Lecture 05
Finite Differences

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Taylor Series Expansion:

$$f(x+\Delta x) = f(x) + f'(x)\Delta x + \frac{f''(x)\Delta(x)^2}{2!} + \frac{f'''(x)\Delta(x)^3}{3!} + \ldots$$

› The well-known finite difference forms are:

i. Forward Difference

$$f'(x) = \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

ii. Backward Difference

$$f'(x) = \frac{f(x) - f(x-\Delta x)}{\Delta x}$$

iii. Central Difference

$$f'(x) = \frac{f(x+\Delta x) - f(x-\Delta x)}{2\Delta x}$$

› Finite Difference Approximation to the Second Derivatives

$$f''(x)=\frac{f(x+\Delta x)-2\,f(x)+f(x-\Delta x)}{\Delta x^2}$$

- The difference between the difference equation and the original differential equation is called the truncation error or order of approximation.

Example

A function is given by $f(x)=5+2x+6x^2+3x^3$

(a)Using the values of the function and the derivatives at x=5, determine the values of the function at x=5.05 with the help of the Taylor's series expansion and compare these values with the actual values of the function.

(b)Taking $\Delta x = 0.05$, determine the % of error of forward, backward and central differences of the first derivatives and of the second derivatives with respect to the point where x=5.

(c)Using the values of the function and its derivatives at x=5, determine the values of the function at x=5.05 and the % of error, if the function is approximated by the first three terms in the Taylor's series.

Solution

$f(x)= 5+2x+6x^2+3x^3$

$f'(x)= 2+12x+9x^2$

$f''(x)= 12+18x$

$f'''(x)= 18$

$f^{iv}(x)= f^{v}(x)= ……..= 0$

$f(5)= 540$

$f'(5)= 287$

$f''(5)= 102$

$f'''(5)= 18$

$f^{iv}(5)= f^{v}(5) ……..= 0$

(a)Using the values of the function and the derivatives at x=5, determine the values of the function at x=5.05 with the help of the Taylor's series expansion and compare these values with the actual values of the function.

The value of the function at x=5.05 is

$$f(5.05)=5+2\times5.05+6\times5.05^2+3\times5.05^3$$
$$=554.477875$$

We have x=5 and $\Delta x=0.05$.

Using the Taylor series expansion

$$f(x+\Delta x) = f(x)+ f'(x)\,\Delta x+ \frac{f''(x)\Delta(x)2}{2!} + \ldots\ldots$$

We get the value of the function at x=5.05 as

$$f(5+0.05) =f(5) +0.05\, f'(5) +\frac{0.05^2}{2!} f''(5) +\frac{0.05^3}{3!} f'''(5)+0$$

$$f(5+0.05) = f(5) + 0.05\, f'(5) + \frac{0.05^2}{2!} f''(5) + \frac{0.05^3}{3!} f'''(5) + 0$$

$$f(5.05) = 540 + 0.05 \times 287 + \frac{0.05^2}{2} \times 102 + \frac{0.05^3}{6} \times 18$$

$$= 554.477875$$

$$\% \text{ of error} = \frac{Approximate\ value - True\ value}{True\ value} \times 100$$

$$\% \text{ of error} = \frac{554.477875 - 554.477875}{554.477875} \times 100$$

$$= 0$$

(b)Taking Δx= 0.05, determine the % of error of forward, backward and central differences of the first derivatives and of the second derivatives with respect to the point where x=5.

Forward Difference:

$$f'(x)=\frac{f(x+\Delta x)-f(x)}{\Delta x}$$

$$=\frac{f(5.05)-f(5)}{0.05}$$

$$=\frac{554.477875-540}{0.05}$$

$$=289.5575$$

But when x=5,

$$f'(x)=f'(5)=287$$

$$\%\ \text{of error}= \frac{Approximate\ value - True\ value}{True\ value} \times 100$$

$$\%\ \text{of error} = \frac{289.5575-287}{287} \times 100$$

$$= +0.8911$$

Backward Difference: $f'(x)= \frac{f(x)-f(x-\Delta x)}{\Delta x}$

$= \frac{f(5)-f(5-0.05)}{0.05}$

$= \frac{f(5)-f(4.95)}{0.05}$ ..... (i)

The value of the function at x=4.95 is

$f(4.95)= 5+2\times 4.95 + 6 \times 4.95^2+3 \times 4.95^3$

› = 525.777125

From equation (i),

$$f'(x) = \frac{540 - 525.777125}{0.05}$$

$$= 284.4575$$

$$\% \text{ of error} = \frac{284.4575 - 287}{287} \times 100$$

$$= -0.8859$$

Central Difference: $f'(x) = \frac{f(x+\Delta x) - f(x-\Delta x)}{2\Delta x}$

$$= \frac{f(5.05) - f(4.95)}{2 \times 0.05}$$

$$= \frac{554.477875 - 525.777125}{0.10}$$

$$= 287.0075$$

$$\% \text{ of error} = \frac{287.0075 - 287}{287} \times 100$$

$$= +0.0026$$

Second Derivative:

$$f''(x)=\frac{f(x+\Delta x)-2\,f(x)+f(x-\Delta x)}{\Delta x^2}$$

$$=\frac{f(5.05)-2f(5)+f(4.95)}{0.05^2}$$

$$=\frac{554.477875-2\times540+525.777125}{0.0025}$$

$$=102$$

But when x=5, $f''(x)= f''(5)=102$

$$\%\text{ of error} = \frac{102-102}{102}\times100$$

$$= 0$$

(c)Using the values of the function and its derivatives at x=5, determine the values of the function at x=5.05 and the % of error, if the function is approximated by the first three terms in the Taylor's series.

When the function is approximated by first 3 terms in the Taylor series, i.e.

$$f(x+\Delta x) = f(x) + f'(x)\,\Delta x + \frac{f''(x)\Delta(x)2}{2!}$$

We have

$$f(5+0.05)=f(5)+0.05\,f'(5) + \frac{0.05^2}{2!}\,f''(5)$$

$$f(5.05)= 540+0.05 \times 287 + \frac{0.05^2}{2!} \times 102$$

$$=554.4775$$

But, the actual value of f(5.05)= 554.477875

$$\% \text{ of error} = \frac{554.4775 - 554.477875}{554.477875} \times 100$$

$$= - 0.0000673$$

Practice Problem

A function is given by $f(x)= 1+2x+3x^2+4x^3$

(a)Using the values of the function and the derivatives at x=1, determine the values of the function at x=1.01,1.10 and 1.50 with the help of the Taylor's series expansion and compare these values with the actual values of the function.

(b)Taking Δx= 0.01,010 and 0.50, determine the % of error of forward, backward and central differences of the first derivative and of the second derivatives with respect to the point where x=1.

(c)Using the values of the function and its derivatives at x=1, determine the values of the function at x=1.01, 1.10 and 1.50 and the % of error, if the function is approximated by the first three terms in the Taylor's series.



THANK YOU