution to the wave equation. This property allows for the development of wave interference patterns and the behavior of waves in complex physical systems.

Derivation of Wave Differential Equation

Let's derive it in the case of waves on strings. We assume as usual that the string is stretched along the x-axis, with the same tension T everywhere. We also assume that the transverse displacements are small. Strictly speaking, the wave equation is only true for strings in this approximation. So we focus on the small segment of string shown in the snapshot at a fixed time t below.

Since the displacement y is small, the shape of the wave is nearly flat, and the angles θ_1 and θ_2 between the tangents to the string at the ends and the horizontal are small; so we can approximate $\sin\theta \approx \tan\theta$ to get the vertical component of the force on the segment, which is entirely due to the tension if the string is light:

$$F_y = T\sin\theta_2 - T\sin\theta_1 \approx T\tan\theta_2 - T\tan\theta_1$$

$$F_y = T\frac{\partial y}{\partial x}\Big|_B - T\frac{\partial y}{\partial x}\Big|_A$$

In the last step, we used the fact that $\tan\theta$ is the slope of the curve, which is also $\partial y/\partial x$, at fixed time.

Next, we use Newton's force law, and the fact that for small displacements the length of the segment is nearly the same as the horizontal component of the displacement between points A and B, $\Delta s \approx \Delta x$:

$$F_y = ma_y = \mu\Delta x\frac{\partial^2 y}{\partial t^2} = T\frac{\partial y}{\partial x}\Big|_B - T\frac{\partial y}{\partial x}\Big|_A$$

$$\frac{\mu}{T}\frac{\partial^2 y}{\partial t^2} = \frac{1}{\Delta x}\left(\frac{\partial y}{\partial x}\Big|_B - \frac{\partial y}{\partial x}\Big|_A\right)$$

As $\Delta x \to 0$, the right hand side of the equation is the x-derivative of $\frac{\partial y}{\partial x}$. Thus,

$$\frac{\mu}{T}\frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 y}{\partial x^2}$$

Define $v = \sqrt{T/\mu}$ (verify that the dimension of v is the dimension of velocity), we get the wave differential equation,

$$\boxed{\frac{\partial^2 y}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2} = 0}$$

Solution of Wave Differential Equation

1. Use Linearity to show that: If f_1 and f_2 are two solutions of the wave differential equation, then $f_1 + f_2$ is also a solution of the wave differential equation.
2. Exercise: Verify that any function of the form $y(x,t) = f(x+vt)$ or $y(x,t) = f(x-vt)$ is a solution of the wave equation.
3. General Solution: Given the result of the exercise just above, and the linearity of the wave equation, it follows that any function of the form,

   $$y(x,t) = f_1(x-vt) + f_2(x+vt)$$

   is a solution of the wave differential equation.

Homework: Show that the solution $f(x-vt)$ represents an arbitrary wave travelling to the right and the solution $f(x+vt)$ represents an arbitrary wave travelling to the left.

Standing Wave

Before we can learn about standing wave patterns, we first need to understand what happens when a wave encounters an end.

• When a wave pulse comes to a fixed end, it will be reflected and inverted.

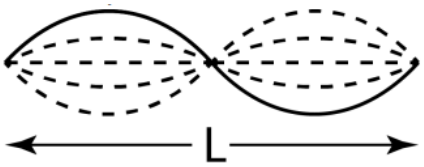
Fixed end = reflection with inversion

• When a wave pulse comes to a free end, it will be reflected, however, it will not be inverted.

Free end = reflection without inversion

Standing wave involves two fixed ends. In other words, the wave pulses are continuously reflected and inverted. One end is at a constant tension and the other end is oscillating up and down in simple harmonic motion at a known frequency. The wave pulses created by the oscillator are sent down the string, reflected and inverted, and then sent back down the string and interfere with the pulses being sent by the oscillator.

Let's start with this visual of a standing wave pattern: The length of the string is L. And you can see there is one wavelength in the length of the string.



• Nodes: Locations where the wave interference causes total destructive interference. There are 3 nodes in the above standing wave.

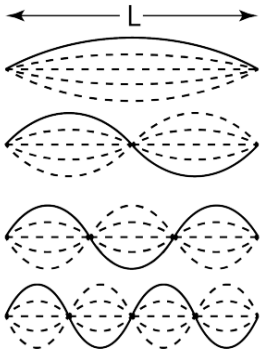
• Anti-nodes: Locations where the wave interference causes constructive interference. There are 2 anti-nodes in the above standing wave.

Standing wave patterns can only be created at specific wavelengths. This is because, in the case of a string, both ends are fixed and are therefore nodes. To the left are some options for possible standing wave patterns in a string. Notice each standing wave pattern is comprised of an integer multiple of half wavelengths.

• The 1st standing wave pattern has 1 half wavelength.

• The 2nd standing wave pattern has 2 half wavelengths.

• The 3rd standing wave pattern has 3 half wavelengths.



It is very important to understand that standing wave patterns will only be created at specific wavelengths and therefore only at specific frequencies. In the example of a string vibrating with two nodes on either end, the standing wave pattern must have an integer multiple of half a wavelength. They are called standing wave patterns because the waves appear to stand still. However, standing waves are not "standing still". They are the constructive and destructive interference of the waves which are traveling back and forth through the medium.

Equation of Displacement of medium in Standing Wave

let's superpose two waves with the same frequency, wavelength, and amplitude, but traveling in opposite directions, which is even fairly easy to do on real strings:

$$y_1 = A\sin(kx - wt)$$

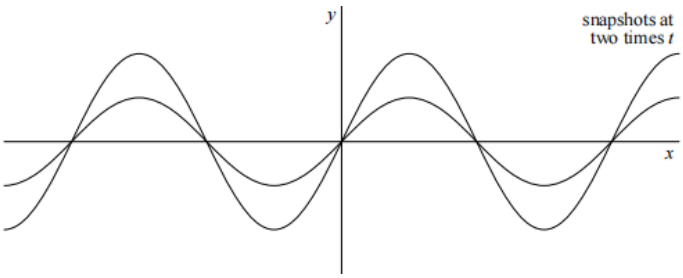
$$y_2 = A\sin(kx + wt)$$

From the superposition principle, the resultant displacement is given by,

$$y = y_1 + y_2 = 2\,A\sin(kx)\cos(\omega t)$$

We know of course that this obeys the wave equation for speed $v = \omega/k$.

Notice that the space and time dependences have factorized. This wave is no longer going anywhere. Below we show two snapshots, one at time $t = 0$, and the other a little later, where$0 < \cos\omega t\ < 1$. Prominent features of the shape such as the peaks and nodes (places where the wave is zero) do not move to the right or left. The antinodes (places where the amplitude has maximum magnitude) move up and down with angular frequency ω. Such waves are called standing waves.



Nodes

The nodes are determined by $\sin(kx) = 0$:

$$kx = \frac{2\pi}{\lambda}x = 0, \pm\pi, \pm 2\pi, \ldots = n\pi$$

$$\boxed{x = n\frac{\lambda}{2}, \qquad n = 0,1,2,\ldots \quad (nodes)}$$

They are separated by half wavelength.

Anti-nodes

The anti-nodes are determined by $\sin(kx) = \pm 1$:

$$kx = \frac{2\pi}{\lambda}x = \pm\frac{\pi}{2}, \pm 3\frac{\pi}{2}, \ldots = (2n+1)\frac{\pi}{2}$$

$$x = (n + \frac{1}{2})\frac{\lambda}{2}, \qquad n = 0,1,2,\ldots \quad (antinodes)$$

They are also separated by half wavelength.

There is stored energy in the up and down motion, but no transmission of energy along the string.

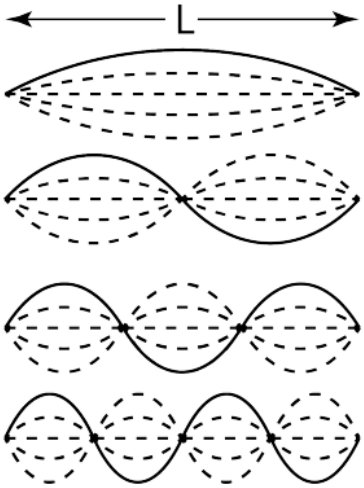
Determining the Speed of a Standing Wave

We demonstrated before, standing wave patterns can only be created at specific wavelengths. To the right are some options for possible standing wave patterns in a string. The equation for the speed of a wave is:

$$v = f\lambda$$

The speed "v" of the wave on the string is constant, therefore: as the frequency of the wave increases, the wavelength of the wave decreases. Because standing wave patterns will only be created at specific wavelengths and therefore only specific frequencies. Frequencies that will create standing wave patterns on this string are:

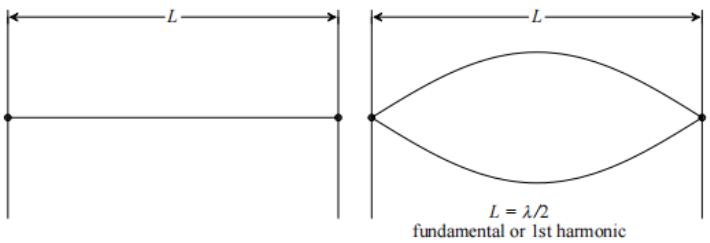
- The 1st standing wave pattern: $L = 1\left(\frac{1}{2}\lambda\right) \Rightarrow \lambda = 2L$
- The 2nd standing wave pattern: $L = 2\left(\frac{1}{2}\lambda\right) \Rightarrow \lambda = L$
- The 3rd standing wave pattern: $L = 3\left(\frac{1}{2}\lambda\right) \Rightarrow \lambda = \frac{2L}{3}$
- The 4th standing wave pattern: $L = 4\left(\frac{1}{2}\lambda\right) \Rightarrow \lambda = \frac{L}{2}$
- The 5th standing wave pattern: $L = 5\left(\frac{1}{2}\lambda\right) \Rightarrow \lambda = \frac{2L}{5}$



Boundary Conditions and Normal Modes

Standing wave nodes and antinodes play a special role in problems with boundary conditions. When the ends of a string, say at $x = 0$ and $x = L$, are tied down so they can't move, they are forced to be nodes. The solutions $y(x,t)$ of the wave equation for such a problem are said to obey the boundary conditions $y(0,t) = y(L,t) = 0$, and the mathematical problem is to find all solutions for $0 \le x \le L$ which obey those boundary conditions.

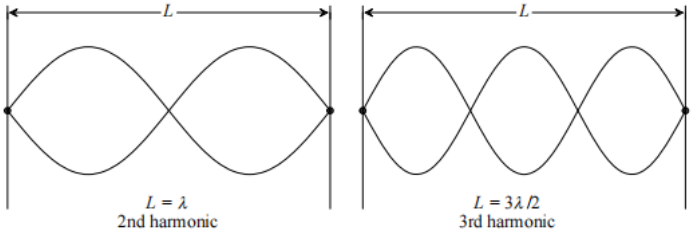
We already know a lot of solutions from our discussion of nodes in harmonic standing waves. Let's classify them by drawing pictures. The first two solutions are shown below, the trivial solution, with $y = 0$, and the fundamental solution or first harmonic, where the first node to the right of $x = 0$ is at the end, $x = L$.



The next two are called the second and third harmonics, with one and two nodes, respectively, between the ends.

The pattern is clear. At each step, we fit one more node corresponding to one more half wavelength between the ends. The rule for the wavelengths and frequencies of the standing waves is,

$$\lambda_n = \frac{2L}{n}, \quad n = 1,2, \ldots$$



$$f_n = \frac{v}{\lambda_n} = \frac{nv}{2L} = \frac{n}{2L}\sqrt{\frac{T}{\mu}}$$

Note that the harmonic frequencies go up in equal steps from the fundamental frequency: $f_n = n\, f_1$. The wavenumbers and angular frequencies are,

$$k_n = \frac{2\pi}{\lambda_n} = \frac{n\pi}{L}$$

$$\omega_n = 2\pi f_n = \frac{n\pi v}{L} = \frac{n\pi}{L}\sqrt{\frac{T}{\mu}}$$

The frequencies f_n are called normal mode frequencies, and the corresponding solutions of the wave equation are called normal mode solutions, or normal modes:

$$y_n = A\sin(k_n x)\cos(\omega_n t) = A\sin(\frac{n\pi}{L}x)\cos(\frac{n\pi v}{L}t)$$

These are in fact all of the solutions, in the sense that any solution can be written as a superposition of these.