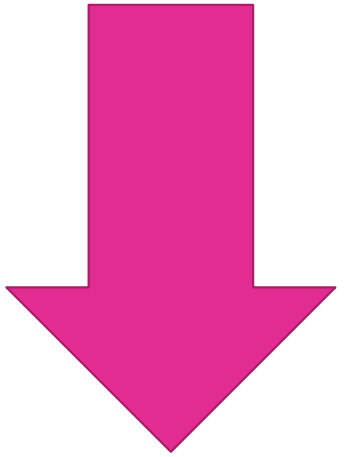
Lecture 04
Finite Differences

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MIST

$$\frac{dy}{dx} = \frac{y}{x}$$

- $\frac{dy}{y} = \frac{dx}{x}$
- $\ln y = \ln x + \ln(A)$
- $\ln y = \ln(xA)$

That's very easy!!!



BUT, what if I ask you to solve,

$$\frac{dy}{dx} = yx^2 + yx^3 + xy^3 + (yx)^{1/2} \ln y$$

Now that is not impossible but hard to solve analytically.

Finite Difference Method

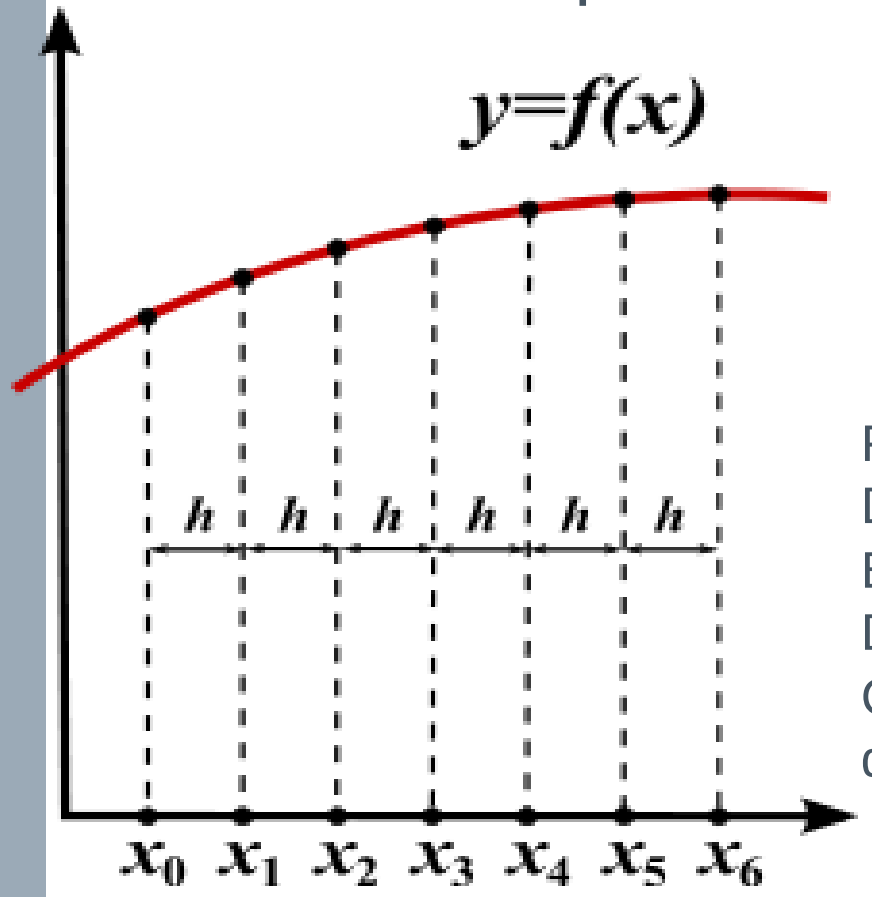
- The finite difference method is a numerical technique of solving differential equations where the derivatives are replaced by finite difference equivalents, thus converting the differential equation into a system of algebraic equations.
- The finite difference techniques are based upon the approximations that permit replacing differential equations by finite difference equations. These finite difference approximations are algebraic in form, and the solutions are related to grid points.

Steps

1. Dividing the solution into grids of nodes.
2. Approximating the given differential equation by finite difference equivalence that relates the solutions to grid points.
3. Solving the difference equations subject to the prescribed boundary conditions and/or initial conditions.

Finite Difference Method

- The concept of FDM is focused on approximating differentials. To use a finite difference method to approximate the solution to a problem, one must first discretize the problem's domain.



Derivative= Slope

$$\frac{\Delta y}{\Delta x}$$

Forward Difference Approximation of the First Derivative

Backward Difference Approximation of the First Derivative

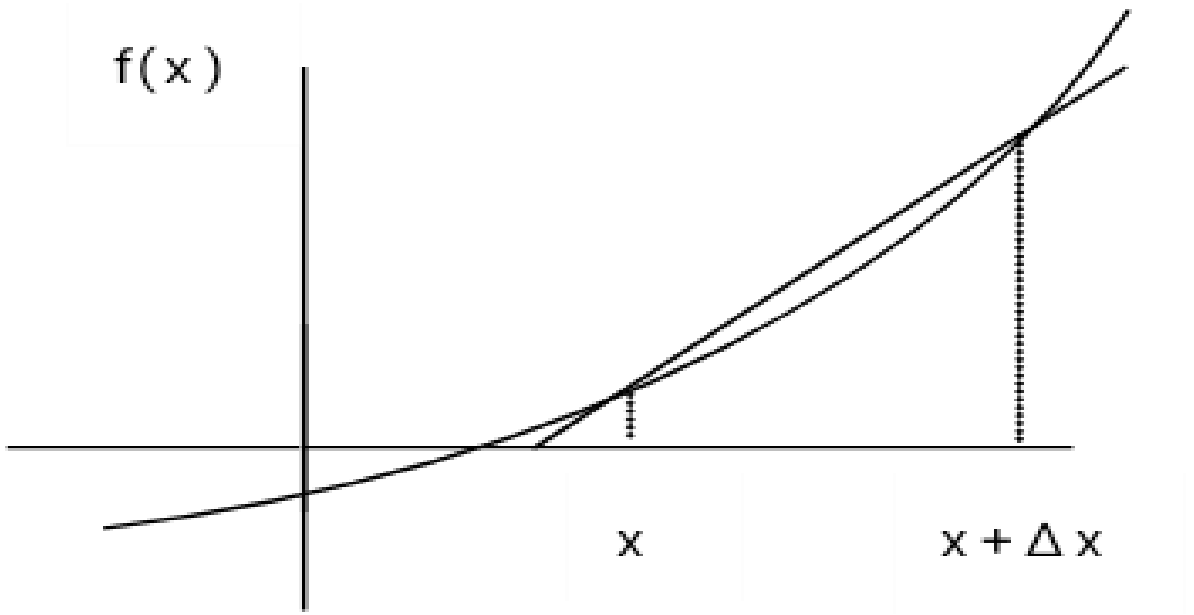
Central difference approximation of the First derivative.

Forward Difference Method

Derivation from Taylor's Theorem

- Taylor's theorem says that if the value of a function 'f' and all its derivatives at a point x are known, provided that the derivatives are continuous between x and x+Δx then,

$$f(x+\Delta x) = f(x) + f'(x)\,\Delta x + \frac{f''(x)\Delta(x)^2}{2!} + \ldots\ldots$$



$$f(x+\Delta x) = f(x) + f'(x)\,\Delta x + \frac{f''(x)\Delta(x)^2}{2!} + \ldots\ldots$$

$$f'(x)\,\Delta x = f(x+\Delta x) - f(x) - \frac{f''(x)\Delta x^2}{2!} - \ldots\ldots\ldots\ldots$$

Dividing by Δx,

$$f'(x) = \frac{f(x+\Delta x) - f(x)}{\Delta x} + K$$

K = Error in approximation

$$f'(x) = \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Forward Difference formula for approximating 1st derivative.

Example 1

The velocity of a rocket is given by

$$v(t) = 2000 \ln\left[\frac{14\times10^4}{14\times10^4 - 2100t}\right] - 9.8t,\ 0 \le t \le 30$$

where v is given in m/s and t is given in seconds. At $t = 16\text{s}$,

a) use the forward difference approximation of the first derivative of $v(t)$ to calculate the acceleration. Use a step size of $\Delta t = 2\text{s}$.

b) find the exact value of the acceleration of the rocket.

c) calculate the absolute relative true error for part (b).

Solution

(a) $$a(t_i) \approx \frac{v(t_{i+1}) - v(t_i)}{\Delta t}$$

$$t_i = 16$$

$$\Delta t = 2$$

$$t_{i+1} = t_i + \Delta t$$

$$= 16 + 2$$

$$= 18$$

$$v(t) = 2000 \ln\left[\frac{14\times10^4}{14\times10^4 - 2100t}\right] - 9.8t$$

$$a(16) \approx \frac{v(18) - v(16)}{2}$$

$$v(18) = 2000 \ln\left[\frac{14\times10^4}{14\times10^4 - 2100(18)}\right] - 9.8(18)$$

$$= 453.02 \text{ m/s}$$

$$v(16) = 2000 \ln\left[\frac{14\times10^4}{14\times10^4 - 2100(16)}\right] - 9.8(16)$$

$$= 392.07 \text{ m/s}$$

Hence

$$a(16) \approx \frac{v(18) - v(16)}{2}$$

$$= \frac{453.02 - 392.07}{2}$$

$$= 30.474\,\mathrm{m/s^2}$$

(b) The exact value of $a(16)$ can be calculated by differentiating

$$v(t) = 2000 \ln\left[\frac{14 \times 10^4}{14 \times 10^4 - 2100t}\right] - 9.8t$$

as

$$a(t) = \frac{d}{dt}[v(t)]$$

Knowing that

$$\frac{d}{dt}[\ln(t)] = \frac{1}{t} \text{ and } \frac{d}{dt}\left[\frac{1}{t}\right] = -\frac{1}{t^2}$$

$$a(t) = 2000\left(\frac{14\times10^4 - 2100t}{14\times10^4}\right)\frac{d}{dt}\left(\frac{14\times10^4}{14\times10^4 - 2100t}\right) - 9.8$$

$$= 2000\left(\frac{14\times10^4 - 2100t}{14\times10^4}\right)(-1)\left(\frac{14\times10^4}{(14\times10^4 - 2100t)^2}\right)(-2100) - 9.8$$

$$= \frac{-4040 - 29.4t}{-200 + 3t}$$

$$a(16) = \frac{-4040 - 29.4(16)}{-200 + 3(16)}$$

$$= 29.674\,\text{m/s}^2$$

(c) The absolute relative true error is

$$|\epsilon_t| = \left|\frac{\text{True Value} - \text{Approximate Value}}{\text{True Value}}\right| \times 100$$

$$= \left|\frac{29.674 - 30.474}{29.674}\right| \times 100$$

$$= 2.6967\%$$



THANK YOU