NP#1. A particle of mass 3 moves along the x-axis attracted toward origin by a force whose magnitude is numerically equal to $12x$. The particle is also subjected to a damping force whose magnitude is numerically equal to 12 times the intaneous speed. If it is initially at rest at $x=10$, find the position and the velocity of the particle at any time.

Soln The equation of motion of the particle is,

$$F_N = F_D + F_H$$

$$\Rightarrow ma = -bV - Kx$$

$$\Rightarrow 3\frac{d^2x}{dt^2} = -12\frac{dx}{dt} - 12x$$

$$\Rightarrow \frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 4x = 0 \quad \text{— (i)}$$

Let the particle is critically damped.

So, $2\beta = 4$ $\quad \omega_0^2 = 4$

$\Rightarrow \beta = 2.$ $\quad \therefore \omega_0 = 2$

The solution of eqn ① is,

$$x = e^{-2t}(A_1 + A_2 t)$$

When $t = 0$, $x = 10$ and $v = 0$.

$A_1 = 10$ and $A_2 = 20$

The position of the particle at any time t is,

$$x = 10e^{-2t}(1 + 2t)$$

The velocity is given by,

$$\frac{dx}{dt} = -20e^{-2t} + 20e^{-2t} - 40te^{-2t}$$

$$\therefore v(t) = -40te^{-2t}$$

Initially vector $v = 0$ and the velocity becomes very small after a long time. The magnitude of the velocity is maximum when $t = 1/2$.

NP#2. A particle of mass 1gm moves along the x-axis under the influence of two forces:

i) a force of attraction toward origin which is numerically equal to 4x dynes,

ii) A damping force whose magnitude in dynes is numerically equal to twice the instaneous speed. Assuming that the particle starts from rest at a distance 10cm from the origin,

(a) Set up the differential equation of motion of the particle.

Soln The differential equation for the motion of the particle is,

$$F_N = F_D + F_H$$

$$\Rightarrow m\frac{d^2x}{dt^2} = -bv - kx$$

$$\Rightarrow 1.\frac{d^2x}{dt^2} = -2\frac{dx}{dt} - 4x$$

$$\therefore \frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 4x = 0 \quad \text{---} (i)$$

Eqn ① is an example where the motion is damped oscillation,

where,

$2\beta = 2$ $\qquad$ $\omega_0^2 = 4$

$\therefore \beta = 1$ $\qquad$ $\therefore \omega_0 = 2$

(b) Find the position of the particle at any time.

Soln The solution of eqn ① is,

$$x(t) = Re^{-t}\cos(\sqrt{3}t - \theta) \quad [\beta^2 < \omega_0^2]$$

[under damped]

$\omega = \sqrt{\omega_0^2 - \beta^2}$

$= \sqrt{2^2 - 1^2}$

$= \sqrt{3}$

since, $x = 10$ cm at $t = 0$,

we find that,

$$R\cos\theta = 10$$

since, $v = 0$ at $t = 0$,

$$-Re^{-t}\cos\theta + \sqrt{3}R\sin\theta . e^{-t} = 0$$

$$\Rightarrow -R\cos\theta + \sqrt{3}R\sin\theta = 0$$

Hence, $\theta = \frac{\pi}{6}$ and $R = \frac{20}{\sqrt{3}}$ cm

Thus, we obtain,

$$x = \frac{20}{\sqrt{3}} e^{-t} \cos\left(\sqrt{3}t - \frac{\pi}{6}\right)$$

(Ans:)

(c) Determine the amplitude, period and frequency of the damped oscillation.

Soln From (b),

$$\text{Amplitude, } A = \frac{20}{\sqrt{3}} e^{-t} \text{ cm}$$

$$\omega = \frac{2\pi}{T}$$

$$\Rightarrow \sqrt{3} = \frac{2\pi}{T}$$

$$\therefore \text{Period, } T = \frac{2\pi}{\sqrt{3}} \text{ s}$$

$$\omega = 2\pi f$$

$$\Rightarrow \sqrt{3} = 2\pi f$$

$$\therefore \text{frequency, } f = \frac{\sqrt{3}}{2\pi} \text{ Hzs.}$$

NP#3. In a damped oscillatory motion an object oscillates with a frequency of 1 Hz and its amplitude of vibration is halved in 5S. Find the differential equation for the oscillation.

Soln

The variation of x with t corresponds to an underdamped decaying oscillation with differential eqn,

$$\frac{d^2x}{dt^2} + 2\beta \frac{dx}{dt} + \omega_0^2 x = 0 \quad -(i)$$

where $\beta^2 < \omega^2$. The solution of this equation is,

$$x = Re^{-\beta t} \cos\left[\sqrt{\omega_0^2 - \beta^2}\, t - \theta\right] = 0$$

It is given that

$$\omega = \sqrt{\omega_0^2 - \beta^2} = 2\pi f$$

$$\therefore \omega = 2\pi . 1 = 2\pi \text{ Rad } S^{-1}$$

Again,

$$Re^{-5\beta} = \frac{R}{2}$$

$$\Rightarrow e^{-5\beta} = \frac{1}{2}$$

$$\Rightarrow -5\beta = -0.693$$

$$\Rightarrow \beta = 0.139$$

$$\Rightarrow 2\beta = 0.278$$

$$\omega^2 = (2\pi)^2 = 39.50 = \omega_0^2 - \beta^2$$

From $\Rightarrow \omega_0^2 = 39.50 + 0.0193$

$$\omega_0^2 = 39.52$$

From eqn (i),

$$\frac{d^2x}{dt^2} + 0.278\frac{dx}{dt} + 39.52x = 0$$

(Ans:)

PNP#4. The energy of recoil of a Rocket launcher of mass = 4500kg is absorbed in a recoil spring. At the end of the recoil, a damping dashpot is arranged in such a way that the launcher returns to the firing position without any oscillation on [@.D]. The launcher recoils 3m with an initial speed of 10m/s. Find

(a) the recoil's spring constand and

(b) the dashpot's co-efficient of critical damping.

(a)

We use the principle of conservation of energy for the rocket launcher and the recoil spring,

$$KE + P.E. = \text{constant}$$

Thus, $\frac{1}{2}mV_0^2 + 0 = 0 + \frac{1}{2}Kx^2_{max}$

$$\Rightarrow K = \frac{mV_0^2}{x^2_{max}}$$

$$= \frac{4500 \times 10^2}{3^2} = 50kNm^{-1}$$

(b)

The co-efficient of critical damping is given by,

$$c = 2m\beta$$
$$= 2m\omega \quad [\beta = \omega]$$
$$= 2\sqrt{mk}$$
$$= 2\sqrt{4500 \cdot 50000}$$
$$= 30000 \text{ N S m}^{-1}$$
$$= 30 \text{ KN S m}^{-1} \quad \text{(Ans)}$$

NP#5. Solve the Problem of simple Pendulum if a damping force Proportional to the instantaneous tangential velocity is taken into account. (over, critically and under damped motion).

Soln The instantaneous tangential velocity of a simple pendulum of length l and mass m is

$$ml\frac{d\psi}{dt}$$

The eqn of motion of the damped simple Pendulum is given by

$$ml\frac{d^2\psi}{dt^2} = -mg\sin\psi - \beta l\frac{d\psi}{dt}$$

Replacing $\sin\psi$ by ψ for small oscillations, we have the equation

$$\frac{d^2\psi}{dt^2} + 2\beta\frac{d\psi}{dt} + \omega^2\psi = 0$$

where, $\beta = \frac{2b}{2m}$ and $\omega = \sqrt{\frac{g}{l}}$

For critically damped motion,

$$\frac{b^2}{4m^2} = \frac{g}{l}$$

$$\psi = e^{\frac{-bt}{2m}}[A_1 + A_2 t]$$

For under damped motion,

$$\frac{b^2}{4m^2} < \frac{g}{l}$$

$$\psi = e^{\frac{-2bt}{2m}}[B_1\cos\omega t + B_2\sin\omega t]$$

where,

$$\omega = \left[\frac{g}{l} - \frac{\beta^2}{4m^2}\right]^{\frac{1}{2}}$$

For over damped motion,

$$\frac{b^2}{4m^2} > \frac{g}{l}$$

$$\psi = e^{\frac{-bt}{2m}}\left[A_1 \cdot e^{\omega t} + A_2 e^{-\omega t}\right]$$

$$\omega = \left[\frac{b^2}{4m^2} - \frac{g}{l}\right]^{1/2}$$

NO#6. (a) A steady force of 50 N is required to lift of a mass of 2kg vertically through water at a constant of 2·5m/s. Assuming that the effect of viscosity can be described by a force proportional to velocity, determine the constant of proportionality.

Soln When the mass is lifted vertically it moves with a constant velocity.

Thus the net force acting on it is zero,

$$50 - mg - bV = 0$$

$$\Rightarrow b = \frac{50 - mg}{V}$$

$$= \frac{50 - 2 \times 9.8}{2.5}$$

$$= 12.16 \text{ NSm}^{-1} \quad \text{(Ans)}$$

(b) The same mass is then suspended in water by a spring with $K = 120 \text{ Nm}^{-1}$. Determine the equilibrium extension of the spring. The mass is further pulled through a small distance below its equilibrium position and released from rest at time $t = 0$. Show that it will vibrate about the equilibrium position according to an equation of the form

$$\frac{d^2x}{dt^2} + 2b\beta \frac{dx}{dt} + \omega^2 x = 0$$

and determine b and ω for this system. Show that the motion is under damped and find its period of oscillation. Find the

in which the amplitude of oscillation fa
by a factor of e.

Soln When the mass is suspended by a spring of spring constant K, it will be stationary at equilibrium and no viscous force will act on it. If x is the extension of the spring. we have,

$$mg = Kx$$

$$\Rightarrow x = \frac{mg}{K}$$

$$= \frac{2 \times 9.8}{120}$$

$$= 0.163 \text{ m}$$

When the mass is pulled through a small distance below its equilibrium position and then released, it will vibrate to and from about equilibrium position. Now a resistive force of amount bv will act on the body. Therefore the equation of motion is given by,

$$m\frac{d^2x}{dt^2} = -b\frac{dx}{dt} - Kx$$

$$\Rightarrow \frac{d^2x}{dt^2} + 2\beta\frac{dx}{dt} + \omega^2 x = 0$$

where,

$$\beta = \frac{b}{2m} = \frac{12.16}{2\times 2} = 3.04\ s^{-1}$$

$$\omega_0 = \sqrt{\frac{K}{m}} = \sqrt{\frac{120}{2}} = 7.746\ rad\,s^{-1}$$

Since $b < \omega_0$, the body will execute damped simple harmonic motion:

$$x(t) = Re^{-Bt}\cos(\omega t - \theta)$$

$$\text{Period, } T = \frac{2\pi}{\omega}$$

$$= \frac{2\pi}{\sqrt{\omega_0^2 - \beta^2}}$$

$$= \frac{2\pi}{\sqrt{7.746^2 - 3.04^2}}$$

$$= 0.88\,s$$

If t is the time in which the amplitude falls by a factor of e, then

$$Re^{-\beta t} = \frac{R}{e}$$

$$\Rightarrow t = \frac{1}{B} = \frac{1}{3.04} = 0.33\,s$$

NP# 7. An automobile suspension system is critically damped and its period of oscillation with no damping is one second. If the system is initially displaced by an amount x_0 and release with zero initial velocity. Find the displacement at $t = 2s$.

Soln The displacement for critically damped motion is given by,

$$x(t) = e^{-\beta t}(A_1 + A_2 t)$$

$$\beta = \omega = \frac{2\pi}{T} = \frac{2\pi}{1} = 2\pi \text{ rad s}^{-1}$$

Now, $\frac{dx(t)}{dt} = -\beta e^{-\beta t}(A_1 + A_2 t) + A_2 e^{-\beta t}$

At $t = 0$, $x_0 = A_1$ and

$$\frac{dx(t)}{dt} = 0$$

$$\Rightarrow -bA_1 + A_2 = 0 \quad [t = 0]$$

— (i)

From ①,

$$-2\pi x_0 + A_2 = 0$$

$$\therefore A_2 = 2\pi x_0$$

So, $x(t) = x_0 (1 + 2\pi t) e^{-2\pi t}$

For, $t = 2S$,

displacement, $x = 0.000047 x_0$

N.P# 8. An LCR circuit has $L = 10mH$, $C = 1\,MF$ and $R = 1\,\Omega$.

(a) After what time t will the amplitude of the charge oscillations drop to one-half of its initial value?

Soln Amplitude of charge oscillation $= q e^{-\beta t}$

we have the condition, $e^{-\beta t} = \frac{1}{2}$

$$\Rightarrow -\beta t = \ln \frac{1}{2}$$

$$\Rightarrow t = \frac{\ln 2}{\beta}$$

$$= \frac{(\ln 2)\,2L}{R}$$

$$= \frac{(\ln 2) \times 2 \times 10 \times 10^{-3}}{1}$$

$$= 13.86 \text{ ms.}$$

(b) To how many periods of oscillation does this correspond?

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{10 \times 10^{-3} \times 10^{-6}}}$$

$$= 10^4 \text{ rad/s}$$

$$\beta = \frac{R}{2L} = 50$$

Since $\omega_0 >> b$. $\omega \approx \omega_0$

The elapsed time, expressed in terms of the period of oscillation, is,

$$N = \frac{t}{T} = \frac{\omega t}{2\pi} \approx 22.$$

Thus the amplitude drops to one-half after about

22 cycles of oscillation.

NP#9. A periodic external force acts on a 3kg mass suspended from the lower end of a vertical spring having $K = 75 Nm^{-1}$. The damping force is proportional to the instantaneous speed of the mass and is 20N when the speed is $1 ms^{-1}$. Find the frequency at which amplitude resonance occurs.

Soln

Natural angular frequency of the spring,

$$\omega_0 = \sqrt{\frac{K}{m}}$$

$$= \sqrt{\frac{75}{3}} = 5 rads^{-1}$$

Damping force, $= bV = 20 \quad | V = 1 ms^{-1}$

$$\Rightarrow b = 20 Nsm^{-1}$$

$$\beta = \frac{b}{2m} = \frac{20}{2 \times 3} = \frac{10}{3} s^{-1}$$

For amplitude resonance, angular frequency,

$$\omega = \sqrt{\omega_0^2 - 2\beta^2} = \sqrt{5^2 - 2\left(\frac{10}{3}\right)^2} = \frac{5}{3} rads^{-1}$$

and

Resonance frequency $= \frac{\omega}{2\pi}$

$= \frac{5}{6\pi}$ HZ

NP#10. The mass on a vertical spring undergoes forced vibrations according to the eqn

$$\frac{d^2x}{dt^2} + \omega^2 x = f \sin \omega t$$

where there is no damping and the impressed frequency is equal to the natural frequency of oscillation.

(a) Obtain the solution of the above differential equation.

Soln The general solution of the eqn is,

$$\frac{d^2x}{dt^2} + \omega^2 x = f \sin \omega t \quad \text{--- (1)}$$

$$x = x_1 + x_2$$

where x_1 is the general solution of the homogeneous equation,

$$\frac{d^2x_1}{dt^2} + \omega_0^2 x_1 = 0$$

and x_2 is the particular integral of eqn. Now, the general solution of Eqn is

$$x_1 = A\cos\omega t + B\sin\omega t$$

The particular solution of eqn has the form,

$$x = t(c_1\cos\omega t + c_2\sin\omega t)$$

which gives,

$$\frac{dx}{dt} = v = (c_1\cos\omega t + c_2\sin\omega t) + t(-\omega c_1\sin\omega t + \omega c_2\cos\omega t)$$

$$\frac{d^2x}{dt^2} = a = 2(-\omega c_1\sin\omega t + \omega c_2\cos\omega t) - t\omega^2 \cdot (c_1\cos\omega t + c_2\sin\omega t)$$

Substituting these in eqn (i), we obtain,

$$2c_2\omega\cos\omega t - 2c_1\omega\sin\omega t = f\sin\omega t$$

from which,

$$C_2 = 0$$

and $C_1 = -\frac{f}{2\omega}$.

Thus the particular integral is,

$$x_2 = -\frac{ft}{2\omega}\cos\omega t$$

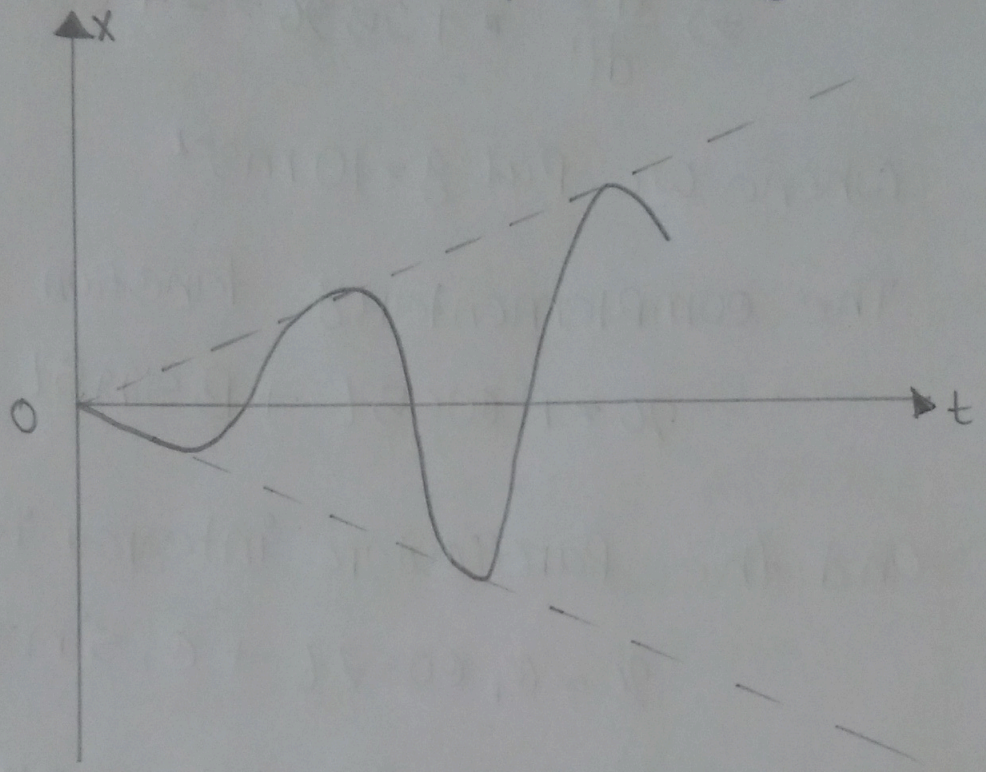
The general solution of eqn is,

$$x = A\cos\omega t + B\sin\omega t - \frac{ft}{2\omega}\cos\omega t. \quad \text{——(i)}$$

(b) Give a physical interpretation.

Soln: The first two terms to eqn (i) are oscillatory with constant amplitude. The last term involving t increase with time to such an extent that spring brea finally. A graph of the last term is show in figure. This example illustrates the Phenomenon of resonance. Here the ral frequency of the spring equals t

frequency of the impressed force.



NP#11. A vertical spring has a spring constant 50 Nm^{-1}. At $t=0$ a force given in newtons by $F(t) = 48\cos 7t$, $t \geq 0$ is applied to a 20N weight which hangs in equilibrium at the end of the spring. Neglecting damping find the position of the weight at any later time t.

soln. We have the equation of motion,

$$\frac{20}{g} \cdot \frac{d^2x}{dt^2} = -50x + 48\cos 7t$$

$$\Rightarrow \frac{d^2x}{dt^2} + 25x = 24\cos 7t \quad \text{— (i)}$$

Where we put $g = 10\,ms^{-2}$

The complementary function of eqn is

$$x = A\cos 5t + B\sin 5t \quad \text{— (ii)}$$

and the particular integral is given by,

$$x = C_1\cos 7t + C_2\sin 7t \quad \text{— (iii)}$$

Substituting eqn (iii) into eqn (i), we get $C_1 = -1$ and $C_2 = 0$.

Thus the general solution is,

$$X = A\cos 5t + B\sin 5t - \cos 7t \quad \text{— (iv)}$$

Using the initial conditions,

$$x = 0,\ v = 0 \text{ at } t = 0$$

we find, $A = 1$, $B = 0$ and thus

$$x = \cos 5t - \cos 7t = 2\sin t.\sin 6t$$