Lecture 13

Numerical Solution of Ordinary Differential Equations

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Runge-Kutta Method

- The Runge-Kutta methods are designed to give greater accuracy and they possess the advantage of requiring only the function values at some selected points without requiring the calculation of higher derivatives.
- Runge-Kutta methods of various orders exist.
- The Euler method is the first-order Runge-Kutta method.

Runge-Kutta Method

- Higher the order of a method, better would be the accuracy of the estimates. Therefore, the selection of order using Runge-Kutta methods depending on the problem under consideration.
- The most popular Runge-Kutta methods are fourth order.
- There are infinite number of versions of fourth-order Runge-Kutta methods. The following is the most commonly used version and is sometimes called the classical fourth-order Runge-Kutta method.

$$y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

Runge-Kutta Method

- The formula of classical fourth-order Runge-Kutta method:

$$\mathbf{y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)}$$

where,

$$k_1 = hf(x_n, y_n)$$

$$k_2 = hf(x_n + \frac{1}{2}h, y_n + \frac{1}{2}k_1)$$

$$k_3 = hf(x_n + \frac{1}{2}h, y_n + \frac{1}{2}k_2)$$

$$k_4 = hf(x_n + h, y_n + k_3)$$

Example 01

Given, $\frac{dy}{dx}$ = y'= y-x with y(0)= 2, estimate y(0.1) by the Runge-Kutta Fourth order formula using h=0.1 .

Solution:

We have y'= y-x, h= 0.1, x_0= 0, y_0=2.

Therefore, $\mathbf{k_1 = hf(x_0, y_0)} = h(y_0 - x_0)$

$= 0.1 \times (2-0) = \mathbf{0.2}$

$\mathbf{k_2 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1\right)}$

$= hf\left(0 + 0.05, 2 + \frac{1}{2} \times 0.2\right)$

$= hf(0.05, 2.1)$

$= 0.1 \times (2.1-0.05) = \mathbf{0.205}$

Example 01

$$\mathbf{k_3 = hf\left(x_0 + \frac{1}{2}h,\ y_0 + \frac{1}{2}k_2\right)}$$

$$= hf\left(0 + 0.05,\ 2 + \frac{1}{2} \times 0.205\right)$$

$$= hf(0.05,\ 2.1025)$$

$$= 0.1 \times (2.1025 - 0.05) = \mathbf{0.20525}$$

$$\mathbf{k_4 = hf(x_0 + h,\ y_0 + k_3)}$$

$$= hf\ (0 + 0.1,\ 2 + 0.20525)$$

$$= hf(0.1,\ 2.20525)$$

$$= 0.1 \times (2.20525 - 0.1) = \mathbf{0.210525}$$

Example 01

$$\mathbf{y\,(0.1) = y_0 + \frac{1}{6}\,(k_1 + 2k_2 + 2k_3 + k_4)}$$

$$= 2 + \frac{1}{6}(0.2 + 2\times 0.205 + 2\times 0.20525 + 0.210525)$$

$$= 2.205170833$$

$$\approx \mathbf{2.2052}$$

Example 02

Given, $\frac{dy}{dx}$ = -2y+ x+ 4 with y(0)= 1, estimate y(0.2) by the Runge-Kutta Fourth order formula using h=0.2 . Exact solution is 1.3472599. Calculate true error.

Solution:

We have , $\frac{dy}{dx}$ = -2y+ x+ 4 , h= 0.2, x_0= 0, y_0=1.

Therefore, $\mathbf{k_1 = hf(x_0, y_0) = h(-2y_0 + x_0 + 4)}$

$= 0.2 \times (-2\times1+0+4)$ **= 0.4**

$\mathbf{k_2 = hf(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1)}$

$= hf\ (0+\frac{1}{2} \times 0.2,\ 1+\frac{1}{2} \times 0.4)$

$= hf(0.1, 1.2)$

$= 0.2 \times \{(-2\times1.2\ )+0.1\ +4)\}$**= 0.34**

Example 02

$$\mathbf{k_3 = hf\left(x_0 + \frac{1}{2}h,\ y_0 + \frac{1}{2}k_2\right)}$$

$$= hf\left(0 + \frac{1}{2} \times 0.2,\ 1 + \frac{1}{2} \times 0.34\right)$$

$$= hf(0.1,\ 1.17)$$

$$= 0.2 \times \{(-2 \times 1.2) + 0.1 + 4\} = \mathbf{0.352}$$

$$\mathbf{k_4 = hf\left(x_0 + h,\ y_0 + k_3\right)}$$

$$= hf\left(0 + 0.2,\ 1 + 0.352\right)$$

$$= hf(0.2,\ 1.352)$$

$$= 0.2\{(-2 \times 1.352) + 0.2 + 4\}$$

$$\mathbf{= 0.2992}$$

Example 02

$$\mathbf{y\,(0.2) = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)}$$

$$= 1 + \frac{1}{6}(0.4 + 2\times0.34 + 2\times0.352 + 0.2992)$$

$$\mathbf{= 1.3472}$$

True error = 1.3472599-1.3472

= 0.0000599

THANK YOU