Numerical Methods
EWCE 205

Lecture-01

Introduction

- ✓The methods that are used to approximate the solution of any problem are called numerical methods. We need approximations because we either cannot solve the procedure analytically

- ✓Any result you get from these methods is **approximated not exact**, they give you the solution faster than normal ones, also these methods are easy to be programmed.

- ✓Numerical methods are algorithms used for computing numeric data. They are used to provide '**approximate**' results for the problems being dealt with and their necessity is felt when it becomes impossible or extremely difficult to solve a given problem analytically.

Why numerical methods are used?

Numerical methods **provide a way to solve problems quickly and easily compared to analytic solutions**.

Whether the goal is integration or solution of complex differential equations, there are many tools available to reduce the solution of what can be sometimes quite difficult analytical math to simple algebra.

What is the difference between numerical and analytical methods?

An analytical solution involves framing the problem in a well-understood form and calculating the exact solution.

A numerical solution means making guesses at the solution and testing whether the problem is solved well enough to stop.

Numerical methods can be used for-

- **finding root(s) of equations** - Bisection method, Newton Raphson, Fixed Point iteration etc.
- **solving ODEs** - Euler method, Improved Euler, RK methods, Mid point method, Predictor Corrector methods etc.
- **finding values of integrals** - Mid point, Trapezoidal, Simpson's rule
- **interpolation** - Lagrange interpolation, Newton interpolation, Spline interpolation etc.

Sometimes integral becomes too complicated, and it is not possible to integrate it with conventional definite integral techniques. See this figure.

$$F_A(0 \leq \alpha \leq \pi/2, \varsigma) = \int_0^{\pi/2} \frac{(a^2 cos^2\alpha + b^2 sin^2\alpha)\, d\alpha}{r^2 + a^2 cos^2\alpha + b^2 sin^2\alpha + 2r\sqrt{a^2 cos^2\alpha + b^2 sin^2\alpha}\, cos\,(\alpha - \Phi) + (1+\varsigma)H^2}$$

It is not possible to obtain an explicit expression for this integral, therefore we can use **numerical integral** (Simpson, trapezoidal).

Application

- Application in Water Resources Engineering:

- Flood Surveying, flood propagation and duration in large river basins.

- Application in Environmental Engineering:

- Using models to estimate the impact of air pollution from an industrial plant, Monitoring of Climate Change.

Numbers and their Accuracy:

There are two kinds of numbers –

1. **Exact Number**: example of exact numbers are 1,2,3….1/2, 1/3..$\sqrt{2}, etc$ written in this manner. They are known with complete certainty.
2. **Approximate Number**: Approximate numbers are those that represent the number to a certain degree of accuracy. Thus an approximate value of $\boldsymbol{\pi}$ **is 3.1416,** or if we desire a better approximation it is 3.14159265. but we can not write the exact value of π.

Significant Digit

- The digits that are used to express a number are called significant digits or significant figures. Thus, the numbers 3.1416, 0.66667 or 4.0867 contains 5 significant digits each.

- However, the number 0.00023 has only two significant digits (2 and 3), since the zeros serve only to determine the position of the decimal.

- Similarly the numbers 0.00145, 0.000145 and 0.000000145 all have three significant digits.

Rounding off

- we come across numbers which have large number of digits, and it will be necessary to cut them to a usable number of figures. This process is called rounding off.
- To round off a number to *n* significant digits, discard all numbers to the right of the *nth* digit and if this discarded number is -

a) Les than half a unit in the *n*th place, leave the *n*th digit unaltered.

b) Greater than half a unit in the *n*th place, increase the *n*th digit by unity.

c) Exactly half a unit in the *n*th place, increase the *n*th digit by unity if it is odd, otherwise leave it unchanged.

Example

Round off the given numbers to **four significant** digits –

1. 1.6583
2. 30.0567
3. 0.859378
4. 3.14159
5. 42.738549
6. 0.00173148
7. 265475

Example

>> Rounded-off to four significant digits:

1. 1.658
2. 30.06
3. 0.8594
4. 3.142
5. 42.74
6. 0.001731
7. 2655

❖ **Mathematical Model:** A mathematical model is a description of a system using mathematical concepts and language. It is when we explain a natural phenomena using mathematical equations.

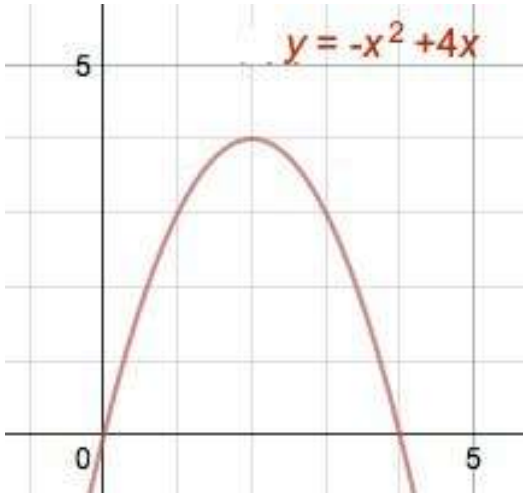
f(x) = 0

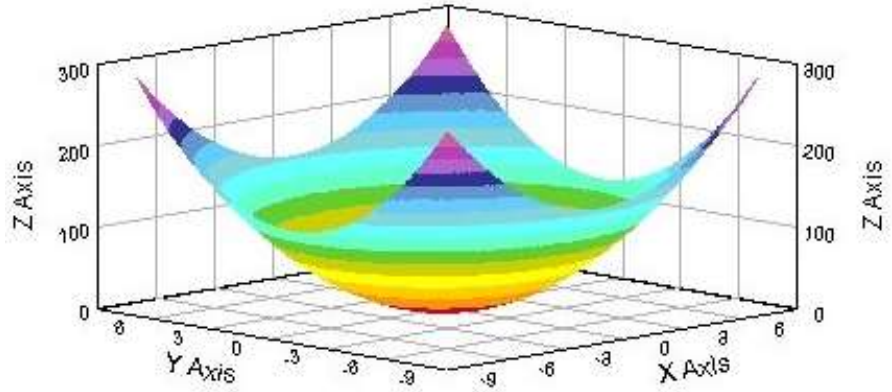
x = root

x is called 'Zero of equation'.

❖ **Linear Function**: f(x) is called to be a linear function If the dependent variable y changes in the direct proportion to the change in independent variable x. for example, $y = 3x + 5$

❖ **Non-linear function**: Any function of one variable which does not graph as a straight line in two dimensions or any functions of two variables which does not graph as a plane in three dimensions are said to be non-linear function.





Types of Equation

- **Algebraic Equation:** An algebraic equation can be defined as a mathematical statement in which two expressions are set equal to each other. Example,

$$3x + 4y - 1 = 0\ (linear)$$
$$2x + 3xy - 2 = 0\ (non - linear)$$
$$x^3 - xy = 0\ (non - linear)$$

- **Polynomial Equation:** The equations formed with variables, exponents and coefficients are called as polynomial equations.

$$x^3 + x^2 - x + 1 = 0$$

Types of Equation

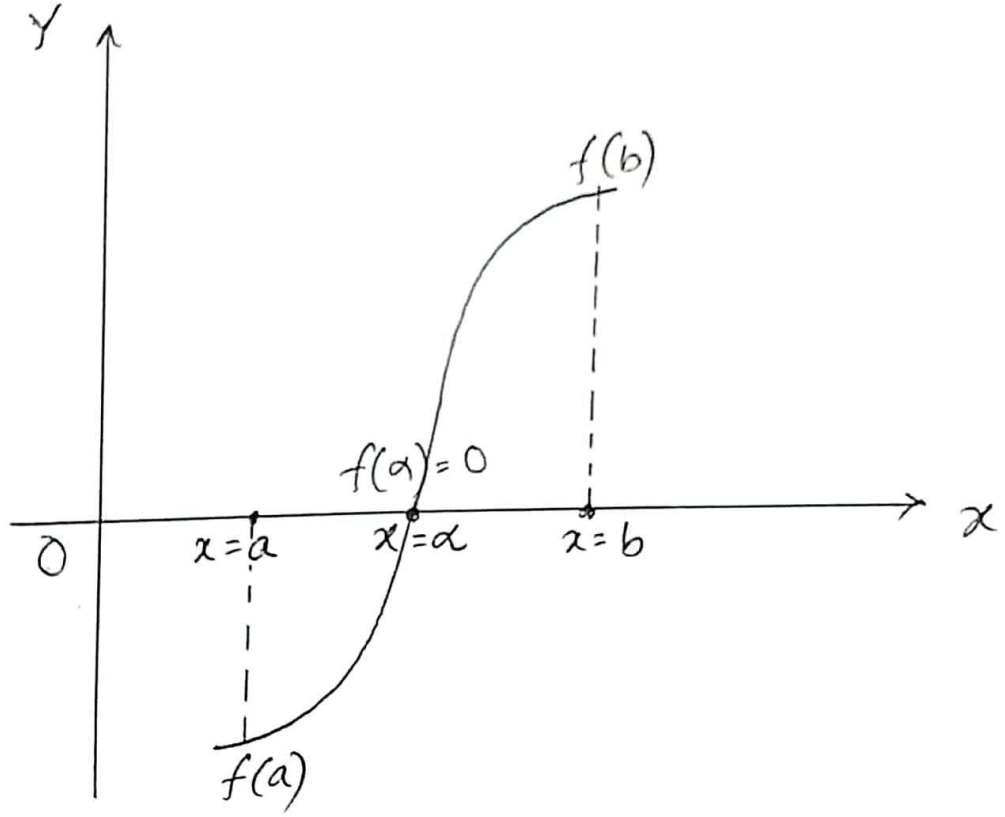
- **Transcendental Equation**: A non-linear equation which includes trigonometric, exponential and logarithmic expression is called transcendental equation.

$$\sin x + \sin y = 0$$
$$e^x sinx - 1 = 0$$
$$\log x + 2 = 0$$

Mathematical Preliminaries

Theorem 1:

If f(x) is continuous in $a \leq x \leq b$
And if f(a) and f(b) are of **opposite signs** then f(α)=0 for at least one number α, such that $a < \alpha < b$.



Mathematical Preliminaries

Theorem 2:

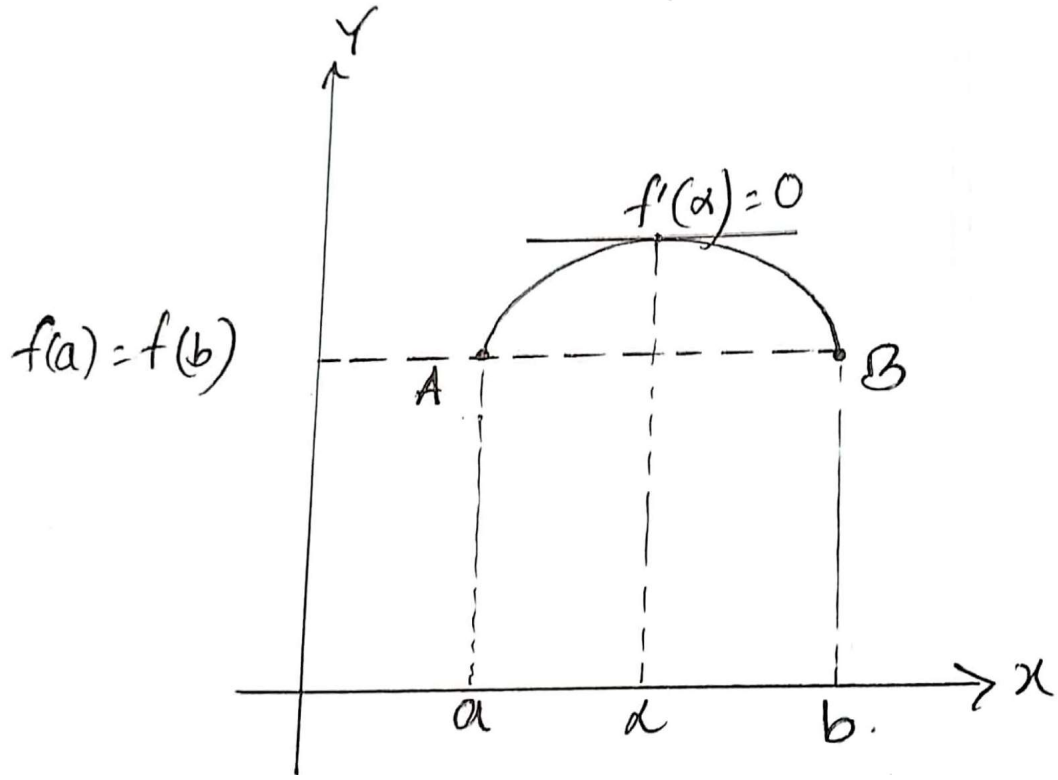
Rolle's Theorem: if-

a) f(x) is continuous between a $\leq x \leq b$

b) f'(x) exists in a $< x < b$ and,

c) f(a) = f(b)

Then there exists at least one value of x, say α such that f'(α)=0 , a $< \alpha < b$



Mathematical Preliminaries

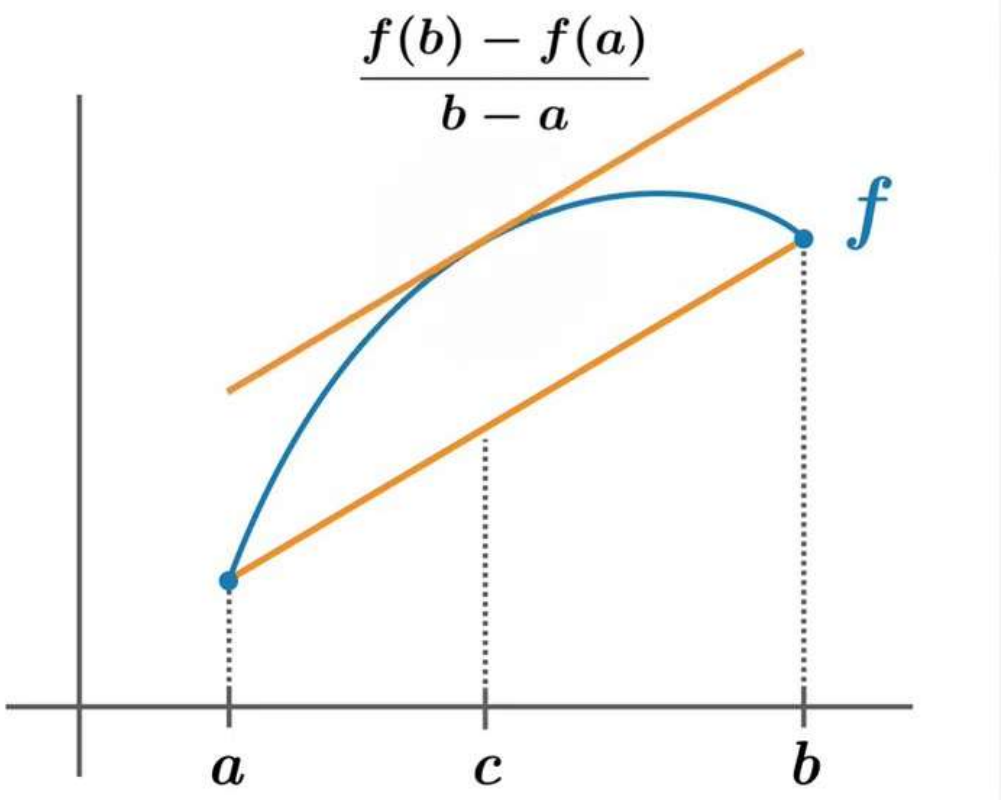
Theorem 3: Intermediate value theorem

Suppose f is continuous on $[a, b]$ and differentiable on (a, b). Then there exists some argument c such that

$$a < c < b$$

and

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$



Method of Solution

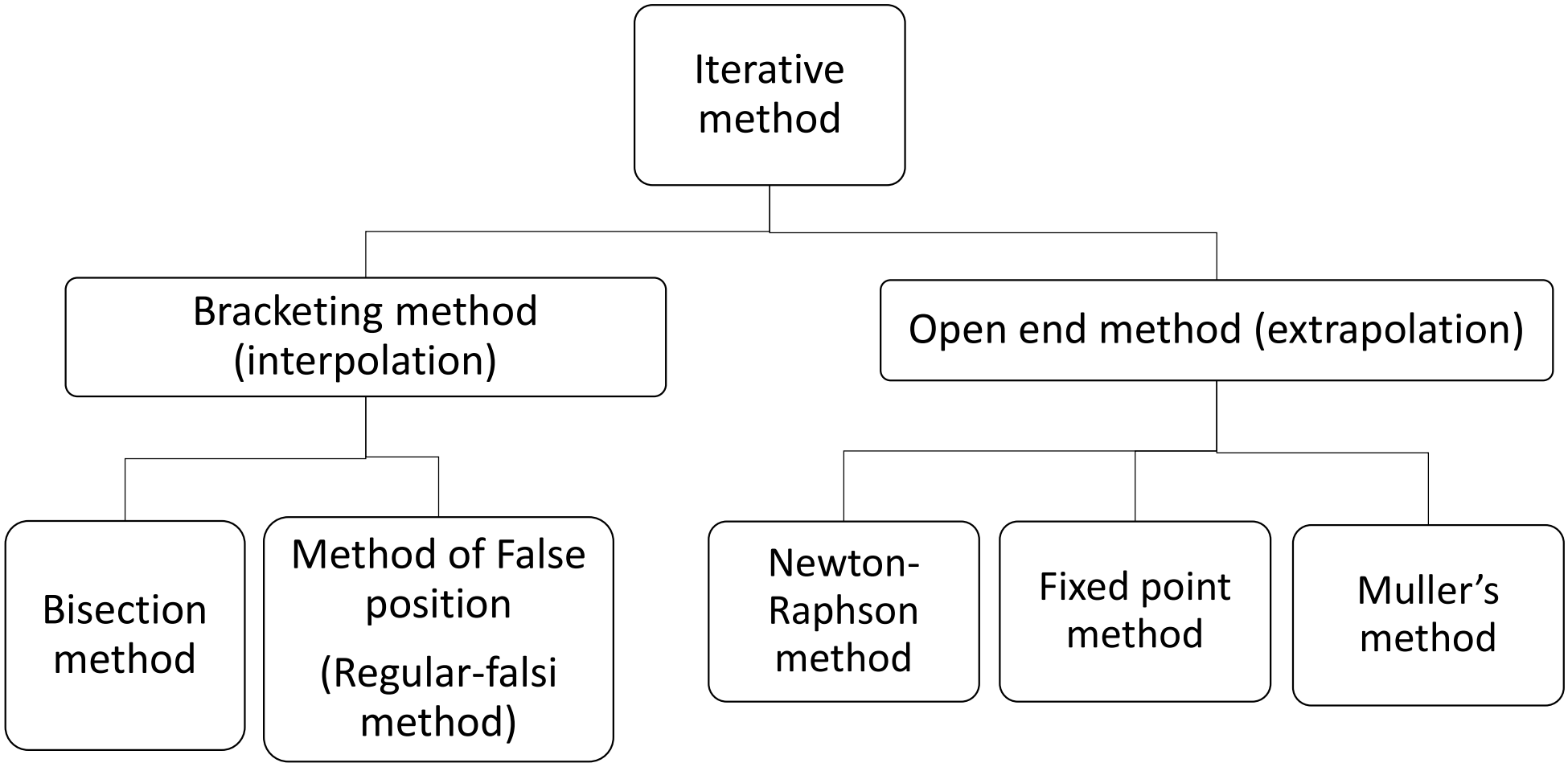
- Direct analytical method
- Graphical method
- Trial and error method
- Iterative method: The **Iterative Method** is a mathematical way of solving a problem which generates a sequence of approximations.

This method is applicable for both linear and nonlinear problems with large number of variables. The word Iterative or Iteration refers to the technique that solve any problems with successive approximation at each step.

Iterative Method

- An **Iterative method** is a mathematical procedure that uses an initial value to generate a sequence of improving approximate solutions for a class of problems, in which the n-th approximation is derived from the previous ones.

- In it, a calculation is repeated multiple times and the answer from each iteration is used as the basis for the next calculation. The answer gets better after each iteration.


Iterative method
Bracketing method (interpolation)
Open end method (extrapolation)
Bisection method
Method of False position
(Regular-falsi method)
Newton-Raphson method
Fixed point method
Muller's method

- **Bracketing Method:** Bracketing methods use two initial guesses and the bounds converge to the solution.

- **Open Method:** open methods require only a single starting value or two starting values that do not necessarily bracket a root. Open methods may diverge as the computation progresses, but when they do converge, they usually do so much faster than bracketing methods.

Bisection Method

Bisection Method

This method is based on Theorem 1.1 which states that if a function $f(x)$ is continuous between a and b, and $f(a)$ and $f(b)$ are of opposite signs, then there exists at least one root between a and b. For definiteness, let $f(a)$ be negative and $f(b)$ be positive. Then the root lies between a and b and let its approximate value be given by $x_0 = (a + b)/2$. If $f(x_0) = 0$, we conclude that x_0 is a root of the equation $f(x) = 0$. Otherwise, the root lies either between x_0 and b, or between x_0 and a depending on whether $f(x_0)$ is negative or positive. We

Then as before we bisect the interval, repeat the process until the root is known to be of desired accuracy.

Bisection Method

The method is shown graphically in Fig. 2.1.

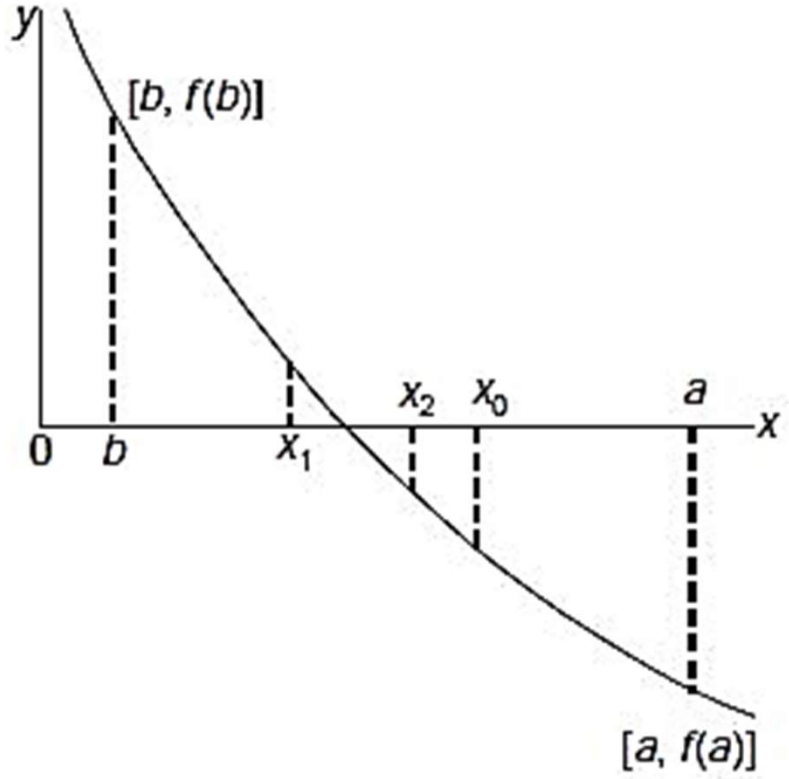


Figure 2.1 Graphical representation of the bisection method.

Steps

1. Choose two real numbers a and b such that $f(a)\, f(b) < 0$.
2. Set $x_r = (a + b)/2$.
3. (a) If $f(a)\, f(x_r) < 0$, the root lies in the interval (a, x_r). Then, set $b = x_r$ and go to step 2 above.

   (b) If $f(a)\, f(x_r) > 0$, the root lies in the interval (x_r, b). Then, set $a = x_r$ and go to step 2.

   (c) If $f(a)\, f(x_r) = 0$, it means that x_r is a root of the equation $f(x) = 0$ and the computation may be terminated.

Bisection Method

Example: 1

Find point a, b *such that*
$a < b$ *and* $f(a).f(b) < 0$

$$xo = (a + b)/2$$

(1) Find the root of the following equation,

$f(x) = x^3 - x - 1$.

desired accuracy. 0.001.

Ans: for, 0. $f(0) = -1$. (-ve value)
1. $f(1) = -1$. (-ve value) ✓
2. $f(2) = 5$ (+ve value). ✓

so we will take,

$a = 1$, $f(a) = -1$.
$b = 2$, $f(b) = 5$
[since, $f(a) \times f(b) < 0$]

$\therefore x_0 = \frac{a+b}{2} = \frac{1+2}{2} = 1.5$

$f(x_0) = 1.5^3 - 1.5 - 1 = 0.875$ (+ve value)

$\therefore x_1 = \frac{1+1.5}{2} = 1.25$

$f(x_1) = 1.25^3 - 1.25 - 1 = -0.2969$ (-ve value).

$\therefore x_2 = \frac{1.25 + 1.5}{2} = 1.375$

$f(x_2) = 1.375^3 - 1.375 - 1$

$= 0.2246$. (+ve value).

Bisection Method

n (number of iteration)	a (-ve)	b (+ve)	Xo = (a+b)/2	f(xo)	update
1	1	2	1.5	0.875	b =Xo
2	1	1.5	1.25	-0.29688	a = Xo
3	1.25	1.5	1.375	0.225	B=xo
4	1.25	1.375	1.3125	-0.0515	A=xo

Bisection method

Example:

n	a(-ve)	b(+ve)	$x_0 = \frac{a+b}{2}$	$f(x_0)$	Sign	Update
1.	1	2	1.5	0.875	+ve	$b = x_0$
2.	1	1.5	1.25	−0.29688	−ve	$a = x_0$
3.	1.25	1.5	1.375	0.22461	+ve	$b = x_0$
4.	1.25	1.375	1.3125	−0.05151	−ve	$a = x_0$
5.	1.3125	1.375	1.34375	0.08261	+ve	$b = x_0$
6.	1.3125	1.34375	1.32812	0.01458	+ve	$b = x_0$
7.	1.3125	1.32812	1.32031	−0.01871	−ve	$a = x_0$
8.	1.32031	1.32812	1.32422	−0.00213	−ve	$a = x_0$
9.	1.32422	1.32812	1.32617	0.00621	+ve	$b = x_0$
10.	1.32422	1.32617	1.3252	0.00204	+ve	$b = x_0$
11.	1.32422	1.3252	1.32471	−0.00005	−ve	$a = x_0$
12	1.3247	1.3252	1.3249	0.0007	+ve	$b = x_0$

∴ Approximate root of the equation. $x^3 - x - 1 = 0$, using bisection method is 1.32471.

(Ans).

Bisection method

Example: 2

Find a root of the equation $f(x) = \sqrt{12}$ using bisection method. Correct to 3 decimal place and mention the iteration number.

Soln:

Here $\sqrt{12}$=0

Let $f(x)=\sqrt{12}$

$x=\sqrt{12}$

$\therefore x^2=12$

$\therefore x^2-12=0$

For 0, f(0) = -12

For 1, f(1) = -11

For 2, f(2) = -8

For 3, f(3)= -3

For 4, f(4) = 4

So, taking a = 3 , f(a) = -3

and b =4, f(b) = 4 [f(a).f(b) <0]

1*st* iteration :

Here $f(3)=-3<0$ and $f(4)=4>0$

∴ Now, Root lies between 3 and 4

$x0=(3+4)/2=3.5$

$f(x0)=f(3.5) = 3.5^2-12 = 0.25>0$

2*nd* iteration : Here $f(3)= -3<0$ and $f(3.5)= 0.25>0$

$\therefore$ Now, Root lies between 3 and 3.5

$x1=3+3.52=3.25$

$f(x1)= f(3.25)= 3.252-12= -1.4375<0$

3*rd* iteration : Here $f(3.25)= -1.4375<0$ and $f(3.5)= 0.25>0$

$\therefore$ Now, Root lies between 3.25 and 3.5

$x2=3.25+3.52=3.375$

$f(x2)=f(3.375)=3.3752-12=-0.6094<0$

n (number of iteration)	a (-ve)	b (+ve)	Xo = (a+b)/2	f(xo)	update

Approximate root of the equation $x^2 - 12 = 0$ using Bisection mehtod is 3.4641 (After 12 iterations)

n	a	$f(a)$	b	$f(b)$	$c = \frac{a+b}{2}$	$f(c)$	**Update**
1	3	-3	4	4	3.5	0.25	$b = c$
2	3	-3	3.5	0.25	3.25	-1.4375	$a = c$
3	3.25	-1.4375	3.5	0.25	3.375	-0.6094	$a = c$
4	3.375	-0.6094	3.5	0.25	3.4375	-0.1836	$a = c$
5	3.4375	-0.1836	3.5	0.25	3.4688	0.0322	$b = c$
6	3.4375	-0.1836	3.4688	0.0322	3.4531	-0.0759	$a = c$
7	3.4531	-0.0759	3.4688	0.0322	3.4609	-0.0219	$a = c$
8	3.4609	-0.0219	3.4688	0.0322	3.4648	0.0051	$b = c$
9	3.4609	-0.0219	3.4648	0.0051	3.4629	-0.0084	$a = c$
10	3.4629	-0.0084	3.4648	0.0051	3.4639	-0.0016	$a = c$
11	3.4639	-0.0016	3.4648	0.0051	3.4644	0.0018	$b = c$
12	3.4639	-0.0016	3.4644	0.0018	3.4641	0.0001	$b = c$