Numerical Methods
EWCE 205

Lecture-02

Iteration Method

Iteration Method

To find the root of the equation $f(x) = 0$ .....(i)

We need to rewrite the equation in the form of $x = \emptyset(x)$ .....(ii)

$x0$ is the initial root of $f(x) = 0$

substituting it for x on the right side of equation (ii) we obtain the first approximation.

$$x1 = \emptyset(x0) \text{ .....(iii)}$$

The successive approximation is given by,

$x2 = \emptyset(x1)$

$x3 = \emptyset(x2)$

. . . . . .

$xn = \emptyset(xn - 1)$

Condition:

if $|\emptyset'(x)| < 1$

Then equation(ii) can be used as iteration formula.

- Iteration method requires **one starting value** of x.

Step-1: First write the equation in the form of $x=\phi(x)$

Step-2: Find points a and b such that $a<b$ and $f(a)\cdot f(b)<0$.

Step-3: If f(a) is more closer to 0 than f(b)
then x0=a else x0=b

Step-4: $x1=\phi(x0)$
$x2=\phi(x1)$
$x3=\phi(x2)$
...
Repeat until $|f(Xn)-f(Xn-1)|\approx 0$

Example-1

(1) Find a real root of the equation $x^3 + x^2 - 1 = 0$ by using iteration method. Correct upto 4 decimal place

Ans: Let $f(x) = x^3 + x^2 - 1 = 0$

$f(0) = 0^3 + 0^2 - 1 = -1$

$f(1) = 1^3 + 1^2 - 1 = 1 > 0.$

so the root of $f(x)$ lies between 0 and 1.

so taking $x_0 = 0.5.$

$|\phi'(x)| < 1.$

Case-1:

$x^3 + x^2 - 1 = 0$

$\Rightarrow x^3 = 1 - x^2$

$\Rightarrow x = (1-x^2)^{\frac{1}{3}}$

$\therefore \phi(x) = (1-x^2)^{\frac{1}{3}}$

$\phi'(x) = \frac{1}{3}(1-x^2)^{\frac{1}{3}-1} \cdot (-2x)$

$\phi'(x) = \frac{-2x}{3(1-x^2)^{2/3}}$

$\phi'(0.5) = \frac{-2 \times 0.5}{3(1-0.5^2)^{2/3}}$

$= -0.4038$

$|\phi'(0.5)| = 0.4038 < 1.$

Case-2:

$x^3 + x^2 - 1 = 0$

$\Rightarrow x^2 = 1 - x^3$

$\Rightarrow x = \sqrt{1-x^3}$

$\therefore \phi(x) = \sqrt{1-x^3}$

$\phi'(x) = \frac{1}{2}(1-x^3)^{1/2-1} \times (-3x^2)$

$\therefore \phi'(x) = \frac{-3x^2}{2(1-x^3)^{1/2}}$

$\therefore \phi'(0.5) = \frac{-3 \times (0.5)^2}{2(1-0.5)^{3/2}}$

$= -0.4009$

$\therefore |\phi'(0.5)| = 0.4009 < 1.$

Case-3:

$x^3 + x^2 - 1 = 0$

$\Rightarrow x^2(x+1) - 1 = 0$

$\Rightarrow x^2 = \frac{1}{x+1}$

$\Rightarrow x = \left(\frac{1}{1+x}\right)^{1/2}$

$\therefore \phi(x) = \left(\frac{1}{(1+x)}\right)^{1/2}$

$\phi'(x) = -\frac{1}{2}(1+x)^{-3/2}$

$\therefore \phi'(x) = -\frac{1}{2(1+x)^{3/2}}$

$\phi'(0.5) = -\frac{1}{2(1+0.5)^{3/2}}$

$= -0.2722$

$\therefore |\phi'(0.5)| = 0.2722 < 1$

here, all 3 cases satisfy the condition $|\phi'(x)| < 1$. but taking case-3, as it has the minimum value.

$$x_n = \sqrt{\frac{1}{x_{n-1}+1}}$$

$$\therefore x_{n+1} = \sqrt{\frac{1}{x_n+1}} \quad \text{[iteration formula]}.$$

$n=0$, $x_1 = \sqrt{\frac{1}{x_0+1}} = \sqrt{\frac{1}{0.5+1}} = 0.81649.$

$n=1$, $x_2 = \sqrt{\frac{1}{x_1+1}} = \sqrt{\frac{1}{0.81649+1}} = 0.74196.$

$n=2$, $x_3 = \sqrt{\frac{1}{x_2+1}} = \sqrt{\frac{1}{0.74196+1}} = 0.75767$

$n=3$, $x_4 = \sqrt{\frac{1}{x_3+1}} = \sqrt{\frac{1}{0.75767+1}} = 0.75427$

$n=4, \quad x_5 = \sqrt{\frac{1}{x_4+1}} = \sqrt{\frac{1}{0.75427+1}} = 0.7550$

$n=5, \quad x_6 = \sqrt{\frac{1}{x_5+1}} = \sqrt{\frac{1}{0.7550+1}} = 0.75485$

$n=6, \quad x_7 = \sqrt{\frac{1}{x_6+1}} = \sqrt{\frac{1}{0.75485+1}} = 0.75488$

Ans: The approximate root of the equation $x^3+x^2-1=0$, using iteration method is 0.7548 (at 7th iteration).

- **Example-2: Find a root of an equation $x^3 - x - 1 = 0$ using Iteration method. Desired accuracy 0.001.**
- **Solution:**

  Let $f(x) = x^3 - x - 1 = 0$

  Here,

x	0	1	2
f(x)	-1	-1	5

So the root lies between 1 and 2.

Taking initial root, x0= 1.5

Condition:

if $| \emptyset'(x) | < 1$

Case-1:

$x3\text{-}x\text{-}1=0$

$\therefore x^3 = 1 + x$

$\therefore x=3\sqrt{(x+1)}$

$\therefore \phi(x)=3\sqrt{(x+1)}$

$\therefore \phi'(x)= \frac{1}{3(1+x)^{3/2}}$

$|\phi'(1.5)|= \frac{1}{3(1+x)^{3/2}}$

$=0.08432<1$

Case-2:

$x3\text{-}x\text{-}1=0$

$x = 1 - x^3$

$\therefore \phi(x)= 1 - x^3$

$\therefore \phi'(x)= -3x^2$

$\therefore \phi'(1.5)= -3 \times (1.5)^2$

$|\phi'(1.5)|= 6.75 > 1$

It does not satisfy the condition.

Case-3:

$x3\text{-}x\text{-}1=0$

$\therefore x3\text{-}x=1$

$\therefore x(x2\text{-}1)=1$

$\therefore x=1/x2\text{-}1$

$\therefore \phi(x)=1/x2\text{-}1$

$x_{n+1} = \sqrt[3]{(x_n + 1)}$ [Iteration formula]

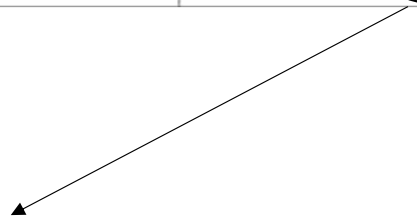
$X1 = \phi(x0) = \phi(1.5) = 1.35721$

$x2 = \phi(x1) = \phi(1.35721) = 1.33086$

$x3 = \phi(x2) = \phi(1.33086) = 1.32588$

X4=

n	x_0	$x_1 = \phi(x_0)$	Update	Difference $x_1 - x_0$
2	1.5	1.35721	$x_0 = x_1$	0.14279
3	1.35721	1.33086	$x_0 = x_1$	0.02635
4	1.33086	1.32588	$x_0 = x_1$	0.00498
5	1.32588	1.32494	$x_0 = x_1$	0.00094
6	1.32494	1.32476	$x_0 = x_1$	0.00018



Ans: 1.325