Lecture 02

System of Linear Algebraic Equations

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Linear Equation

- Linear equations are equations having variables with power 1.
- Ax + b = 0 is an example with one variable where x is the variable, a and b are real numbers.
- Example:

$$2x+3y=7$$

$$4a+3b+5c=12$$

- If the number of unknowns equals the number of equations, the system has a unique solution.

$$2x+3y=7$$

$$3x+5y=11$$

Linear Equation

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots = b_3$$
$$\vdots$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots \\ a_{21} & a_{22} & a_{23} & \cdots \\ a_{31} & a_{32} & a_{33} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \end{bmatrix}$$

AX=B

Linear Equation

* A system of equations is called homogeneous if all the constant terms $b_1, b_2, \ldots\ldots, b_n$ are zero, i.e.

$$b_1 = b_2 = b_3 = \ldots\ldots b_n = 0$$

Otherwise, the system is non-homogeneous.

AX=B

where A is an n×n matrix, known as the coefficient matrix, B is an n vector and X is a vector of n unknowns known as the solution vector.

Methods of Solution

Direct or elimination methods

- Matrix Inversion method
- Gauss Elimination method
- Gauss-Jordan method
- LU Decomposition or Triangularization or Factorization method

Iteration or indirect methods

- Gauss-Seidel method
- Jacobi method
- Relaxation method

Existence of Solution

There are four possibilities:

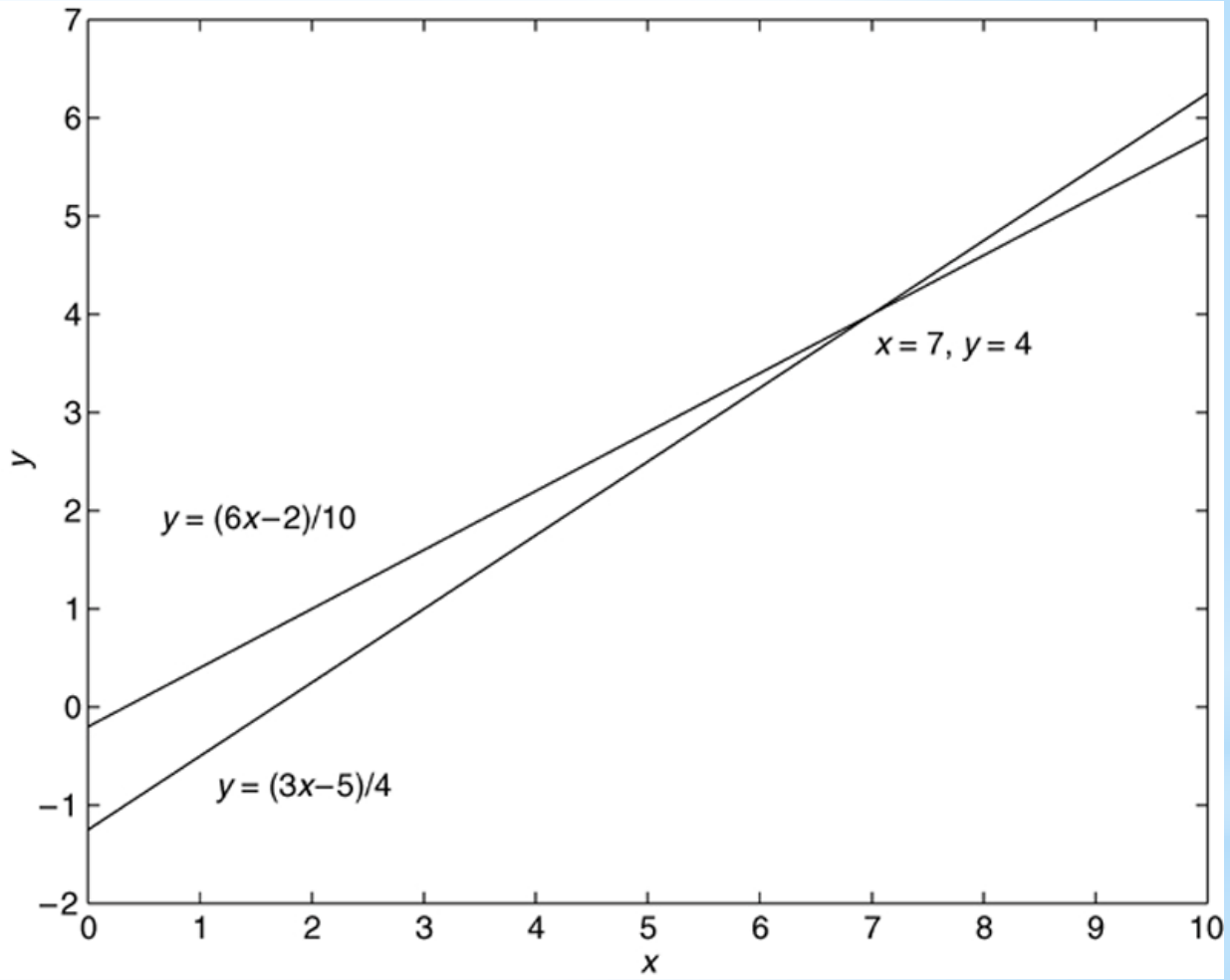
1. System has a unique solution
2. System has no solution
3. System has infinite solutions
4. System is ill conditioned

Unique Solution

$$3x - 4y = 5$$
$$6x - 10y = 2$$

- The equations have graphs that intersect at the solution

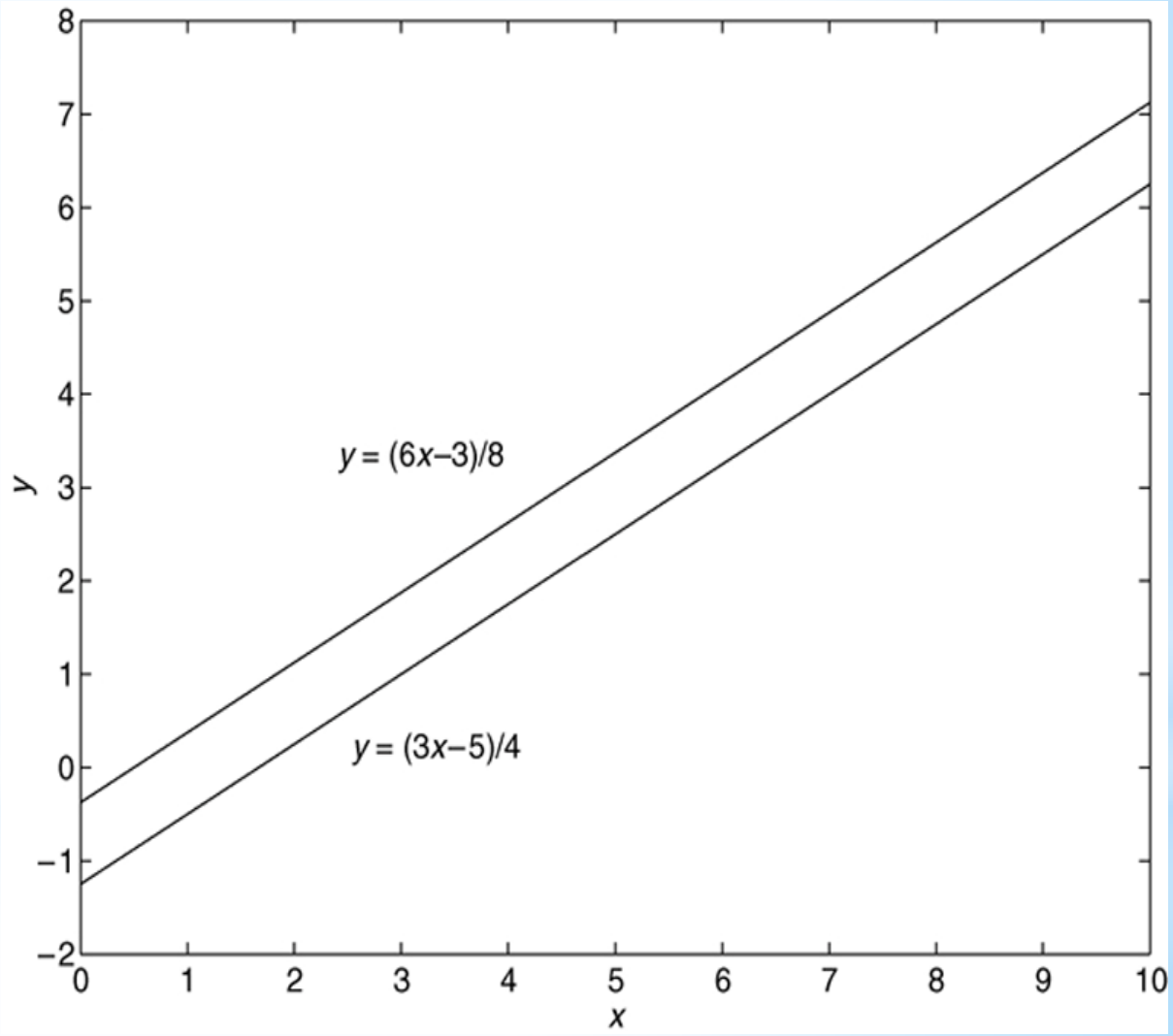
  y = 4, x = 7.



No Solution

$$3x - 4y = 5$$
$$6x - 8y = 3$$

- The set is singular. The graphs of these two equations are distinct but *parallel.* Because they do no intersect, no solution exists.



Infinite Solutions

$$3x - 4y = 5$$

$$6x - 8y = 10$$

* The set is singular and has no unique solution because the second equation is identical to the first equation, multiplied by 2. The graphs of these two equations are identical. All we can say is that the solution must satisfy $y = (3x - 5)/4$, which describes an infinite number of solutions.
* The systems of equations which represent either parallel lines (having no solution)or the same line (having an infinite number of solutions) are known as singular systems ($|A|=0$).

Ill-Conditioned systems

* A system of equations in which small changes in the coefficients of the system produces large changes in the solution is said to be ill-conditioned or unstable.
* On the other hand, if small changes in the coefficients produce small changes in the solution, the system is well-conditioned or stable.
* The system

$$2x+y=2$$

$$2x+1.01y=2.01$$

has the solution x=.5 and y=1

Ill-Conditioned systems

$$2x+y=2$$

$$2.01x+y=2.05$$

has the solution $x=5$ and $y=-8$

Conditions

- Number of equations=number of unknowns → **Unique Solution**
- Number of equations<number of unknowns → **Under-determined**
- Number of equations>number of unknowns → **Over-determined**

Matrix Inversion Methods

* Consider the system AX=B of n linear equations in n unknowns. Let A be non-singular ($|A| \neq 0$) so that A^{-1} exists. Then multiplying both sides by A^{-1},we obtain

$$A^{-1} AX = A^{-1} B$$

$$X = A^{-1} B$$

since $AA^{-1} = I$ and $IX = X$. If A^{-1} is known, then the solution vector X can be found out from the above matrix relation.

Example 1.1

Solve the equations by matrix inversion method:

$$3x+y+2z=3$$

$$2x-3y-z=-3$$

$$x+2y+z=4$$

Solution:

$$|A|=\begin{vmatrix}3 & 1 & 2\\ 2 & -3 & -1\\ 1 & 2 & 1\end{vmatrix}$$

$= 3(-3+2)-1(2+1)+2(4+3)$

$= -3-3+14$

$= 8$

$$\text{Adj } (A)=\begin{bmatrix}-1 & 3 & 5\\ -3 & 1 & 7\\ 7 & -5 & -11\end{bmatrix}$$

(Show full calculation here)

$$A^{-1} = \frac{1}{8}\begin{bmatrix}-1 & 3 & 5\\ -3 & 1 & 7\\ 7 & -5 & -11\end{bmatrix}$$

Example 1.1

$$X=\begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1} B$$

$$=\frac{1}{8}\begin{bmatrix} -1 & 3 & 5 \\ -3 & 1 & 7 \\ 7 & -5 & -11 \end{bmatrix}\begin{bmatrix} 3 \\ -3 \\ 4 \end{bmatrix}$$

$$=\frac{1}{8}\begin{bmatrix} 8 \\ 16 \\ -8 \end{bmatrix}$$

$$=\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

x=1, y=2 and z=-1

Methods Using Determinants: Cramer's Rule

* Consider a 3×3 square matrix AX=B where

$$A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad X=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad \text{and} \quad B=\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Let A_1, A_2 and A_3 are the determinants obtained from A by replacing the first, second and third columns by $[b_1, b_2, b_3]$,

$$A_1=\begin{bmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{bmatrix}, \quad A_2=\begin{bmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{bmatrix} \quad \text{and} \quad A_3=\begin{bmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{bmatrix}$$

Methods Using Determinants: Cramer's Rule

* Then according to Cramer's rule,

$$x_1 = \frac{|A_1|}{|A|},$$

$$x_2 = \frac{|A_2|}{|A|} \text{ and}$$

$$x_3 = \frac{|A_3|}{|A|}$$

Example 1.2

* Solve the equations by the Cramer's rule:

$$3x+y+2z=3$$

$$2x-3y-z=-3$$

$$x+2y+z=4$$

Solution:

$$|A|=\begin{vmatrix}3 & 1 & 2\\ 2 & -3 & -1\\ 1 & 2 & 1\end{vmatrix}$$

$$= 3(-3+2)-1(2+1)+2(4+3)$$

$$= -3-3+14$$

$$= 8$$

$$|A_1|=\begin{vmatrix}3 & 1 & 2\\ -3 & -3 & -1\\ 4 & 2 & 1\end{vmatrix}$$

$$= 3(-3+2)-1(-3+4)+2(-6+12)$$

$$= -3-1+12$$

$$= 8$$

Example 1.2

$$|A_2| = \begin{vmatrix} 3 & 3 & 2 \\ 2 & -3 & -1 \\ 1 & 4 & 1 \end{vmatrix}$$

$= 3(-3+4)-3(2+1)+2(8+3)$

$= 3-9+22$

$=16$

$$|A_3| = \begin{vmatrix} 3 & 1 & 3 \\ 2 & -3 & -3 \\ 1 & 2 & 4 \end{vmatrix}$$

$= 3(-12+6)-1(8+3)+3(4+3)$

$= -18-11+21$

$= -8$

$$x = \frac{|A_1|}{|A|} = \frac{8}{8} = 1$$

$$y = \frac{|A_2|}{|A|} = \frac{16}{8} = 2$$

$$z = \frac{|A_3|}{|A|} = \frac{-8}{8} = -1$$

Practice Problems:

1.3 Solve the equations by the matrix inversion method:

$$3x+2y+z=10$$

$$2x+3y+2z=14$$

$$x+2y+3z=14$$

1.4 Solve the equations by the Cramer's rule:

$$2x+3y+z=9$$

$$x+2y+3z=6$$

$$3x+y+2z=8$$

Gauss Elimination Method

- The Gaussian elimination method is known as the row reduction algorithm for solving linear equations systems. It consists of a sequence of operations performed on the corresponding matrix of coefficients.
- To perform row reduction on a matrix, we have to complete a sequence of elementary row operations to transform the matrix till we get 0s (i.e., zeros) on the lower left-hand corner of the matrix as much as possible. That means the obtained matrix should be an upper triangular matrix. There are three types of elementary row operations; they are:

→Swapping two rows and this can be expressed using the notation ↔, for example, R2 ↔ R3

→Multiplying a row by a nonzero number, for example, R2 → kR2 where k is some nonzero number

→Adding a multiple of one row to another row, for example, R2 → R2 + 3R1

Gauss Elimination Method

The basic Gauss Elimination method for solving a system of linear equations involves two phases:

1. **Elimination phase**: This phase eliminates some unknowns successively from the system of equations using the forward elimination approach and reduces the system of equations to an equivalent upper triangular form.
2. **Back substitution phase**: In this phase, the reduced upper triangular system is solved by the back substitution to obtain the values of unknowns.

Gauss Elimination Method

Example 1.5

Solve the equations:

$$3x+y+2z= 3 \quad \ldots\ldots\ldots\ldots\ldots\ldots(i)$$

$$2x-3y-z= -3 \quad \ldots\ldots\ldots\ldots\ldots\ldots(ii)$$

$$x+2y+z=4 \quad \ldots\ldots\ldots\ldots\ldots\ldots(iii)$$

Solution:

Eliminating x from 2nd and 3rd equations:

(ii)- $\frac{2}{3} \times$ (i)

$$2x- 3y- z- 2x- \frac{2}{3} y- \frac{4}{3}z=-3-2$$

$$\text{or, } \frac{-11}{3} y - \frac{7}{3} z= -5 \quad \ldots\ldots\ldots\ldots\ldots\ldots(iv)$$

$$\begin{bmatrix} 3 & 1 & 2 \\ 2 & -3 & -1 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 1 & 2 \\ 2 & -3 & -1 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \\ 4 \end{bmatrix}$$

Gauss Elimination Method

(iii)- $\frac{1}{3} \times$ **(i)**

$$x+ 2y+ z- x- \frac{1}{3} y- \frac{2}{3}z = 4-1$$

$$\text{or, } \frac{5}{3} y + \frac{1}{3} z= 3 \quad \ldots\ldots\ldots\ldots\ldots\ldots(v)$$

Now eliminating y:

(iv)+ $\frac{11}{5} \times$ **(v)**

$$\frac{-11}{3} y - \frac{7}{3} z + \frac{11}{3} y + \frac{11}{15} z =-5+ \frac{33}{5}$$

$$\text{or, } \frac{-24}{15} z = \frac{8}{5}$$

$$z=-1$$

$$\begin{bmatrix} 3 & 1 & 2 \\ 0 & \frac{-11}{3} & \frac{-7}{3} \\ 0 & \frac{5}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -5 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 1 & 2 \\ 0 & \frac{-11}{3} & \frac{-7}{3} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -5 \\ 1 \end{bmatrix}$$

Gauss Elimination Method

From (v), since z=-1,

$$\frac{5}{3}y+\frac{1}{3}\times -1=3$$

$$\text{or, } \frac{5}{3}y=3+\frac{1}{3}$$

$$\text{or, } \frac{5}{3}y=\frac{10}{3}$$

$$y=2$$

From (iii), since y= 2 and z= -1 ,

$$x+(2\times 2)+(-1)=4$$

$$x=1$$

Gauss Jordan Method

* This is a modification of the Gauss elimination method.
* The method does not require back substitution to obtain the solution.

Gauss Jordan Method

Example 1.6

* Solve the equations:

$3x+y+2z=3$ ……..(i)

$2x-3y-z=-3$ ……(ii)

$x+2y+z=4$ …….(iii)

Solution:

Eliminating x from 2nd and 3rd equations:

$\frac{-11}{3}y-\frac{7}{3}z=-5$ …….(iv)

$\frac{5}{3}y+\frac{1}{3}z=3$ …….(v)

$$\begin{bmatrix} 3 & 1 & 2 \\ 2 & -3 & -1 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 1 & 2 \\ 0 & \frac{-11}{3} & \frac{-7}{3} \\ 0 & \frac{5}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -5 \\ 3 \end{bmatrix}$$

Gauss Jordan Method

$$3x+y+2z=3 \quad \text{……..(i)}$$

$$\frac{-11}{3}y - \frac{7}{3}z = -5 \quad \text{…….(iv)}$$

$$\frac{5}{3}y + \frac{1}{3}z = 3 \quad \text{…….(v)}$$

Now eliminating y from 1st and 5th equations:

(i)+(iv) $\times \frac{3}{11}$

$$3x+y+2z - y - \frac{7}{11}z = 3 - \frac{15}{11}$$

$$\text{or, } 3x + \frac{15}{11}z = \frac{18}{11} \quad \text{…….(vi)}$$

$$\begin{bmatrix} 3 & 1 & 2 \\ 0 & \frac{-11}{3} & \frac{-7}{3} \\ 0 & \frac{5}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -5 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 0 & \frac{15}{11} \\ 0 & \frac{-11}{3} & \frac{-7}{3} \\ 0 & \frac{5}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{18}{11} \\ -5 \\ 3 \end{bmatrix}$$

Gauss Jordan Method

(v) +(iv) $\times \frac{5}{11}$

$$\frac{5}{3}y + \frac{1}{3}z - \frac{5}{3}y - \frac{35}{33}z = 3 - \frac{25}{11}$$

$$\text{or, } \frac{-24}{33}z = \frac{8}{11}$$

$$z = -1$$

$$\begin{bmatrix} 3 & 0 & \frac{15}{11} \\ 0 & \frac{-11}{3} & \frac{-7}{3} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{18}{11} \\ -5 \\ -1 \end{bmatrix}$$

From (v), since z=-1 ,

$$\frac{5}{3}y + \frac{1}{3} \times (-1) = 3$$

$$\text{or, } 5y - 1 = 9$$

$$y = 2$$

From (vi), z= -1 ,

$$3x + \frac{15}{11}(-1) = \frac{18}{11}$$

$$\text{or, } x = 1$$

Gauss Jordan Method

Example 1.7

* Solve the equations:

$$x_1+2x_2-3x_3= -4 \quad \text{.....(i)}$$
$$x_1+3x_2+x_3= 10 \quad \text{.....(ii)}$$
$$2x_1-4x_2-2x_3= -12 \quad \text{.....(iii)}$$

Solution:

Eliminating x_1 from 2nd and 3rd equations:

(ii)- (i)

$$x_1+3x_2+x_3 -x_1-2x_2+3x_3 = 10+4$$
$$x_2+ 4x_3 = 14\text{.....(iv)}$$

(iii)- (i)×2

$$2x_1-4x_2-2x_3 -2x_1-4x_2+6x_3 = -12+8$$
$$-8x_2+ 4x_3 = -4\text{.....(v)}$$

Gauss Jordan Method

$$x_1+2x_2-3x_3= -4 \quad \ldots\ldots\ldots\ldots\ldots\ldots(i)$$

$$x_2+ 4x_3 = 14 \quad \ldots\ldots\ldots\ldots\ldots\ldots(iv)$$

$$-8x_2+ 4x_3 = -4 \ldots\ldots\ldots\ldots\ldots\ldots(v)$$

Eliminating x_2 from 1st and 5th equations:

(i)- (iv)×2

$$x_1+2x_2-3x_3-2x_2-8x_3 = -4-28$$

$$\text{or, } x_1-11x_3 = -32\ldots\ldots\ldots\ldots\ldots(vi)$$

(v)+8(iv)

$$-8x_2+4x_3 +8x_2+32x_3 = -4+112$$

$$\text{or, } 36x_3 = 108$$

$$\text{or, } x_3 = 3\ldots\ldots\ldots(vii)$$

Gauss Jordan Method

$$x_1 - 11x_3 = -32 \ldots\ldots\ldots\ldots\ldots\ldots (vi)$$

$$x_2 + 4x_3 = 14 \ldots\ldots\ldots\ldots\ldots\ldots (iv)$$

$$x_3 = 3 \ldots\ldots\ldots\ldots\ldots\ldots (vii)$$

From (iv), since $x_3 = 3$,

$$x_2 + (4 \times 3) = 14$$

$$\text{or, } x_2 = 2$$

From (vi), since $x_3 = 3$,

$$x_1 - 33 = -32$$

$$\text{or, } x_1 = 1$$

THANK YOU