LECTURE 08

NUMERICAL INTEGRATION

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INTRODUCTION

- Many practical problems involve integrals which can not be solved by mathematical analysis or the function may be too complicated for efficient computation.
- For example, speed v km/h measured at different time interval t hours is given. If the distance travelled in this time has to be calculated we have to use the equation:

$$s=\int_0^t v dt$$

But we do not know v in terms of t.

INTRODUCTION

- Differentiation and integration of functions are important mathematical operation.

- Very often the derivative of a function can be expressed explicitly in terms of the variable which can be integrated directly and exactly. In many cases, although it may be expressed in terms of x, but can not be integrated. It also may be that the function is not known as an explicit expression in terms of x, but only as a tabulation of values.

- Numerical integration is a process for determining the approximate value of the integrals.

WHEN USED?

- We need to seek the help of numerical integration techniques in the following situations:

1. Functions that do not possess analytical solutions.
2. Analytical solutions exist but are complex and difficult to use for calculations.
3. Data for variables are available as a table, but no mathematical relationship between them is known, as is often the case with experimental data.

GENERAL QUADRATURE FORMULA

$$\int_{x_0}^{x_0+nh} y\,dx = h\int_0^n y\,ds$$

$$\int_{x_0}^{x_0+nh} y\,dx = h\left[nf_0 + \frac{n^2}{2!}\Delta f_0 + \left(\frac{n^3}{3} - \frac{n^2}{2}\right)\frac{\Delta^2 f_0}{2!} + \left(\frac{n^4}{4} - n^3 + n^2\right)\frac{\Delta^3 f_0}{3!} + \ldots.\right]$$

This is the General Quadrature Formula.

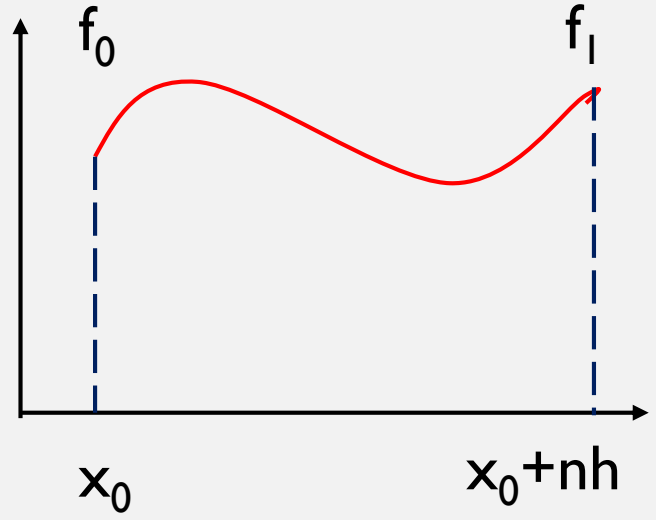
TRAPEZIUM RULE

- In this rule, we approximate f(x) in the interval (x_0, x_{0+nh}) by a straight line, i.e. it is assumed that the function f(x) varies linearly between x_0 and x_{0+nh} .
- Higher order differences are negligible for small h. Neglecting the second and higher order differences and putting n=1, we obtain,

$$\int_{x_0}^{x_0+h} y\,dx = h\left(f_0 + \frac{1}{2}\Delta f_0\right)$$

$$= \frac{h}{2}\left(f_1 + f_0\right)$$

$f_1 = f_0 + \Delta f_0$



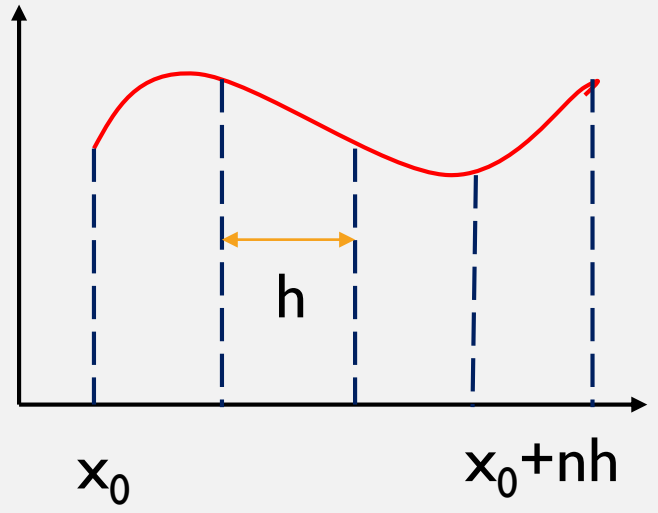
TRAPEZOIDAL RULE

- Let the area between the ordinates at x_0 and x_0 +nh be divided into n strips at x_0, x_0+h, x_0+2h, ….. , x_0+nh. Applying trapezium rule to each strip and summing up, we get

$$\int_{x_0}^{x_0+h} y\,dx = \frac{h}{2}(f_1 + f_0) + \frac{h}{2}(f_1 + f_2) + \ldots\ldots + \frac{h}{2}(f_{n-1} + f_n)$$

$$= \frac{h}{2}(f_0 + f_n) + h\,(f_1 + f_2 + \ldots\ldots + f_{n-1})$$

$$\int_{x_0}^{x_0+h} y\,dx = \frac{h}{2}(f_0 + f_n + 2f_1 + 2f_2 + \ldots\ldots\ldots + 2f_{n-1})$$



This is known as the trapezoidal rule.

EXAMPLE 4.1

- Evaluate $\int_{1.8}^{3.4} e^x$ dx using the trapezoidal rule correct to two decimal places with (i) h= 0.2 and (ii) h=0.4.

Solution:

With h=0.2, the values of x and f(x)=e^x are tabulated.

x	f(x)	x	f(x)
1.8	6.050	2.8	16.445
2.0	7.389	3.0	20.086
2.2	9.014	3.2	24.533
2.4	11.023	3.4	29.964
2.6	13.464		

EXAMPLE 4.1 (CONTD.)

(i) For h=0.2

$$\int_{x_0}^{x_0+h} f(x)dx = \frac{h}{2}\,(f_0 + f_n + 2f_1 + 2f_2 + \ldots\ldots\ldots + 2f_{n-1})$$

$$I = \frac{0.2}{2}\,[6.050 + 29.964 + 2(7.389 + 9.025 + 11.023 + 13.464 + 16.445 + 20.086 + 24.533)]$$

$$= 23.994$$

$$= 23.99$$

EXAMPLE 4.1 (CONTD.)

ii. For h=0.4

$$\int_{x_0}^{x_0+h} f(x)dx = \frac{h}{2}\,(f_0 + f_n + 2f_1 + 2f_2 + \ldots\ldots\ldots + 2f_{n-1})$$

$$I = \frac{0.4}{2}\,[6.050 + 29.964 + 2(9.025 + 13.464 + 20.086)]$$

$$= 24.233$$

$$= 24.23$$

EXAMPLE 4.2

- Evaluate numerically $\int_0^{\pi/4} cosxdx$ by dividing the interval 4 strips.

Solution: f(x)=cosx

here h= $\frac{\pi}{16}$ =0.19635

x	f(x)
0	I
0.19635	0.980785
0.39269	0.923883
0.58905	0.8314688
0.7854	0.707105

EXAMPLE 4.2 (CONTD.)

Using trapezoidal rule,

$$\int_0^{\pi/4} cosx dx = \frac{h}{2} (f_0 + f_n + 2f_1 + 2f_2 + \ldots\ldots\ldots + 2f_{n-1})$$

$$= \frac{0.19635}{2} \{1 + 0.707105 + 2(0.980785 + 0.923883 + .8314688)\}$$

$$= 0.704835$$

SIMPSON'S $\frac{1}{3}$ RULE

- Let n=2 , then we have,

$$\int_{x_0}^{x_0+2h} y\,dx = \int_0^h y\,ds = h[2f_0 + \frac{2^2}{2!}\Delta f_0 + (\frac{2^3}{3} - \frac{2^2}{2})\frac{\Delta^2 f_0}{2!} + \ldots]$$

Here the third and higher-order differences have been neglected. By replacing Δf_0 and $\Delta^2 f_0$ in terms of functional value, we have

$$\int_{x_0}^{x_0+2h} y\,dx = \frac{h}{3}(f_0 + 4f_1 + f_2)$$

In general , interval is divided into an even numbers of equal interval.

$$\int_{x_0}^{x_0+nh} y\,dx = \frac{h}{3}[f_0 + f_n + 4(f_1 + f_3 + \ldots\ldots\ldots + f_{n-1}) + 2(f_2 + f_4 + \ldots\ldots\ldots + f_{n-2})]$$

This formula is called Simpson's $\frac{1}{3}$ rule.

SIMPSON'S $\frac{3}{8}$ RULE

$$\int_{x_0}^{x_0+nh} y\,dx = \frac{3h}{8}\,[f_0 + f_n + 3(f_1 + f_2 + f_4 + f_5 \ldots + f_{n-1}) + 2(f_3 + f_6 + \ldots\ldots + f_{n-3})]$$

Here n is multiple of 3

n= No of intervals

EXAMPLE 4.3

- Evaluate numerically $\int_1^7 \frac{1}{x} dx$ where h=1.

Solution:

x	$\frac{1}{x}$
1	1
2	0.5
3	0.333
4	0.25
5	0.2
6	0.167
7	0.143

Simpson's $\frac{1}{3}$ rule:

$$\int_{x_0}^{x_0+nh} y\,dx = \frac{h}{3}\,[f_0 + f_n + 4(f_1 + f_3 + \ldots\ldots\ldots + f_{n-1}) + 2(f_2 + f_4 + \ldots\ldots\ldots + f_{n-2})]$$

$$\int_1^7 \frac{1}{x}\,dx = \frac{1}{3}\,(1+0.143+ 4(0.5+0.25+0.167)+ 2\,(0.333+0.2))$$

$$= 1.959$$

Simpson's $\frac{3}{8}$ Rule:

$$\int_{x_0}^{x_0+nh} y\,dx = \frac{3h}{8}\,[f_0 + f_n + 3(f_1 + f_2 + f_4 + f_5 \ldots + f_{n-1}) + 2(f_3 + f_6 + \ldots\ldots + f_{n-3})]$$

$$\int_1^7 \frac{1}{x}\,dx = \frac{3*1}{8}\,[1+0.143+ 3(0.5+0.333+0.2+0.167)+ 2\,(0.25)]$$

$$=1.966125$$

EXAMPLE 4.4

Evaluate by Simpson's rule $\int_0^1 \sqrt{(1-x^2)(2-x)}\,dx$; dividing the interval into four equal parts.

Solution: $h = \frac{1-0}{4}$

$= 0.25$

x	$\sqrt{(1-x^2)(2-x)}$
0	1.41421
0.25	1.28087
0.50	1.06067
0.75	0.73951
1	0

EXAMPLE 4.4

Simpson's $\frac{1}{3}$ rule:

$$\int_{x_0}^{x_0+nh} y\,dx = \frac{h}{3}\,[f_0 + f_n + 4(f_1 + f_3 + \ldots\ldots\ldots + f_{n-1}) + 2(f_2 + f_4 + \ldots\ldots\ldots + f_{n-2})]$$

$$\int_0^1 \sqrt{(1-x^2)(2-x)}\,dx = \frac{0.25}{3}\,(1.41421 + 0 + 4(1.28087 + 0.73591) + 2\times 1.06067)$$

$$= 0.96689$$

PRACTICE PROBLEM

4.5 Using trapezoidal rule compute $\int_0^1 \frac{sinx}{2} dx$ where h=0.1 .

4.6 Evaluate by Simpson's rule

$$\int_0^1 \frac{dx}{1+x^2} \text{ ; h=0.1}$$

4.7 The table below shows the temperature f(t) as a function of time:

Time, t	1	2	3	4	5	6	7
Temperature, f(t)	81	75	80	83	78	70	60

Use Simpson's 1/3 rule to estimate $\int_1^7 f(t)dt$.

THANK YOU