Lecture 07
Numerical Differentiation

Simlia Satter Songi
Lecturer
EWCE Department
MIST

The Difference Formulae

i. **With one term**

$$f'(x_0) = \frac{1}{h}\Delta f_0 = \frac{1}{h}(f_1 - f_0)$$

This is the 2-point difference formula.

ii. **With two terms**

$$f'(x_0) = \frac{1}{h}\left[\Delta f_0 - \frac{1}{2}\Delta^2 f_0\right]$$

$$= \frac{1}{h}\left[(f_1 - f_0) - \frac{1}{2}(f_2 - 2f_1 + f_0)\right]$$

$$= \frac{1}{2h}\left[-3f_0 + 4f_1 - f_2\right]$$

This is the 3-point difference formula.

The Difference Formulae (Contd.)

iii. **With three terms**

$$f'(x_0) = \frac{1}{h}\left[\Delta f_0 - \frac{1}{2}\Delta^2 f_0 + \frac{1}{3}\Delta^3 f_0\right]$$

$$= \frac{1}{h}\left[(f_1 - f_0) - \frac{1}{2}(f_2 - 2f_1 + f_0) + \frac{1}{3}(f_3 - 3f_2 + 3f_1 - f_0) \right]$$

$$= \frac{1}{6h}\left[2f_3 - 9f_2 + 18f_1 - 11f_0 \right]$$

This is the 4-point difference formula.

The Difference Formulae (Contd.)

iv. **With four terms**

$$f'(x_0)= \frac{dy}{dx} = \frac{1}{h}\left[\Delta f_0 - \frac{1}{2}\Delta^2 f_0 + \frac{1}{3}\Delta^3 f_0 - \frac{1}{4}\Delta^4 f_0\right]$$

$$= \frac{1}{h}\left[(f_1-f_0) - \frac{1}{2}(f_2-2f_1+f_0) + \frac{1}{3}(f_3-3f_2+3f_1-f_0) - \frac{1}{4}(f_4-4f_3+6f_2-4f_1+f_0)\right]$$

$$= -\frac{1}{12h}\left[3f_4-16f_3+36f_2-48f_1+25f_0\right]$$

This is the 5-point difference formula.

Example 3.6

- From the following table find f'(1.4) using difference table, 2-point and 3-point difference formulae. Compare the result with the correct result f'(1.4)= cosh(1.4)=2.1509

x	**1.2**	**1.3**	**1.4**	**1.5**	**1.6**
y	1.5095	1.6984	1.9043	2.1293	2.3756

i) The difference table is:

x	**y= f(x)**	Δ **f**	Δ^2**f**	Δ^3**f**	Δ^4**f**
1.2	1.5095				
		0.1889			
1.3	1.6984		0.017		
		0.2059		0.0021	
1.4	1.9043		0.0191		0.0001
		0.2250		0.0022	
1.5	2.1293		0.0213		
		0.2463			
1.6	2.3756				

We have $x=x_0=1.4$, $h=0.1$ and $s=0$.

$$\mathbf{f'(x_0)= \frac{1}{h} [\Delta f_0 - \frac{1}{2}\Delta^2 f_0 + \frac{1}{3}\Delta^3 f_0 - \frac{1}{4}\Delta^4 f_0 + \frac{1}{5}\Delta^5 f_0 - \frac{1}{6}\Delta^6 f_0 + \ldots\ldots]}$$

$$f'(1.4)= \frac{1}{0.1} [\, 0.2250 - \frac{1}{2} \times 0.0213]$$

$$= \mathbf{2.1435}$$

ii) 2-point difference formula:

$$\mathbf{f'(x_0) = \frac{1}{h} \Delta f_0 = \frac{1}{h} (f_1 - f_0)}$$

Here, $f_0=1.9043$, $f_1=2.1293$, $h=0.1$

$$f'(1.4)= \frac{2.1293-1.9043}{0.1}$$

$$\mathbf{= 2.25}$$

iii) 3-point difference formula:

$$\mathbf{f'(x_0) = \frac{1}{2h}\,[-3f_0 + 4f_1 - f_2]}$$

$$= \frac{1}{2\times 0.1}\,[-3 \times 1.9043 + 4 \times 2.1293 - 2.3756]$$

$$\mathbf{= 2.1435}$$

Example 3.7

- The following velocities were measured along a vertical in a channel. Determine the velocity gradient du/dz at z=0 using the 5-point difference formulae.

Distance from channel bed, z (cm)	0	2	4	6	8
Velocity, u (cm/s)	0	25	32	36	40

Example 3.7 (Contd.)

Solution:

h= 2cm= 0.02m

Using 5-point difference formula

$$\frac{du}{dz} = -\frac{1}{12h}[3f_4-16f_3+36f_2-48f_1+25f_0]$$

$$= -\frac{1}{12\times\ 0.2}(3\times 0.4 -16\times 0.36+36\times 0.32-48\times 0.25+25\times\ 0)$$

$$=2.1\frac{m/s}{m}$$

Practice Problems

3.8 A rod is rotating in a plane. The following table gives the angle θ (in radians) through which the rod has turned for various values of the time t in seconds. Find the angular velocity and the angular acceleration of the rod when t= 0.

t (s)	0	0.2	0.4	0.6	0.8	1.0	1.2
θ (rad)	0	0.122	0.493	1.123	2.022	3.200	4.666

Practice Problems

3.9 The following velocities were measured along a vertical in a channel. Determine the velocity gradient du/dz at z=0 using the (i) 2-point, (ii) 3-point, (iii) 4-point and (iv) 5-point difference formulae.

Distance from channel bed, z (cm)	0	2	4	6	8
Velocity, u (cm/s)	0	20	30	36	40

Differentiating Tabulated Function

Newton-Gregory Backward Difference Formula:

$$f'(x)=\frac{dy}{dx}=\frac{1}{h}\left[\nabla f_0+\frac{2s+1}{2}\nabla^2 f_0+\frac{3s^2+6s+2}{6}\nabla^3 f_0+\ldots\ldots\right]$$

However, for tabulated values of x, the formula takes a simple form, for setting $x=x_0$ and hence the equation becomes

$$f'(x_0)=\frac{dy}{dx}=\frac{1}{h}\left[\nabla f_0+\frac{1}{2}\nabla^2 f_0+\frac{1}{3}\nabla^3 f_0+\frac{1}{4}\nabla^4 f_0+\frac{1}{5}\nabla^5 f_0+\frac{1}{6}\nabla^6 f_0+\ldots\right]$$

Differentiating Tabulated Function

Differentiating once again,

$$f''(x)= \frac{1}{h^2}\left[\nabla^2 f_0 + (s+1)\nabla^3 f_0 + \frac{12s^2+36s+22}{24}\nabla^4 f_0 + \ldots\ldots\right]$$

From which we obtain by setting $x=x_0$ and $s=0$ the equation

$$f''(x_0)= \frac{1}{h^2}\left[\nabla^2 f_0 + \nabla^3 f_0 + \frac{11}{12}\nabla^4 f_0 + \frac{5}{6}\nabla^5 f_0 + \frac{137}{180}\nabla^6 f_0 + \frac{7}{10}\nabla^7 f_0 + \frac{363}{360}\nabla^8 f_0 + ..\right]$$

Example 3.10

Find f'(2.2) and f ''(2.2) from the following table:

x	**1.4**	**1.6**	**1.8**	**2.0**	**2.2**
y	4.0552	4.9530	6.0496	7.3891	9.025

The difference table is:

x	**y= f(x)**	**∇f**	**$\nabla^2 f$**	**$\nabla^3 f$**	**$\nabla^4 f$**
1.4	4.0552				
		0.8978			
1.6	4.9530		0.1988		
		1.0966		0.0441	
1.8	6.0496		0.2429		0.0094
		1.3395		0.0535	
2.0	7.3891		0.2964		
		1.6359			
2.2	9.0250				

We have $x=x_0=2.2$, $h=0.2$ and $s=0$.

$$\mathbf{f'(x_0)= \frac{1}{h} [\nabla f_0 + \frac{1}{2}\nabla^2 f_0 + \frac{1}{3}\nabla^3 f_0 + \frac{1}{4}\nabla^4 f_0 + \frac{1}{5}\nabla^5 f_0 + \frac{1}{6}\nabla^6 f_0 + \dots]}$$

$$f'(2.2)= \frac{1}{0.2} [1.6359 + \frac{1}{2} \times 0.2964 + \frac{1}{3} \times 0.0535 + \frac{1}{4} \times 0.0094]$$

$$= 9.0214$$

$$\mathbf{f''(x_0)= \frac{1}{h^2} [\nabla^2 f_0 + \nabla^3 f_0 + \frac{11}{12}\nabla^4 f_0 + \frac{5}{6}\nabla^5 f_0 + \frac{137}{180}\nabla^6 f_0 + \frac{7}{10}\nabla^7 f_0 + \frac{363}{360}\nabla^8 f_0 + ..]}$$

$$f''(2.2)= \frac{1}{0.04} [0.2964+ 0.0535 + \frac{11}{12} \times 0.0094]$$

$$= 8.9629$$

Differentiation based on Lagrange's Interpolation Formulae

- Applicable when observations are not equally spaced.
- Derivatives are found using Lagrange's interpolation formula for divided difference.

The divided difference formula is:

$$y = y_0 + (x-x_0)[x_0,x_1] + (x-x_0)(x-x_1)[x_0,x_1,x_2] + (x-x_0)(x-x_1)(x-x_2)[x_0,x_1,x_2,x_3] + \ldots\ldots + (x-x_0)(x-x_1)\ldots\ldots(x-x_{n-1})[x_0,x_1,x_2,\ldots\ldots,x_n]$$

Example 3.11

Find f'(0.12) from the following data:

x	0.05	0.1	0.2	0.26
f(x)	0.05	0.0999	0.1987	0.2571

Solution:

The divided difference formula is:

$$y = y_0 + (x-x_0)[x_0,x_1] + (x-x_0)(x-x_1)[x_0,x_1,x_2] + (x-x_0)(x-x_1)(x-x_2)[x_0,x_1,x_2,x_3]$$
$$+\ldots + (x-x_0)(x-x_1)\ldots(x-x_{n-1})[x_0,x_1,x_2,\ldots\ldots,x_n]$$

The difference table is:

x	y= f(x)	Order 1	Order 2	Order 3
0.05	0.05			
		0.998		
0.1	0.0999		-0.0667	
		0.988		-0.1198
0.2	0.1987		-0.091875	
		0.9733		
0.26	0.2571			

$$y = y_0 + (x-x_0)[x_0,x_1] + (x-x_0)(x-x_1)[x_0,x_1,x_2] + (x-x_0)(x-x_1)(x-x_2)[x_0,x_1,x_2,x_3] + \ldots\ldots + (x-x_0)(x-x_1)\ldots\ldots(x-x_{n-1})[x_0,x_1,x_2,\ldots\ldots,x_n]$$

The polynomial fitting the data is given by:

$f(x) = 0.05 + (x-0.05)(0.998) + (x-0.05)(x-0.1)(-0.667) + (x-0.05)(x-0.1)(x-0.2)(-0.1198)$

$= -0.032\times10^{-3} + 1.003811x - 0.02477\,x^2 - 0.1198x^3$

Differentiating with respect to x, we get

$f'(x) = 1.003811 - 0.04954x - 0.3594\,x^2$

$f'(0.12) = 0.9927$

Practice Problems

3.12 The table below gives the values of distances travelled by a car at various time intervals using the initial running.

Time, t(s)	5	6	7	8	9
Distance, s(km)	10.0	14.5	19.5	25.5	32.0

Estimate the velocity at time t= 5,7 and 9s using Newton-Gregory difference formula. Also, calculate the acceleration at t=7s.

Practice Problems

3.13 Given the following set of data points, obtain the table of divide differences. Use the table to estimate the value of f'(3.25).

x	1.3	2.4	3.8	4.1
f(x)	2	19	36	53

THANK YOU