Numerical Differentiation

Numerical Differentiation

x	y
x_0	y_0
x_1	y_1
x_2	y_2
x_3	y_3
x_4	y_4

for, $x = x_i$,

↳ $y'(x_i)$, $y''(x_i)$

(1) Equal Interval.

(2) Unequal Interval.

Newton's forward Interpolation formula.

$$y(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{\lfloor 2}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{\lfloor 3}\Delta^3 y_0 + \cdots$$

where,

$$p = \frac{x - x_0}{h} \Rightarrow x = x_0 + ph.$$

$$\Rightarrow y(x_0 + ph) = y_0 + p\Delta y_0 + \frac{p(p-1)}{\lfloor 2}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{\lfloor 3}\Delta^3 y_0 + \cdots$$

Differentiating with respect to 'p',

$$y'(x).h = \frac{dy}{dx}.h = \Delta y_0 + \frac{p^2 - p}{\lfloor 2}.\Delta^2 y_0 + \frac{p^3 - 3p^2 + 2p}{\lfloor 3}\Delta^3 y_0 + \cdots$$

$$\Rightarrow \frac{dy}{dx} = y'(x) = \frac{1}{h}\left[\Delta y_0 + \frac{2p-1}{\lfloor 2}\Delta^2 y_0 + \frac{3p^2 - 6p + 2}{\lfloor 3}\Delta^3 y_0 + \cdots\right]$$

Differentiating again with respect to 'p'.

$$\Rightarrow \frac{d^2y}{dx^2} = y''(x) = \frac{1}{h^2}\left[\Delta^2 y_0 + (p-1)\Delta^3 y_0 + \cdots\right]$$

When $x = x_0$, so $p = 0$, we have:

① $$\frac{dy}{dx} = y'(x_0) = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \cdots\right].$$

② $$\frac{d^2y}{dx^2} = y''(x_0) = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 + \cdots\right].$$

Example- 1: Find $\frac{dy}{dx}$ at $x = 1.5$ from following table.

x :	1.5	2.0	2.5	3.0	3.5	4.0
y :	3.375	7.0	13.625	24	38.875	59

Solution: The forward difference table for the given data:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
$x_0 = 1.5$	3.375				
		3.625			
2	7		3		
		6.625		0.75	
2.5	13.625		3.75		0
		10.375		0.75	
3	24		4.5		0
		14.875		0.75	
3.5	38.875		2.25		
		20.125			
4	59				

Given, $x_0 = 1.5$, $x = 1.5$, $h = 0.5$, $p = 0$.

from the formula,

$$\left(\frac{dy}{dx}\right)_{x=1.5} = y'(1.5) = \frac{1}{0.5}\left[3.625 - \frac{1}{2}\cdot 3 + \frac{1}{3}\cdot 0.75\right]$$

$$y'(1.5) = 4.75 \quad \checkmark \text{(Ans)}$$

Example:2

Use Newton's forward/backward differentiation method to find solution for x=0

x	f(x)
0.0	1.0000
0.1	0.9975
0.2	0.9900
0.3	0.9776
0.4	0.8604

Solution:

Newton's forward differentiation table is as follows.

Newton's forward differentiation table is as follows.

X	Y(X)	ΔY	$\Delta^2 Y$	$\Delta^3 Y$	$\Delta^4 Y$
0	1				
		-0.0025			
0.1	0.9975		**-0.005**		
		-0.0075		**0.0001**	
0.2	0.99		-0.0049		**-0.1**
		-0.0124		-0.0999	
0.3	0.9776		-0.1048		
		-0.1172			
0.4	0.8604				

Solution for $y'(0)=0.25033$

Solution for $y''(0)= -9.67667$

Example 3: Find Solution of an Equation $2x^3 - 4x + 1$

(x1 = 2 and x2 = 4, Step value (h) = 0.25)

For x = 2.25, use Newton's Forward Difference formula.

Solution:

Equation is $f(x) = 2x^3 - 4x + 1$.

Numerical differentiation method to find solution.

The value of table for *x* and *y*

x	2	2.25	2.5	2.75	3	3.25	3.5	3.75	4
y	9	14.7812	22.25	31.5938	43	56.6562	72.75	91.4688	113

Newton's forward differentiation table is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
2	9				
		5.7812			
2.25	**14.7812**		1.6875		
		7.4688		0.1875	
2.5	22.25		**1.875**		0
		9.3438		**0.1875**	
2.75	31.5938		2.0625		**0**
		11.4062		0.1875	
3	43		2.25		0
		13.6562		0.1875	
3.25	56.6562		2.4375		0
		16.0938		0.1875	
3.5	72.75		2.625		0
		18.7188		0.1875	
3.75	91.4688		2.8125		
		21.5312			
4	113				

The value of x at you want to find $f(x)$:$x0=2.25$

$h=x1-x0=2.25-2=0.25$

$[dy/dx] =1/h\cdot(\Delta y0 -1/2\cdot\Delta 2y0 +1/3\cdot\Delta 3y0 -1/4\cdot\Delta 4y0)$

$\therefore[dy/dx]=1/0.25\cdot(7.4688 -1/2\times1.875 +1/3\times0.1875 -1/4\times0)$

$\therefore[dy/dx] =26.375$

$[d2y/dx2] =1/h2\cdot(\Delta 2y0 -\Delta 3y0 +11/12\cdot\Delta 4y0)$

$\therefore[d2y/dx2] =1/0.0625\cdot(1.875 -0.1875 +11/12\times0)$

$\therefore[d2y/dx2] =27$

$\therefore y'(2.25)=26.375$ and $y''(2.25)=27$

Newton's Backward Difference formula

Newton's Backward Difference formula

① For any value of x,

$$\frac{dy}{dx} = y'(x) = \frac{1}{h}\left[\nabla y_n + \frac{2p+1}{\lfloor 2}\nabla^2 y_n + \frac{3p^2+6p+2}{\lfloor 3}\nabla^3 y_n + \cdots\right]$$

$$\frac{d^2y}{dx^2} = y''(x) = \frac{1}{h^2}\left[\nabla^2 y_n + (p+1)\nabla^3 y_n + \frac{12p^2+36p+22}{24}\nabla^4 y_n + \cdots\right]$$

② when, $x = x_n$, $\rightarrow p = 0$.

$$\frac{dy}{dx} = y'(x_n) = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \frac{1}{4}\nabla^4 y_n + \cdots\right]$$

$$\frac{d^2y}{dx^2} = y''(x_n) = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \cdots\right]$$

Example: Use Newton's backward differentiation method to find solution for x= 2.2

x	f(x)
1.4	4.0552
1.6	4.9530
1.8	6.0496
2.0	7.3891
2.2	9.0250

Solution: Numerical differentiation method to find solution.

The value of table for *x* and *y*

x	1.4	1.6	1.8	2	2.2
y	4.0552	4.953	6.0496	7.3891	9.025

Newton's backward differentiation table is

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
1.4	4.0552				
		0.8978			
1.6	4.953		0.1988		
		1.0966		0.0441	
1.8	6.0496		0.2429		**0.0094**
		1.3395		**0.0535**	
2	7.3891		**0.2964**		
		1.6359			
2.2	**9.025**				

The value of x at you want to find $f(x)$:xn=2.2

h=x1-x0=1.6-1.4=0.2

$[dy/dx]x=xn=1/h\cdot(\nabla yn+1/2\cdot\nabla 2yn+1/3\cdot\nabla 3yn+1/4\cdot\nabla 4yn)$

$\therefore[dy/dx]x=2.2=1/0.2\times(1.6359+1/2\times0.2964+1/3\times0.0535+1/4\times0.0094)$

$\therefore[dy/dx]x=2.2=9.02142$

$[d2y/dx2]x=xn=1/h2\cdot(\nabla 2yn+\nabla 3yn+11/12\cdot\nabla 4yn)$

$\therefore[d2y/dx2]x=2.2=1/0.04\cdot(0.2964+0.0535+11/12\times0.0094)$

$\therefore[d2y/dx2]x=2.2=8.96292$

$\therefore y'(2.2)=9.02142$ and $y''(2.2)=8.96292$

Example 2: Find Solution of an Equation $2x^3 - 4x + 1$

(x1 = 2 and x2 = 4)

For x = 3.5, Step value (h) = 0.25 .

Solution:

Equation is $f(x)= 2x^3 - 4x + 1$.

Numerical differentiation method to find solution.

The value of table for *x* and *y*

x	2	2.25	2.5	2.75	3	3.25	3.5	3.75	4
y	9	14.7812	22.25	31.5938	43	56.6562	72.75	91.4688	113

Answer: $y'(3.5)=69.5$ and $y''(3.5)=42$