Interpolation

Newton's Formula for Interpolation

Interpolation

- In the mathematical field of numerical analysis, interpolation is a type of estimation, a method of constructing new data points within the range of a discrete set of known data points.
- **can be used to estimate data points in-between the given ones.**
- In engineering and science, one often has a number of data points, obtained by sampling or experimentation, which represent the values of a function for a limited number of values of the independent variable. It is often required to interpolate, i.e., estimate the value of that function for an intermediate value of the independent variable.

Given data

X	Y
X_o	Y_o
X_1	Y_1
X_2	Y_2
X_3	Y_3
X_4	Y_4
X_5	Y_5
.....	.....
X_n	Y_n

We need to Find $X = \text{Xi} = \text{Y(Xi)}$
Xi lies between X_0 to X_n.
(interpolation)

Extrapolation: when Xi isn't between X_0 to X_n.

3.3 FINITE DIFFERENCES

Assume that we have a table of values (x_i, y_i), $i = 0, 1, 2, \ldots, n$ of any function $y = f(x)$, the values of x being equally spaced, i.e., $x_i = x_0 + ih$, $i = 0, 1, 2, \ldots, n$. Suppose that we are required to recover the values of $f(x)$ for some intermediate values of x, or to obtain the derivative of $f(x)$ for some x in the range $x_0 \leq x \leq x_n$. The methods for the solution to these problems are based on the concept of the 'differences' of a function which we now proceed to define.

3.3.1 Forward Differences

If $y_0, y_1, y_2, \ldots, y_n$ denote a set of values of y, then $y_1 - y_0, y_2 - y_1, \ldots, y_n - y_{n-1}$ are called the *differences* of y. Denoting these differences by $\Delta y_0, \Delta y_1, \ldots, \Delta y_{n-1}$ respectively, we have

$$\Delta y_0 = y_1 - y_0, \qquad \Delta y_1 = y_2 - y_1, \ \ldots, \qquad \Delta y_{n-1} = y_n - y_{n-1},$$

where Δ is called the *forward difference operator* and $\Delta y_0, \Delta y_1, \ldots,$ are called *first forward differences*. The differences of the first forward differences are called *second forward differences* and are denoted by $\Delta^2 y_0, \Delta^2 y_1, \ldots$ Similarly, one can define *third forward differences*, *fourth forward differences*, etc.

$$\Delta^2 y_0 = \Delta y_1 - \Delta y_0 = y_2 - y_1 - (y_1 - y_0)$$
$$= y_2 - 2y_1 + y_0,$$
$$\Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_0 = y_3 - 2y_2 + y_1 - (y_2 - 2y_1 + y_0)$$
$$= y_3 - 3y_2 + 3y_1 - y_0$$
$$\Delta^4 y_0 = \Delta^3 y_1 - \Delta^3 y_0 = y_4 - 3y_3 + 3y_2 - y_1 - (y_3 - 3y_2 + 3y_1 - y_0)$$
$$= y_4 - 4y_3 + 6y_2 - 4y_1 + y_0.$$

It is, therefore, clear that any higher-order difference can easily be expressed in terms of the ordinates, since the coefficients occurring on the right side are the binomial coefficients.

Table 3.1 shows how the forward differences of all orders can be formed:

Table 3.1 Forward Difference Table

x	y_0	Δ	Δ^2	Δ^3	Δ^4	Δ^5	Δ^6
x_0	y_0						
		Δy_0					
x_1	y_1		$\Delta^2 y_0$				
		Δy_1		$\Delta^3 y_0$			
x_2	y_2		$\Delta^2 y_1$		$\Delta^4 y_0$		
		Δy_2		$\Delta^3 y_1$		$\Delta^5 y_0$	
x_3	y_3		$\Delta^2 y_2$		$\Delta^4 y_1$		$\Delta^6 y_0$
		Δy_3		$\Delta^3 y_2$		$\Delta^5 y_1$	
x_4	y_4		$\Delta^2 y_3$		$\Delta^4 y_2$		
		Δy_4		$\Delta^3 y_3$			
x_5	y_5		$\Delta^2 y_4$				
		Δy_5					
x_6	y_6						

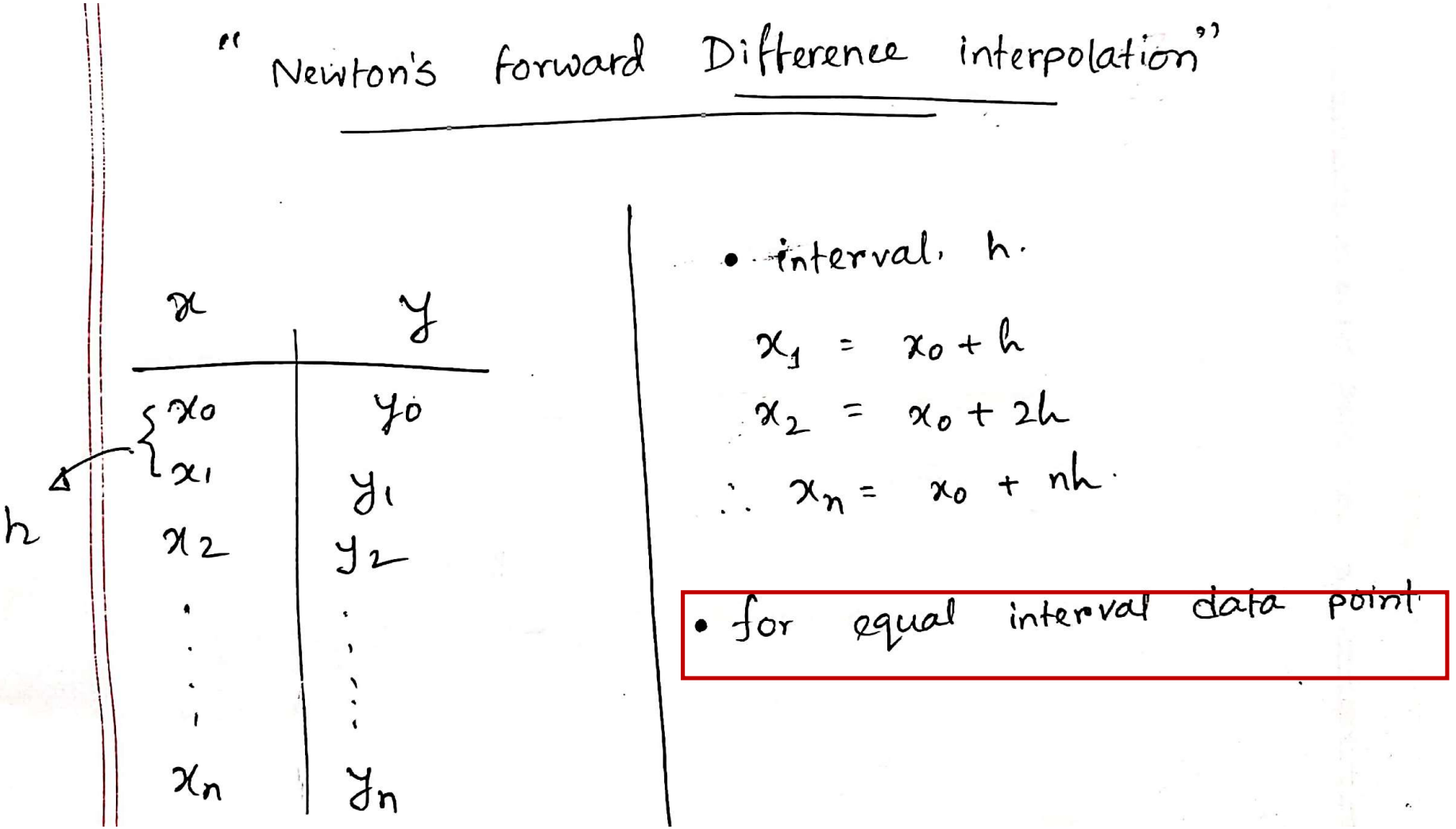
$y1 - yo$

$y2 - y1$

$y5 - y4$

$\Delta^5 y1 - \Delta^5 y0$

Interpolation – Newton's forward difference formula



Formula:

$$y_n(x) = f(x_n) = y_0 + p\Delta y_0 + \frac{p(p-1)}{\lfloor 2}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{\lfloor 3}\Delta^3 y_0 + \cdots\cdots + \frac{p(p-1)(p-2)\cdots(p-n+1)}{\lfloor n}\Delta^n y_0$$

where, $p = \dfrac{x - x_0}{h}$ $\longrightarrow$ $x = x_0 + ph$.

h → interval

Question : Find the cubic polynomial which takes the following values :

x :	0	1	2	3
$f(x)$:	1	2	1	10

Also find $f(4)$ and $f'(4)$.

Soln: The ~~two~~ forward difference table for given data points:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
0	1			
		$\Delta y_0 = (2-1) = 1$		
1	2		$\Delta^2 y_0 = (-1-1) = -2$	
		$\Delta y_1 = (1-2) = -1$		$\Delta^3 y_0 = 12$
2	1		$\Delta^2 y_1 = 10$	
		$\Delta y_2 = 9$		
3	10			

here, $x_0 = 0$, $y_0 = 1$, interval, $h = 1$.

$$\therefore p = \frac{x - x_0}{h} = \frac{x - 0}{1} = x.$$

from Newton's forward interpolation formula,

$$y_n(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{\lfloor 2}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{\lfloor 3}\Delta^3 y_0$$

$$\Rightarrow f(x) = 1 + x \cdot 1 + \frac{x(x-1)}{2}(-2) + x\frac{(x-1)(x-2)}{6}(12)$$

$$\therefore f(x) = 2x^3 - 7x^2 + 6x + 1 \checkmark$$

$$\therefore f'(x) = 6x^2 - 14x + 6.$$

now we need to find $f(4)$ & $f'(4)$

$$\therefore f(4) = 2(4)^3 - 7(4)^2 + 6 \cdot 4 + 1$$
$$= 41 \checkmark$$

$$\therefore f'(4) = 6 \cdot (4)^2 - 14 \cdot (4) + 6$$
$$= 46 \checkmark$$

Example-2:

Question: From the table, estimate the number of students who obtained marks between 40 and 45.

marks:	30-40	40-50	50-60	60-70	70-80
No. of students:	31	42	51	35	31.

Soln: The forward difference table (rearranging the given data):

Marks below (x)	No. of stud. (y)	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
$x_0 = 40$	$y_0 = 31$				
		$\Delta y_0 = 42$			
50	73		$\Delta^2 y_0 = 9$		
		$\Delta y_1 = 51$		-25	
60	124		-16		37
		$\Delta y_2 = 35$		12	
70	159		-4		
		$\Delta y_3 = 31$.			
80	190				

Let $x = 45$, $x_0 = 40$, $h = 10$,

$$\Rightarrow p = \frac{x - x_0}{h} = \frac{5}{10} = 0.5$$

By Newton's forward interpolation formula -

$$f(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{\lfloor 2}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{\lfloor 3}\Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{\lfloor 4}\cdot\Delta^4 y_0 .$$

$$= 31 + 0.5 \times 42 + \frac{0.5(-0.5)}{2}(9) + \frac{0.5(-0.5)(-1.5)}{6}(-25) + \frac{0.5(-0.5)(-1.5)(-2.5)}{24}\cdot(37)$$

$\therefore f(45) = 47.87$

≈ 48. (No. of stud. having marks below 45).

$\therefore$ No. of students obtaining marks between 40 and 45 $= 48 - 31$

$= 17$ ✓