EWCE-211
Concept of Strain

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Concept of Strain

- When a deformable body is subjected to stresses, it experiences deformations and distortions. The term deformations refers to the geometric changes that take place in the dimensions (extensions or contractions) of the body, while the term distortions represents geometric changes in its shape.
- Any line element in a stressed body is said to undergo deformation if its length increases or decreases. The term average strain of a line element is used to signify its unit deformation. This unit deformation is known as linear or normal strain and denoted by the symbol ε_n

NORMAL STRAIN UNDER AXIAL LOADING

Let us consider a rod BC, of length L and uniform cross-sectional area A, which is suspended from B (Fig. 2.1*a*). If we apply a load **P** to end C, the rod elongates (Fig. 2.1*b*). Plotting the magnitude P of the load against the deformation δ (Greek letter delta), we obtain a certain load-deformation diagram (Fig. 2.2). While this diagram contains information useful to the analysis of the rod under consideration, it cannot be used directly to predict the deformation of a rod of the same material but of different dimensions. Indeed, we observe that, if a deformation δ is produced in rod BC by a load **P**, a load $2\mathbf{P}$ is required to cause the same deformation in a rod $B'C'$ of the same length L, but of cross-sectional area $2A$ (Fig. 2.3). We note that, in both cases, the value of the stress is the same: $\sigma = P/A$. On the other hand, a load **P** applied to a rod $B''C''$, of the same cross-sectional area A, but of length $2L$, causes a deformation 2δ in that rod (Fig. 2.4), i.e., a deformation twice as large as the deformation δ it produces in rod BC. But in both cases the ratio of the deformation over the length of the rod is the same; it is equal to δ/L. This observation brings us to introduce the concept of *strain:* We define the *normal strain* in a rod under axial loading as the *deformation per unit length* of that rod. Denoting the normal strain

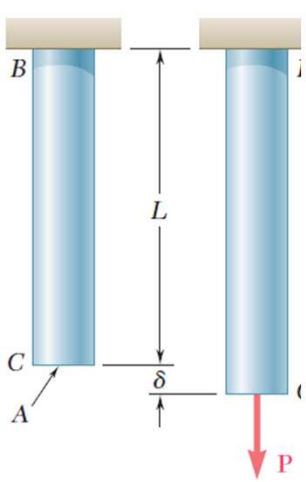


Fig. 2.1 Deformation of axially-loaded rod.

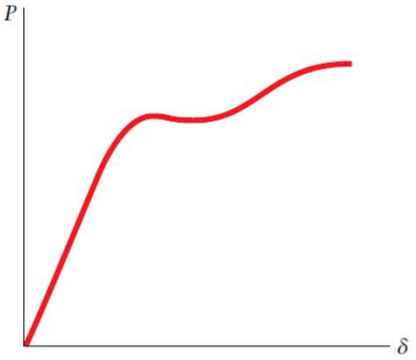


Fig. 2.2 Load-deformation diagram.

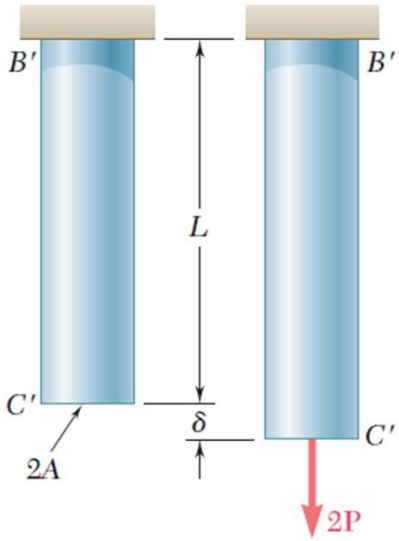


Fig. 2.3

- Denoting the normal strain by ε (Greek letter epsilon), we write

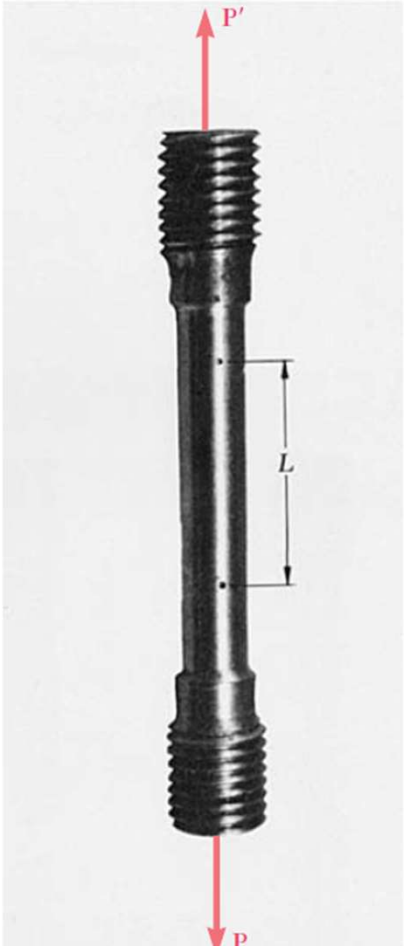
$$\epsilon = \frac{\delta}{L}$$

- Plotting the stress $\sigma = P/A$ against the strain $\sigma\varepsilon = \delta/L$ we obtain a curve that is characteristic of the properties of the material and does not depend upon the dimensions of the particular specimen used. This curve is called a stress-strain diagram.

Since the rod BC considered in the preceding discussion had a uniform cross section of area A, the normal stress σ could be assumed to have a constant value P/A throughout the rod. Thus, it was appropriate to define the strain ϵ as the ratio of the total deformation δ over the total length L of the rod. In the case of a member of variable cross-sectional area A, however, the normal stress $\sigma = P/A$ varies along the member, and it is necessary to define the strain at a given point Q by considering a small element of undeformed length Δx (Fig. 2.5). Denoting by $\Delta\delta$ the deformation of the element under the given loading, we define the *normal strain at point* Q as

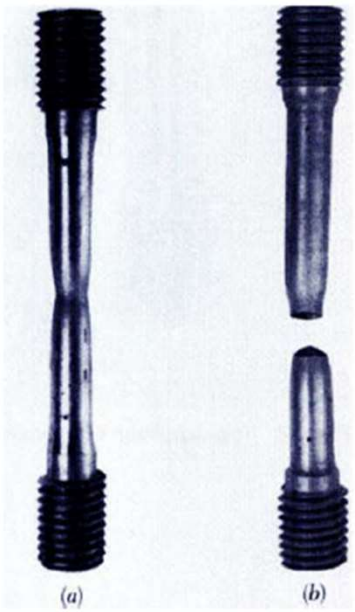
$$\epsilon = \lim_{\Delta x \to 0} \frac{\Delta\delta}{\Delta x} = \frac{d\delta}{dx} \tag{2.2}$$

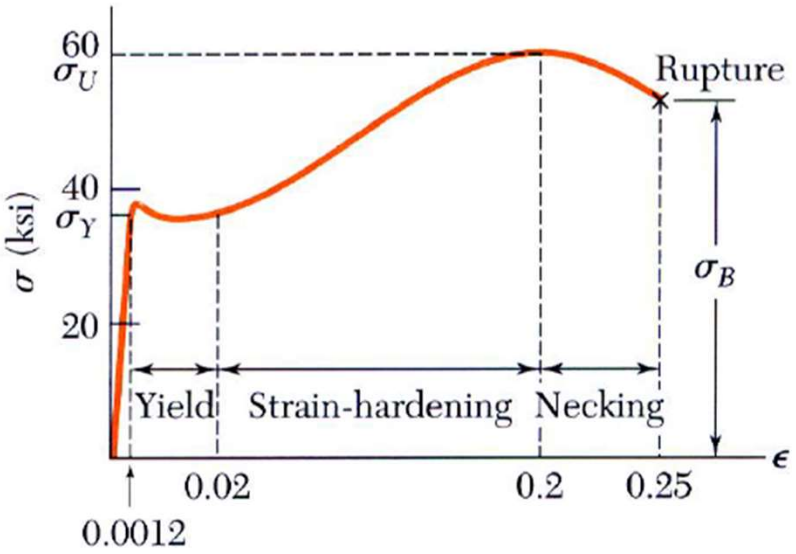
Stress-Strain Test



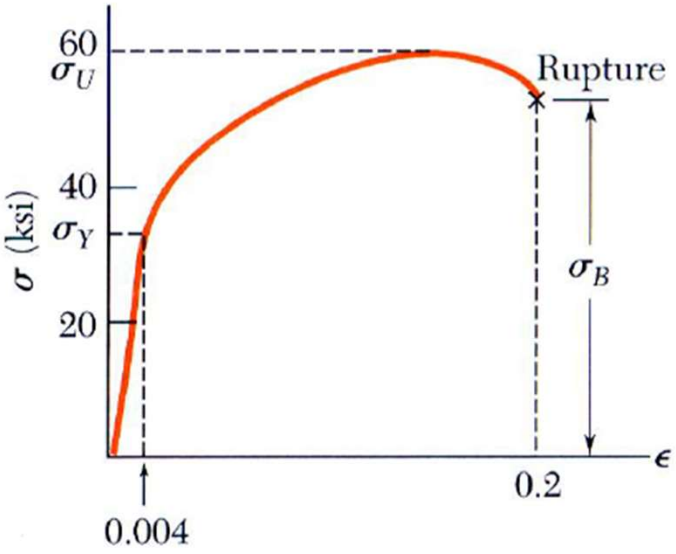


Stress-Strain Diagram: Ductile Materials





(*a*) Low-carbon steel



(*b*) Aluminum alloy

HOOKE'S LAW; MODULUS OF ELASTICITY

Most engineering structures are designed to undergo relatively small deformations, involving only the straight-line portion of the corresponding stress-strain diagram. For that initial portion of the diagram (Fig. 2.6), the stress σ is directly proportional to the strain ϵ, and we can write

$$\sigma = E\epsilon \tag{2.4}$$

This relation is known as *Hooke's law*, after Robert Hooke (1635–1703), an English scientist and one of the early founders of applied mechanics. The coefficient E is called the *modulus of elasticity* of the material involved, or also *Young's modulus*, after the English scientist Thomas Young (1773–1829). Since the strain ϵ is a dimensionless quantity, the modulus E is expressed in the same units as the stress σ, namely in pascals or one of its multiples if SI units are used, and in psi or ksi if U.S. customary units are used.

Some of the physical properties of structural metals, such as strength, ductility, and corrosion resistance, can be greatly affected by alloying, heat treatment, and the manufacturing process used. For example, we note from the stress-strain diagrams of pure iron and of three different grades of steel (Fig. 2.11) that large variations in the yield strength, ultimate strength, and final strain (ductility) exist among these four metals. All of them, however, possess the same modulus of elasticity; in other words, their "stiffness," or ability to resist a deformation within the linear range, is the same. Therefore, if a high-strength steel is substituted for a lower-strength steel in a given structure, and if all dimensions are kept the same, the structure will have an increased load-carrying capacity, but its stiffness will remain unchanged.

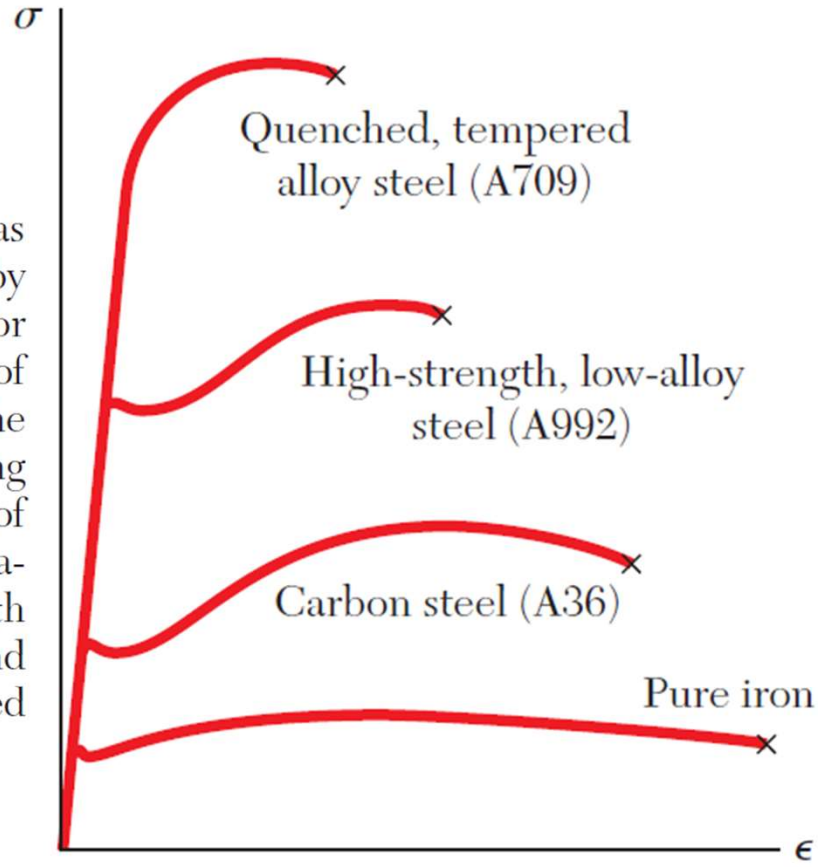


Fig. 2.11 Stress-strain diagrams for iron and different grades of steel.

Elastic vs. Plastic Behavior

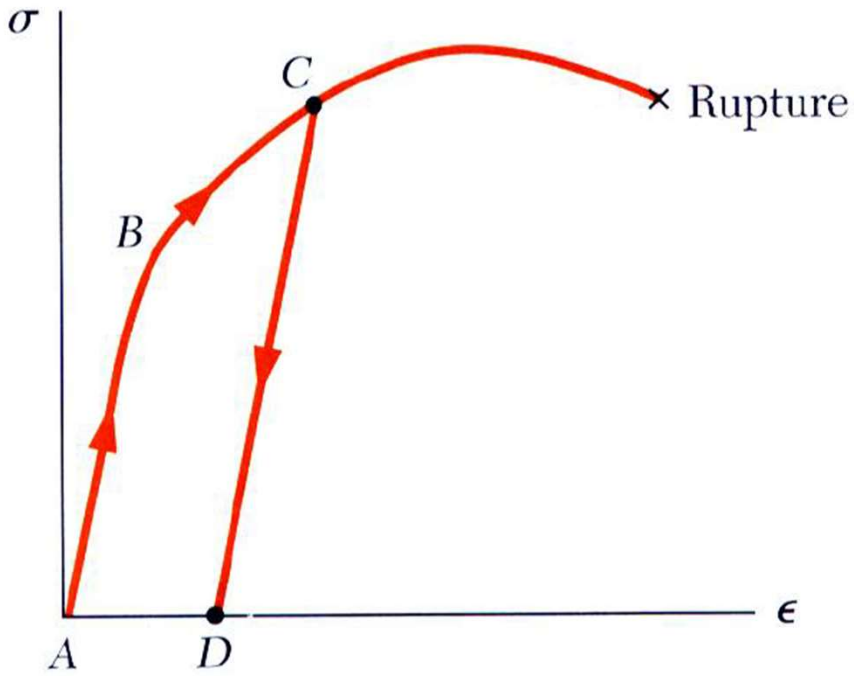


Fig. 2.18

- If the strain disappears when the stress is removed, the material is said to behave *elastically*.
- The largest stress for which this occurs is called the *elastic limit*.
- When the strain does not return to zero after the stress is removed, the material is said to behave *plastically*.

DEFORMATIONS OF MEMBERS UNDER AXIAL LOADING

Consider a homogeneous rod BC of length L and uniform cross section of area A subjected to a centric axial load **P** (Fig. 2.17). If the resulting axial stress $\sigma = P/A$ does not exceed the proportional limit of the material, we may apply Hooke's law and write

$$\sigma = E\epsilon \tag{2.4}$$

from which it follows that

$$\epsilon = \frac{\sigma}{E} = \frac{P}{AE} \tag{2.5}$$

Recalling that the strain ϵ was defined in Sec. 2.2 as $\epsilon = \delta/L$, we have

$$\delta = \epsilon L \tag{2.6}$$

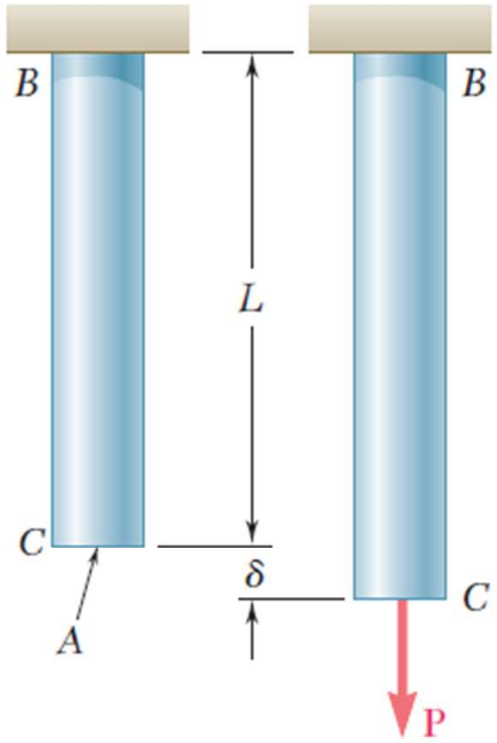


Fig. 2.17 Deformation of axially loaded rod.

and, substituting for ϵ from (2.5) into (2.6):

$$\delta = \frac{PL}{AE} \tag{2.7}$$

Equation (2.7) may be used only if the rod is homogeneous (constant E), has a uniform cross section of area A, and is loaded at its ends. If the rod is loaded at other points, or if it consists of several portions of various cross sections and possibly of different materials, we must divide it into component parts that satisfy individually the required conditions for the application of formula (2.7). Denoting, respectively, by P_i, L_i, A_i, and E_i the internal force, length, cross-sectional area, and modulus of elasticity corresponding to part i, we express the deformation of the entire rod as

$$\delta = \sum_i \frac{P_i L_i}{A_i E_i} \tag{2.8}$$

We recall from Sec. 2.2 that, in the case of a member of variable cross section (Fig. 2.18), the strain ϵ depends upon the position of the point Q where it is computed and is defined as $\epsilon = d\delta/dx$. Solving for $d\delta$ and substituting for ϵ from Eq. (2.5), we express the deformation of an element of length dx as

$$d\delta = \epsilon\, dx = \frac{P\, dx}{AE}$$

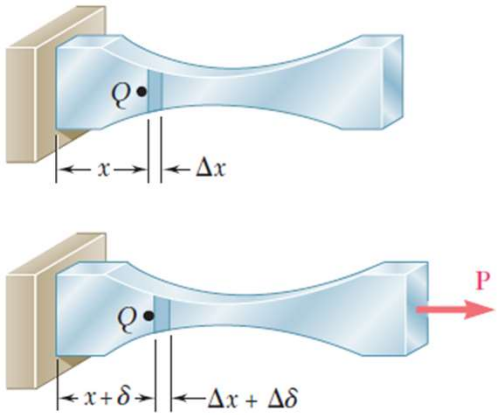


Fig. 2.18 Deformation of axially loaded member of variable cross-sectional area.

The total deformation δ of the member is obtained by integrating this expression over the length L of the member:

$$\delta = \int_0^L \frac{P\, dx}{AE} \tag{2.9}$$

EXAMPLE

Determine the deformation of the steel rod shown in Fig. 2.19*a* under the given loads ($E = 29 \times 10^6$ psi).

We divide the rod into three component parts shown in Fig. 2.19*b* and write

$$L_1 = L_2 = 12 \text{ in.} \qquad L_3 = 16 \text{ in.}$$
$$A_1 = A_2 = 0.9 \text{ in}^2 \qquad A_3 = 0.3 \text{ in}^2$$

To find the internal forces P_1, P_2, and P_3, we must pass sections through each of the component parts, drawing each time the free-body diagram of the portion of rod located to the right of the section (Fig. 2.19*c*). Expressing that each of the free bodies is in equilibrium, we obtain successively

$$P_1 = 60 \text{ kips} = 60 \times 10^3 \text{ lb}$$
$$P_2 = -15 \text{ kips} = -15 \times 10^3 \text{ lb}$$
$$P_3 = 30 \text{ kips} = 30 \times 10^3 \text{ lb}$$

Carrying the values obtained into Eq. (2.8), we have

$$\delta = \sum_i \frac{P_i L_i}{A_i E_i} = \frac{1}{E}\left(\frac{P_1 L_1}{A_1} + \frac{P_2 L_2}{A_2} + \frac{P_3 L_3}{A_3}\right)$$
$$= \frac{1}{29 \times 10^6}\left[\frac{(60 \times 10^3)(12)}{0.9} + \frac{(-15 \times 10^3)(12)}{0.9} + \frac{(30 \times 10^3)(16)}{0.3}\right]$$
$$\delta = \frac{2.20 \times 10^6}{29 \times 10^6} = 75.9 \times 10^{-3} \text{ in.}$$

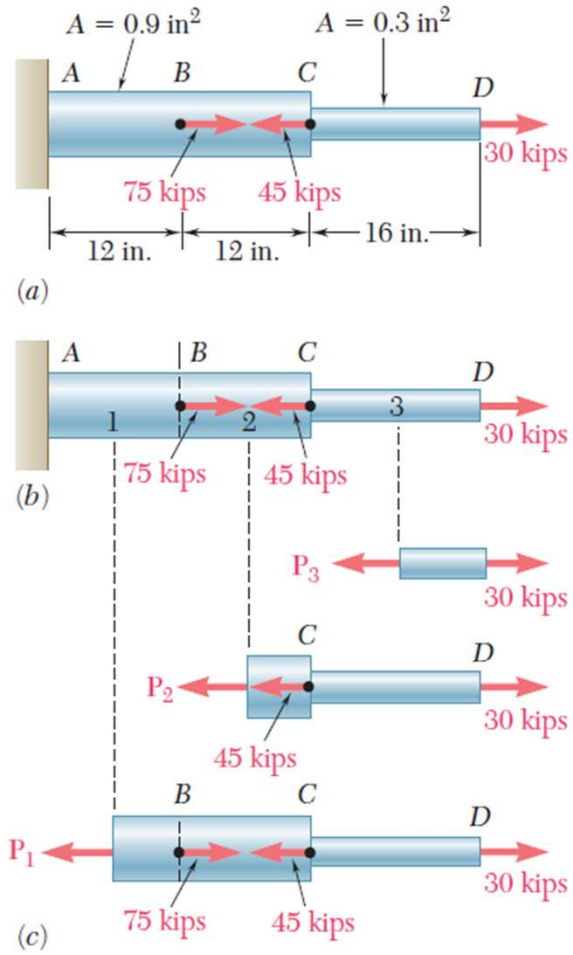
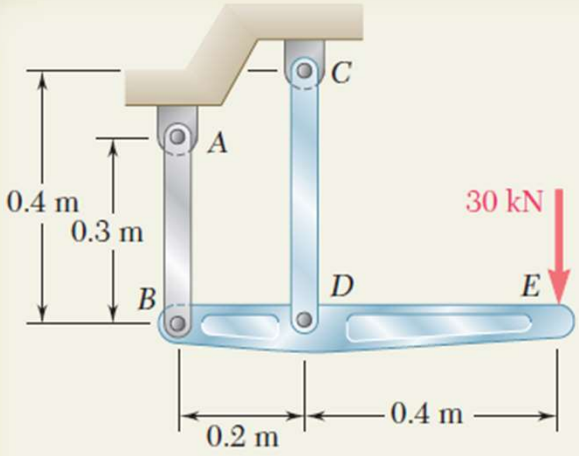


Fig. 2.19

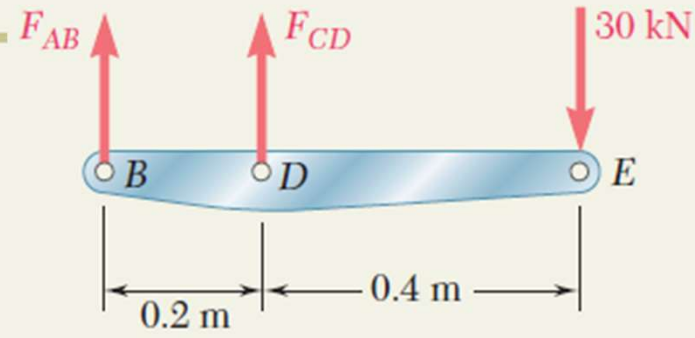


SAMPLE PROBLEM 2.1

The rigid bar BDE is supported by two links AB and CD. Link AB is made of aluminum ($E = 70$ GPa) and has a cross-sectional area of 500 mm^2; link CD is made of steel ($E = 200$ GPa) and has a cross-sectional area of 600 mm^2. For the 30-kN force shown, determine the deflection (*a*) of B, (*b*) of D, (*c*) of E.

SOLUTION

Free Body: Bar *BDE*



$+\circlearrowleft \Sigma M_B = 0$: $\quad -(30\text{ kN})(0.6\text{ m}) + F_{CD}(0.2\text{ m}) = 0$

$F_{CD} = +90\text{ kN} \qquad F_{CD} = 90\text{ kN}$ *tension*

$+\circlearrowleft \Sigma M_D = 0$: $\quad -(30\text{ kN})(0.4\text{ m}) - F_{AB}(0.2\text{ m}) = 0$

$F_{AB} = -60\text{ kN} \qquad F_{AB} = 60\text{ kN}$ *compression*

a. Deflection of *B*. Since the internal force in link AB is compressive, we have $P = -60$ kN

$$\delta_B = \frac{PL}{AE} = \frac{(-60 \times 10^3\ \text{N})(0.3\ \text{m})}{(500 \times 10^{-6}\ \text{m}^2)(70 \times 10^9\ \text{Pa})} = -514 \times 10^{-6}\ \text{m}$$

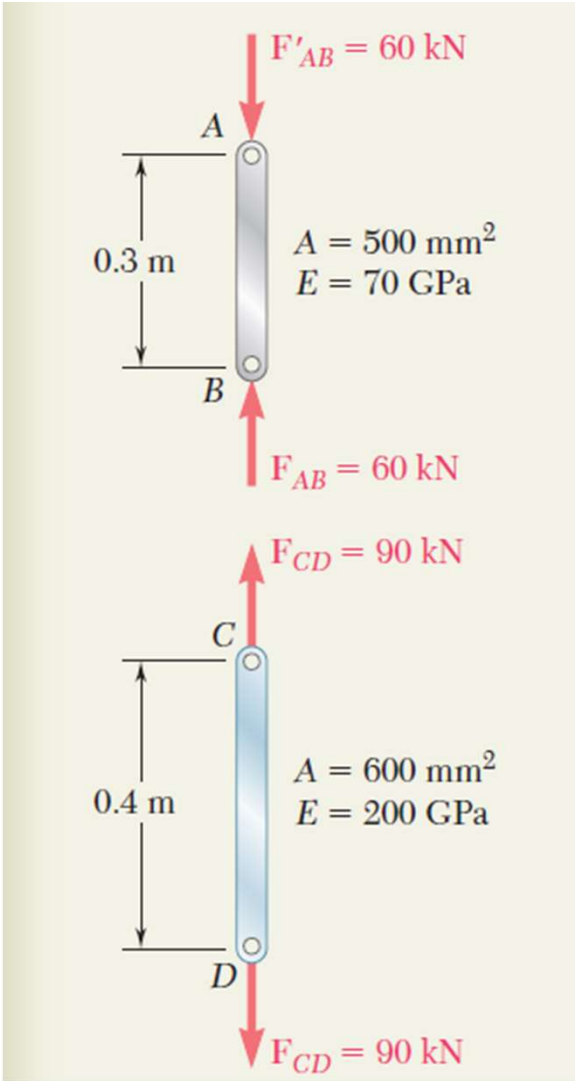
The negative sign indicates a contraction of member AB, and, thus, an upward deflection of end B:

$$\delta_B = 0.514\ \text{mm} \uparrow$$

b. Deflection of *D*. Since in rod CD, $P = 90$ kN, we write

$$\delta_D = \frac{PL}{AE} = \frac{(90 \times 10^3\ \text{N})(0.4\ \text{m})}{(600 \times 10^{-6}\ \text{m}^2)(200 \times 10^9\ \text{Pa})}$$

$$= 300 \times 10^{-6}\ \text{m} \qquad \delta_D = 0.300\ \text{mm} \downarrow$$

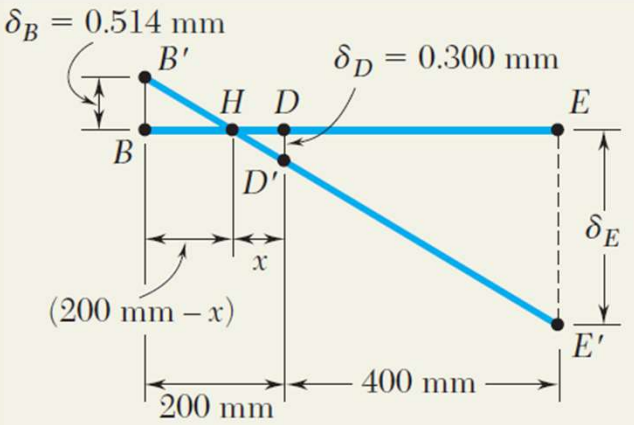


c. Deflection of *E*. We denote by B' and D' the displaced positions of points B and D. Since the bar BDE is rigid, points B', D', and E' lie in a straight line and we write

$$\frac{BB'}{DD'} = \frac{BH}{HD} \qquad \frac{0.514\text{ mm}}{0.300\text{ mm}} = \frac{(200\text{ mm}) - x}{x} \qquad x = 73.7\text{ mm}$$

$$\frac{EE'}{DD'} = \frac{HE}{HD} \qquad \frac{\delta_E}{0.300\text{ mm}} = \frac{(400\text{ mm}) + (73.7\text{ mm})}{73.7\text{ mm}}$$

$$\delta_E = 1.928\text{ mm} \downarrow$$



STATICALLY INDETERMINATE PROBLEMS

A bar AB of length L and uniform cross section is attached to rigid supports at A and B before being loaded. What are the stresses in portions AC and BC due to the application of a load P at point C (Fig. 2.22*a*)?

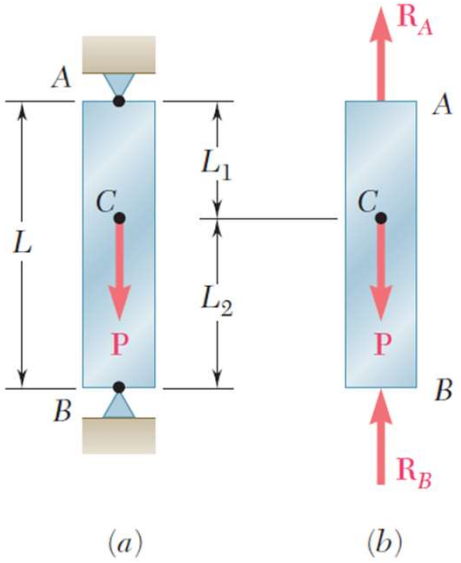
Drawing the free-body diagram of the bar (Fig. 2.22*b*), we obtain the equilibrium equation

$$R_A + R_B = P \tag{2.14}$$

Since this equation is not sufficient to determine the two unknown reactions R_A and R_B, the problem is statically indeterminate.

However, the reactions may be determined if we observe from the geometry that the total elongation δ of the bar must be zero. Denoting by δ_1 and δ_2, respectively, the elongations of the portions AC and BC, we write

$$\delta = \delta_1 + \delta_2 = 0$$



or, expressing δ_1 and δ_2 in terms of the corresponding internal forces P_1 and P_2:

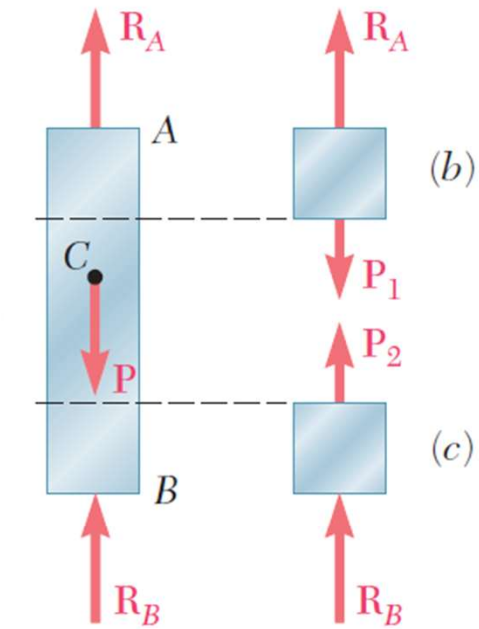
$$\delta = \frac{P_1 L_1}{AE} + \frac{P_2 L_2}{AE} = 0 \tag{2.15}$$

But we note from the free-body diagrams shown respectively in parts b and c of Fig. 2.23 that $P_1 = R_A$ and $P_2 = -R_B$. Carrying these values into (2.15), we write

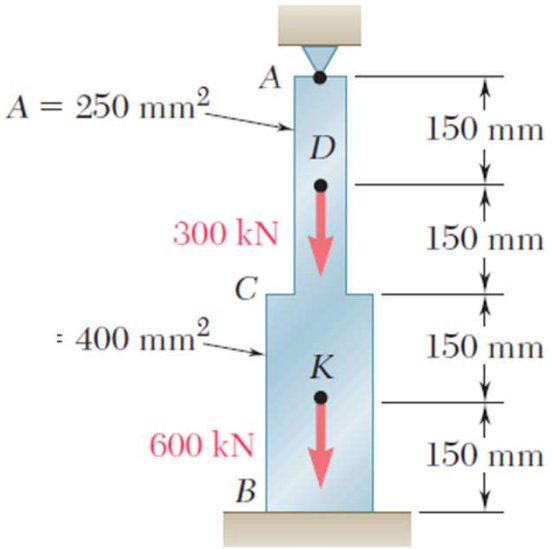
$$R_A L_1 - R_B L_2 = 0 \tag{2.16}$$

Equations (2.14) and (2.16) can be solved simultaneously for R_A and R_B; we obtain $R_A = PL_2/L$ and $R_B = PL_1/L$. The desired stresses σ_1 in AC and σ_2 in BC are obtained by dividing, respectively, $P_1 = R_A$ and $P_2 = -R_B$ by the cross-sectional area of the bar:

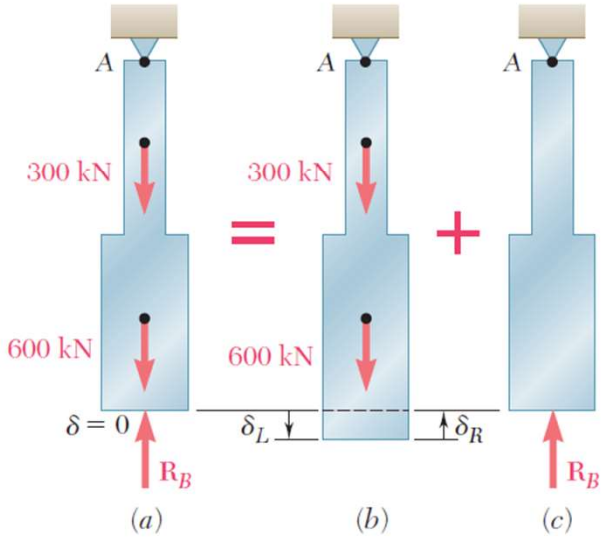
$$\sigma_1 = \frac{PL_2}{AL} \qquad \sigma_2 = -\frac{PL_1}{AL}$$



Determine the reactions at A and B for the steel bar and loading shown in Fig. 2.24, assuming a close fit at both supports before the loads are applied.



We consider the reaction at B as redundant and release the bar from that support. The reaction $\mathbf{R}_B$ is now considered as an unknown load (Fig. 2.25*a*) and will be determined from the condition that the deformation δ of the rod must be equal to zero. The solution is carried out by considering separately the deformation δ_L caused by the given loads (Fig. 2.25*b*) and the deformation δ_R due to the redundant reaction $\mathbf{R}_B$ (Fig. 2.25*c*).



The deformation δ_L is obtained from Eq. (2.8) after the bar has been divided into four portions, as shown in Fig. 2.26. Following the same procedure as in Example 2.01, we write

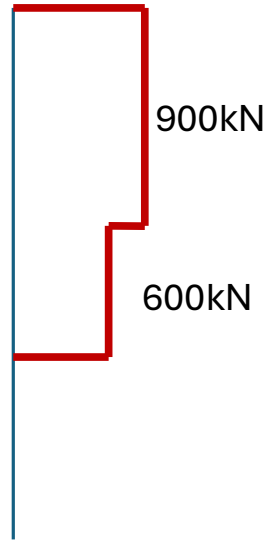
$$P_1 = 0 \qquad P_2 = P_3 = 600 \times 10^3\ \text{N} \qquad P_4 = 900 \times 10^3\ \text{N}$$
$$A_1 = A_2 = 400 \times 10^{-6}\ \text{m}^2 \qquad A_3 = A_4 = 250 \times 10^{-6}\ \text{m}^2$$
$$L_1 = L_2 = L_3 = L_4 = 0.150\ \text{m}$$

Substituting these values into Eq. (2.8), we obtain

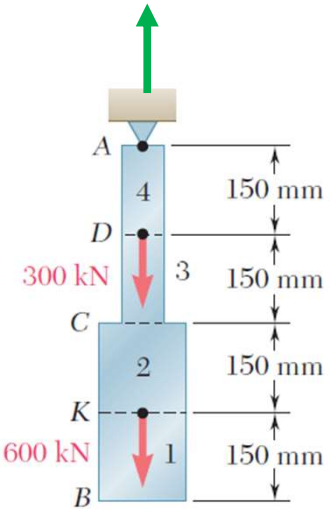
$$\delta_L = \sum_{i=1}^{4} \frac{P_i L_i}{A_i E} = \left(0 + \frac{600 \times 10^3\ \text{N}}{400 \times 10^{-6}\ \text{m}^2} + \frac{600 \times 10^3\ \text{N}}{250 \times 10^{-6}\ \text{m}^2} + \frac{900 \times 10^3\ \text{N}}{250 \times 10^{-6}\ \text{m}^2} \right) \frac{0.150\ \text{m}}{E}$$

$$\delta_L = \frac{1.125 \times 10^9}{E} \qquad (2.17)$$

Reaction at A =900kN



Axial Force Diagram

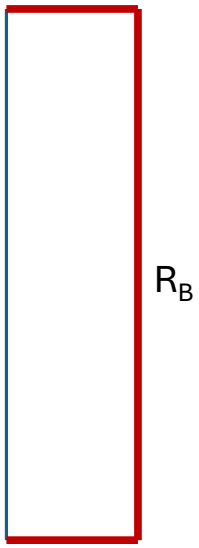


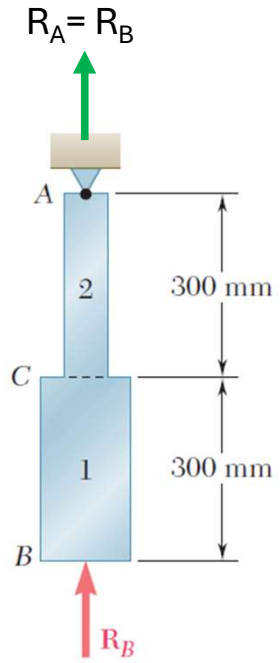
Considering now the deformation δ_R due to the redundant reaction R_B, we divide the bar into two portions, as shown in Fig. and write

$$P_1 = P_2 = -R_B$$

$$A_1 = 400 \times 10^{-6}\ \text{m}^2 \qquad A_2 = 250 \times 10^{-6}\ \text{m}^2$$

$$L_1 = L_2 = 0.300\ \text{m}$$





Substituting these values into Eq. (2.8), we obtain

$$\delta_R = \frac{P_1 L_1}{A_1 E} + \frac{P_2 L_2}{A_2 E} = -\frac{(1.95 \times 10^3) R_B}{E} \tag{2.18}$$

Expressing that the total deformation δ of the bar must be zero, we write

$$\delta = \delta_L + \delta_R = 0 \tag{2.19}$$

and, substituting for δ_L and δ_R from (2.17) and (2.18) into (2.19),

$$\delta = \frac{1.125 \times 10^9}{E} - \frac{(1.95 \times 10^3) R_B}{E} = 0$$

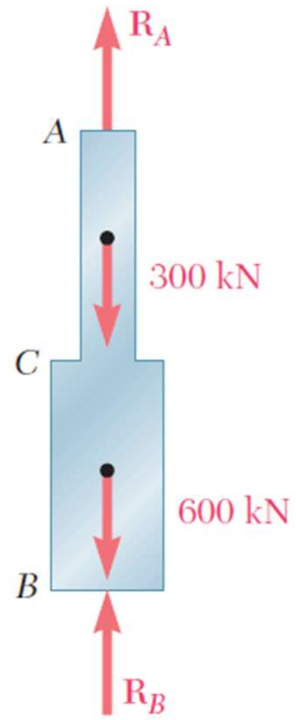
Solving for R_B, we have

$$R_B = 577 \times 10^3 \text{ N} = 577 \text{ kN}$$

The reaction R_A at the upper support is obtained from the free-body diagram of the bar (Fig. 2.28). We write

$$+\uparrow \Sigma F_y = 0: \qquad R_A - 300 \text{ kN} - 600 \text{ kN} + R_B = 0$$

$$R_A = 900 \text{ kN} - R_B = 900 \text{ kN} - 577 \text{ kN} = 323 \text{ kN}$$



PROBLEMS INVOLVING TEMPERATURE CHANGES

All of the members and structures that we have considered so far were assumed to remain at the same temperature while they were being loaded. We are now going to consider various situations involving changes in temperature.

Let us first consider a homogeneous rod AB of uniform cross section, which rests freely on a smooth horizontal surface (Fig. 2.30*a*). If the temperature of the rod is raised by ΔT, we observe that the rod elongates by an amount δ_T which is proportional to both the temperature change ΔT and the length L of the rod (Fig. 2.30*b*). We have

$$\delta_T = \alpha(\Delta T)L \tag{2.21}$$

where α is a constant characteristic of the material, called the *coefficient of thermal expansion*. Since δ_T and L are both expressed in units of length, α represents a quantity *per degree C*, or *per degree F*, depending whether the temperature change is expressed in degrees Celsius or in degrees Fahrenheit.

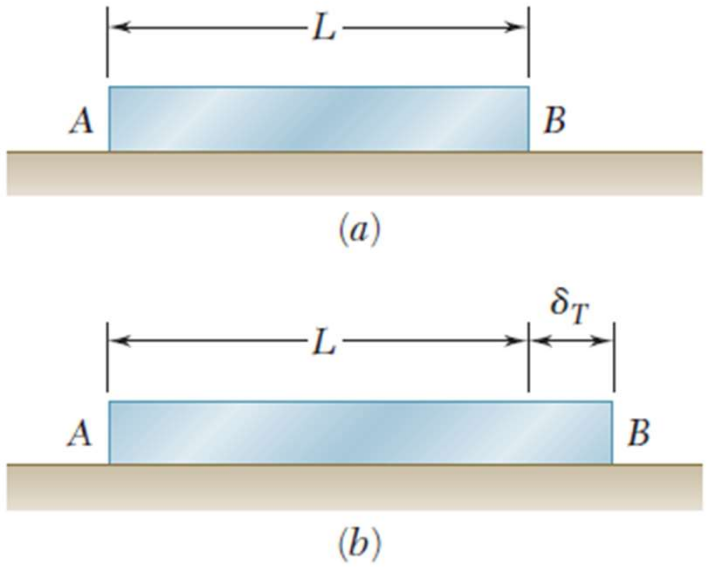


Fig. 2.30 Elongation of rod due to temperature increase.

With the deformation δ_T must be associated a strain $\epsilon_T = \delta_T/L$. Recalling Eq. (2.21), we conclude that

$$\epsilon_T = \alpha \Delta T \tag{2.22}$$

The strain ϵ_T is referred to as a *thermal strain*, since it is caused by the change in temperature of the rod. In the case we are considering here, there is *no stress associated with the strain* ϵ_T.

Let us now assume that the same rod AB of length L is placed between two fixed supports at a distance L from each other (Fig. 2.31*a*). Again, there is neither stress nor strain in this initial condition. If we raise the temperature by ΔT, the rod cannot elongate because of the restraints imposed on its ends; the elongation δ_T of the rod is thus zero. Since the rod is homogeneous and of uniform cross section, the strain ϵ_T at any point is $\epsilon_T = \delta_T/L$ and, thus, also zero. However, the supports will exert equal and opposite forces $\mathbf{P}$ and $\mathbf{P}'$ on the rod after the temperature has been raised, to keep it from elongating (Fig. 2.31*b*). It thus follows that a state of stress (with no corresponding strain) is created in the rod.

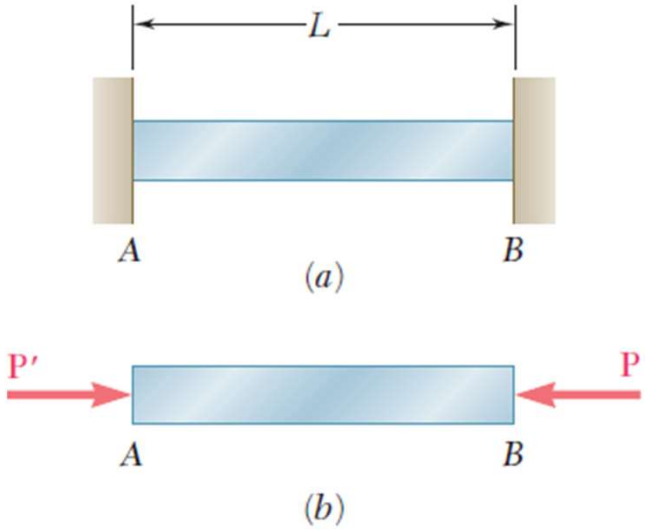


Fig. 2.31 Rod with ends restrained against thermal expansion.

As we prepare to determine the stress σ created by the temperature change ΔT, we observe that the problem we have to solve is statically indeterminate. Therefore, we should first compute the magnitude P of the reactions at the supports from the condition that the elongation of the rod is zero. Using the superposition method described in Sec. 2.9, we detach the rod from its support B (Fig. 2.32*a*) and let it elongate freely as it undergoes the temperature change ΔT (Fig. 2.32*b*). According to formula (2.21), the corresponding elongation is

Applying now to end B the force **P** representing the redundant reaction, and recalling formula (2.7), we obtain a second deformation (Fig. 2.32*c*)

$$\delta_P = \frac{PL}{AE}$$

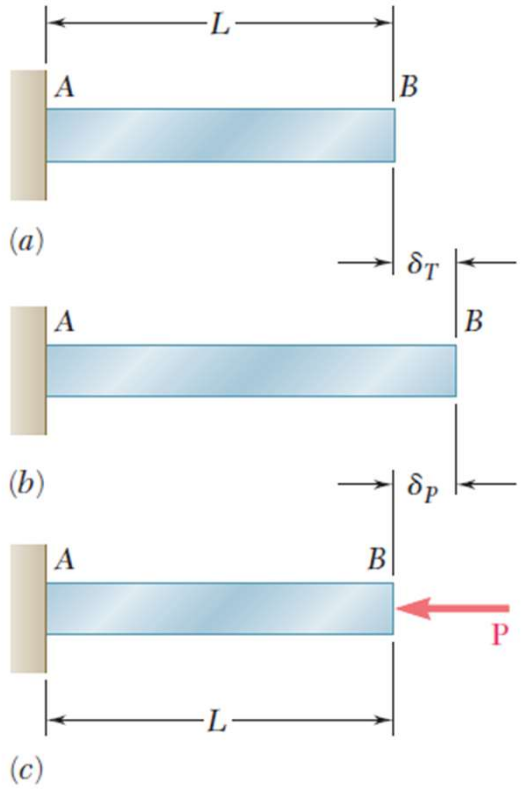


Fig. 2.32 Superposition method applied to rod restrained against thermal expansion.

Expressing that the total deformation δ must be zero, we have

$$\delta = \delta_T + \delta_P = \alpha(\Delta T)L + \frac{PL}{AE} = 0$$

from which we conclude that

$$P = -AE\alpha(\Delta T)$$

and that the stress in the rod due to the temperature change ΔT is

$$\sigma = \frac{P}{A} = -E\alpha(\Delta T) \tag{2.23}$$

It should be kept in mind that the result we have obtained here and our earlier remark regarding the absence of any strain in the rod *apply only in the case of a homogeneous rod of uniform cross section.*

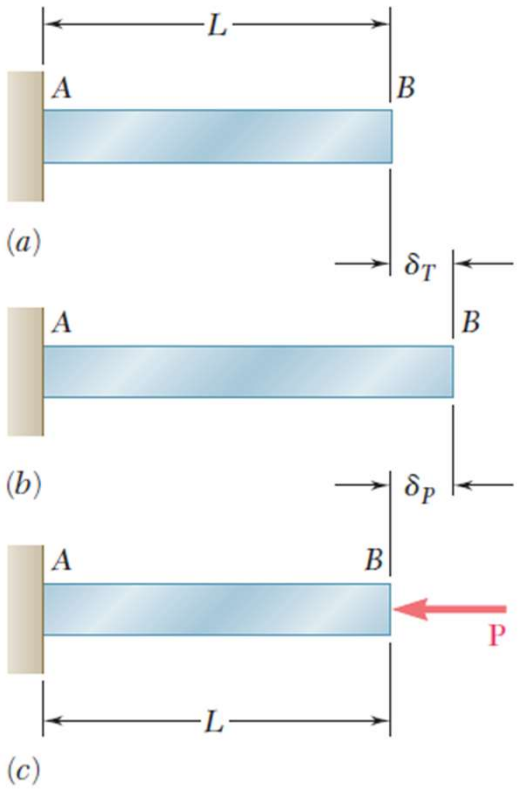
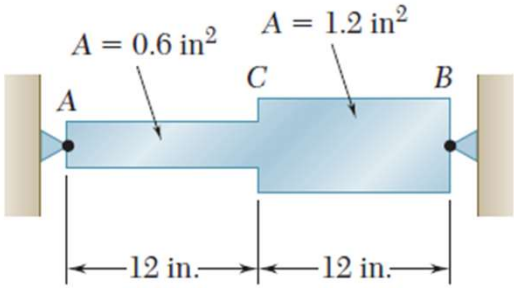


Fig. 2.32 Superposition method applied to rod restrained against thermal expansion.

Determine the values of the stress in portions AC and CB of the steel bar shown (Fig. 2.33) when the temperature of the bar is $-50°\text{F}$, knowing that a close fit exists at both of the rigid supports when the temperature is $+75°\text{F}$. Use the values $E = 29 \times 10^6$ psi and $\alpha = 6.5 \times 10^{6}/°\text{F}$ for steel.

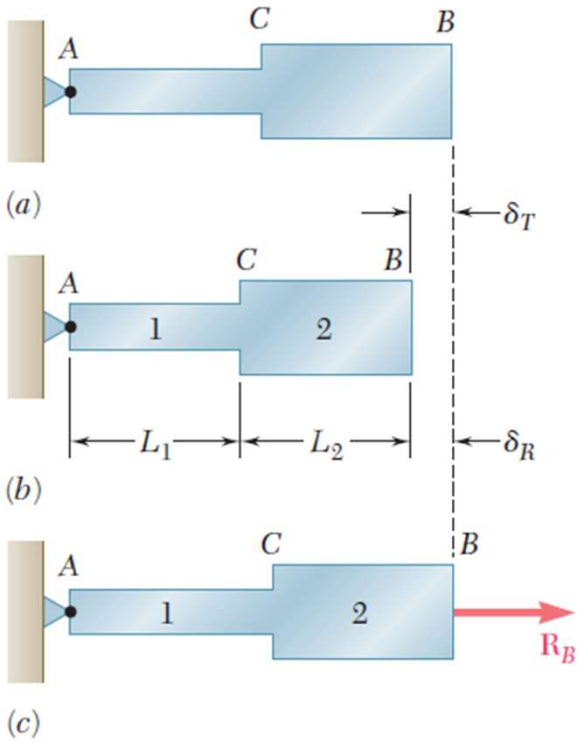


We first determine the reactions at the supports. Since the problem is statically indeterminate, we detach the bar from its support at B and let it undergo the temperature change

$$\Delta T = (-50°\text{F}) - (75°\text{F}) = -125°\text{F}$$

The corresponding deformation (Fig. 2.34*b*) is

$$\delta_T = \alpha(\Delta T)L = (6.5 \times 10^{-6}/°\text{F})(-125°\text{F})(24\text{ in.})$$
$$= -19.50 \times 10^{-3}\text{ in.}$$



Applying now the unknown force $\mathbf{R}_B$ at end B (Fig. 2.34*c*), we use Eq. (2.8) to express the corresponding deformation δ_R. Substituting

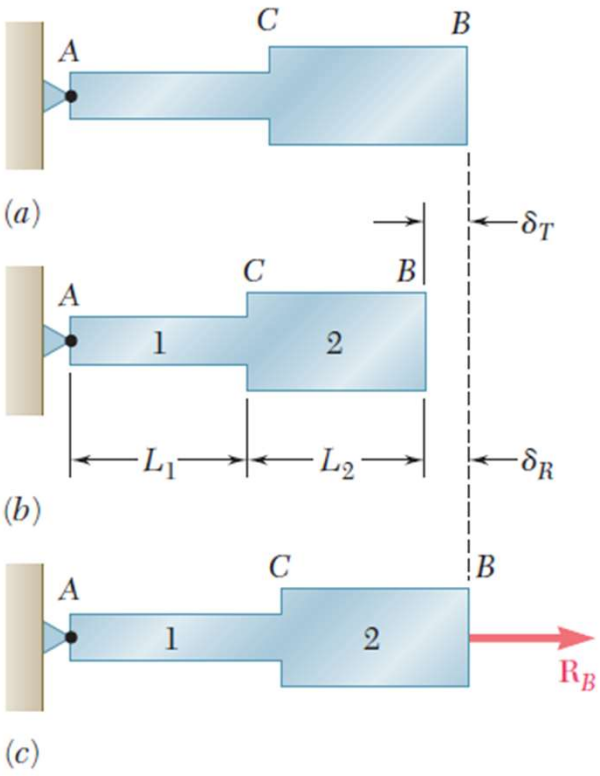
$$L_1 = L_2 = 12 \text{ in.}$$
$$A_1 = 0.6 \text{ in}^2 \qquad A_2 = 1.2 \text{ in}^2$$
$$P_1 = P_2 = R_B \qquad E = 29 \times 10^6 \text{ psi}$$

into Eq. (2.8), we write

$$\begin{aligned}\delta_R &= \frac{P_1 L_1}{A_1 E} + \frac{P_2 L_2}{A_2 E} \\ &= \frac{R_B}{29 \times 10^6 \text{ psi}} \left(\frac{12 \text{ in.}}{0.6 \text{ in}^2} + \frac{12 \text{ in.}}{1.2 \text{ in}^2} \right) \\ &= (1.0345 \times 10^{-6} \text{ in./lb}) R_B\end{aligned}$$

Expressing that the total deformation of the bar must be zero as a result of the imposed constraints, we write

$$\begin{aligned}\delta &= \delta_T + \delta_R = 0 \\ &= -19.50 \times 10^{-3} \text{ in.} + (1.0345 \times 10^{-6} \text{ in./lb}) R_B = 0\end{aligned}$$



from which we obtain

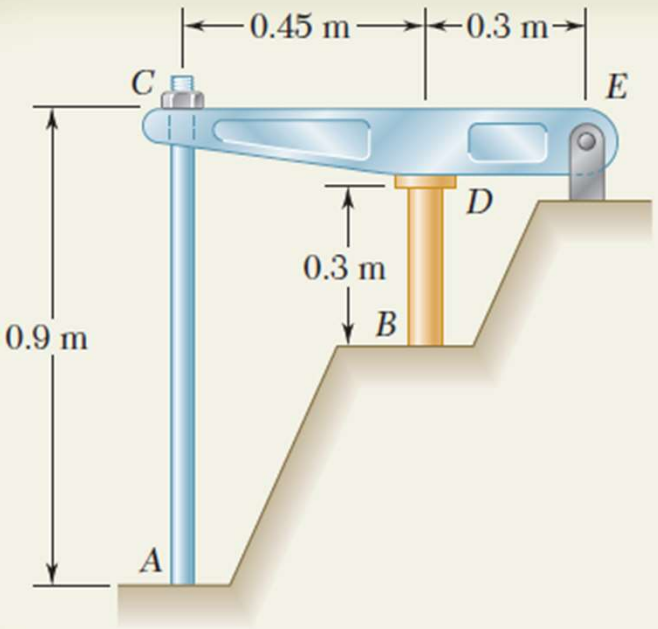
$$R_B = 18.85 \times 10^3 \text{ lb} = 18.85 \text{ kips}$$

The reaction at A is equal and opposite.

Noting that the forces in the two portions of the bar are $P_1 = P_2 = 18.85$ kips, we obtain the following values of the stress in portions AC and CB of the bar:

$$\sigma_1 = \frac{P_1}{A_1} = \frac{18.85 \text{ kips}}{0.6 \text{ in}^2} = +31.42 \text{ ksi}$$

$$\sigma_2 = \frac{P_2}{A_2} = \frac{18.85 \text{ kips}}{1.2 \text{ in}^2} = +15.71 \text{ ksi}$$

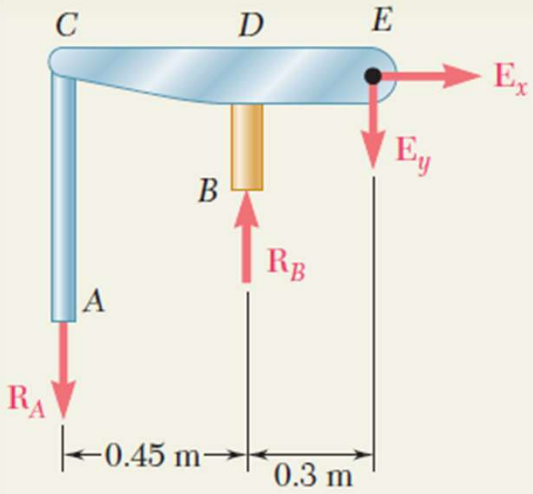


SAMPLE PROBLEM 2.4

The rigid bar CDE is attached to a pin support at E and rests on the 30-mm-diameter brass cylinder BD. A 22-mm-diameter steel rod AC passes through a hole in the bar and is secured by a nut which is snugly fitted when the temperature of the entire assembly is 20°C. The temperature of the brass cylinder is then raised to 50°C while the steel rod remains at 20°C. Assuming that no stresses were present before the temperature change, determine the stress in the cylinder.

Rod AC: Steel
$E = 200$ GPa
$\alpha = 11.7 \times 10^{-6}/°C$

Cylinder BD: Brass
$E = 105$ GPa
$\alpha = 20.9 \times 10^{-6}/°C$



SOLUTION

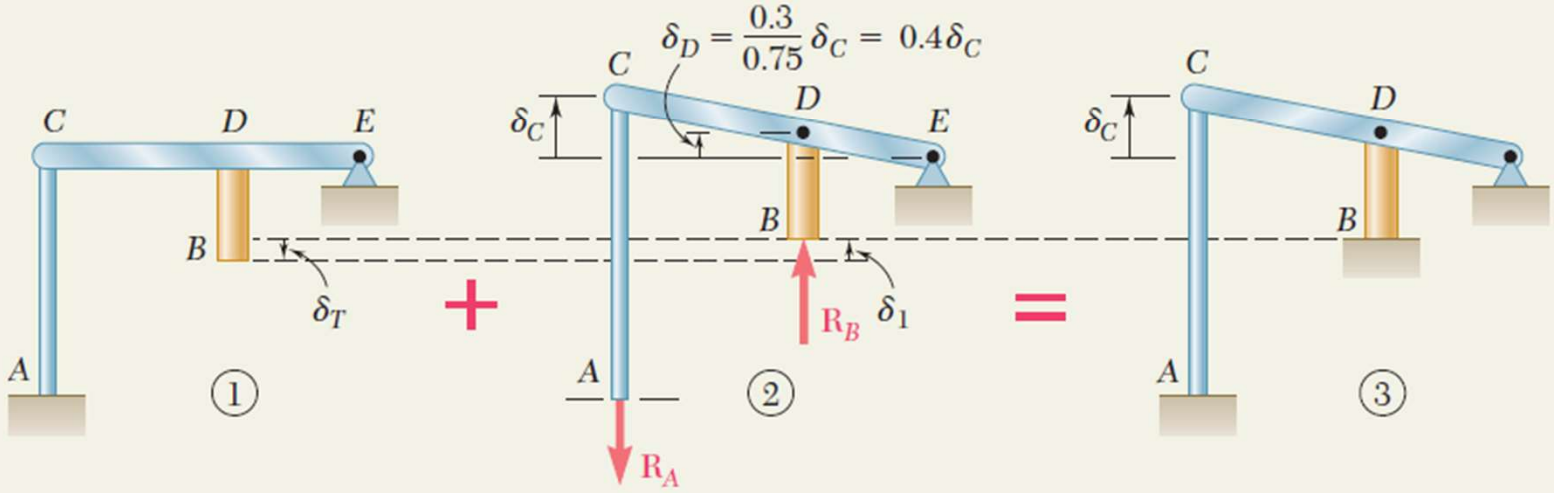
Statics. Considering the free body of the entire assembly, we write

$$+\circlearrowleft \Sigma M_E = 0: \qquad R_A(0.75\text{ m}) - R_B(0.3\text{ m}) = 0 \qquad R_A = 0.4R_B \tag{1}$$

Deformations. We use the method of superposition, considering $\mathbf{R}_B$ as redundant. With the support at B removed, the temperature rise of the cylinder causes point B to move down through δ_T. The reaction $\mathbf{R}_B$ must cause a deflection δ_1 equal to δ_T so that the final deflection of B will be zero (Fig. 3).

Deflection δ_T. Because of a temperature rise of $50° - 20° = 30°\text{C}$, the length of the brass cylinder increases by δ_T.

$$\delta_T = L(\Delta T)\alpha = (0.3\text{ m})(30°\text{C})(20.9 \times 10^{-6}/°\text{C}) = 188.1 \times 10^{-6}\text{ m} \downarrow$$



Deflection δ_1. We note that $\delta_D = 0.4\delta_C$ and $\delta_1 = \delta_D + \delta_{B/D}$.

$$\delta_C = \frac{R_A L}{AE} = \frac{R_A(0.9\text{ m})}{\frac{1}{4}\pi(0.022\text{ m})^2(200\text{ GPa})} = 11.84 \times 10^{-9} R_A \uparrow$$

$$\delta_D = 0.40\delta_C = 0.4(11.84 \times 10^{-9} R_A) = 4.74 \times 10^{-9} R_A \uparrow$$

$$\delta_{B/D} = \frac{R_B L}{AE} = \frac{R_B(0.3\text{ m})}{\frac{1}{4}\pi(0.03\text{ m})^2(105\text{ GPa})} = 4.04 \times 10^{-9} R_B \uparrow$$

We recall from (1) that $R_A = 0.4R_B$ and write

$$\delta_1 = \delta_D + \delta_{B/D} = [4.74(0.4R_B) + 4.04R_B]10^{-9} = 5.94 \times 10^{-9} R_B \uparrow$$

But $\delta_T = \delta_1$: $\qquad 188.1 \times 10^{-6}\text{ m} = 5.94 \times 10^{-9} R_B \qquad R_B = 31.7\text{ kN}$

Stress in Cylinder: $\sigma_B = \dfrac{R_B}{A} = \dfrac{31.7\text{ kN}}{\frac{1}{4}\pi(0.03\text{ m})^2}$ $\qquad \sigma_B = 44.8\text{ MPa}$ ◀

2.11 POISSON'S RATIO

We saw in the earlier part of this chapter that, when a homogeneous slender bar is axially loaded, the resulting stress and strain satisfy Hooke's law, as long as the elastic limit of the material is not exceeded. Assuming that the load **P** is directed along the x axis (Fig. 2.35*a*), we have $\sigma_x = P/A$, where A is the cross-sectional area of the bar, and, from Hooke's law,

$$\epsilon_x = \sigma_x/E \tag{2.24}$$

where E is the modulus of elasticity of the material.

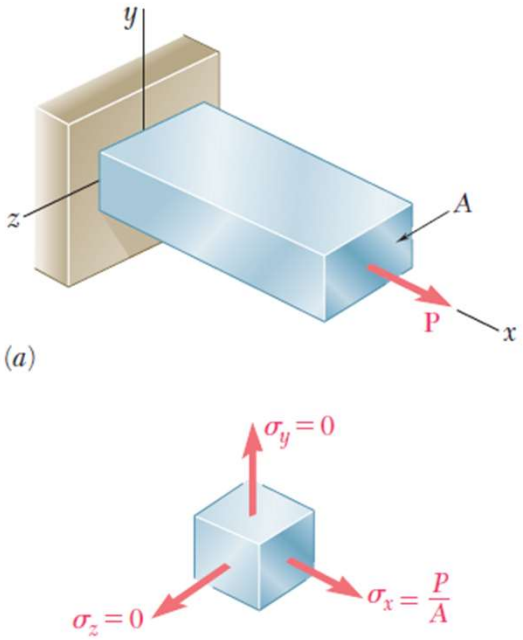


Fig. 2.35 Stresses in an axially-loaded bar.

We also note that the normal stresses on faces respectively perpendicular to the y and z axes are zero: $\sigma_y = \sigma_z = 0$ (Fig. 2.35*b*). It would be tempting to conclude that the corresponding strains ϵ_y and ϵ_z are also zero. This, however, *is not the case*. In all engineering materials, the elongation produced by an axial tensile force **P** in the direction of the force is accompanied by a contraction in any transverse direction (Fig. 2.36).† In this section and the following sections (Secs. 2.12 through 2.15), all materials considered will be assumed to be both *homogeneous* and *isotropic*, i.e., their mechanical properties will be assumed independent of both *position* and *direction*. It follows that the strain must have the same value for any transverse direction. Therefore, for the loading shown in Fig. 2.35 we must have $\epsilon_y = \epsilon_z$. This common value is referred to as the *lateral strain*. An important constant for a given material is its *Poisson's ratio*, named after the French mathematician Siméon Denis Poisson (1781–1840) and denoted by the Greek letter ν (nu). It is defined as

$$\nu = -\frac{\text{lateral strain}}{\text{axial strain}} \tag{2.25}$$

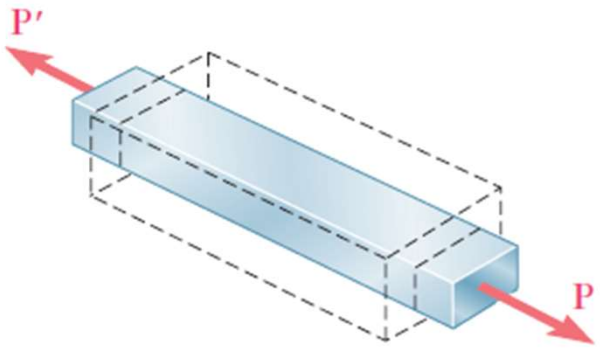


Fig. 2.36 Transverse contraction of bar under axial tensile force.

$$\nu = -\frac{\epsilon_y}{\epsilon_x} = -\frac{\epsilon_z}{\epsilon_x} \tag{2.26}$$

for the loading condition represented in Fig. 2.35. Note the use of a minus sign in the above equations to obtain a positive value for ν, the axial and lateral strains having opposite signs for all engineering materials.‡ Solving Eq. (2.26) for ϵ_y and ϵ_z, and recalling (2.24), we write the following relations, which fully describe the condition of strain under an axial load applied in a direction parallel to the x axis:

$$\epsilon_x = \frac{\sigma_x}{E} \qquad \epsilon_y = \epsilon_z = -\frac{\nu\sigma_x}{E} \tag{2.27}$$

Isotropic, Orthropic and Anisotropic Materials

Isotropic Materials

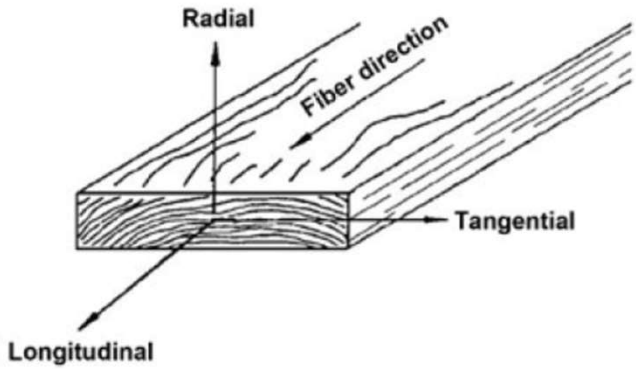
- Materials which have same or identical properties in all directions.
- Metal or glass
- 2 independent elastic constants required to define materials.

Anisotropic Materials

- Materials which changes their properties with directions.
- Wood, Composites, Biological tissue etc.
- 21 independent elastic constants required to define materials.

Orthrotopic Materials

- A subset of anisotropic materials in which properties will change in mutually perpendicular directions.
- Wood, composites.
- 9 independent elastic constants required to define materials.



Wood is Orthotropic – properties being directionally dependent

EXAMPLE 2.07

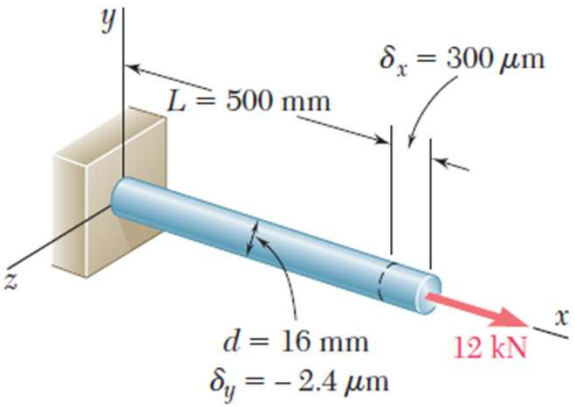


Fig. 2.37

A 500-mm-long, 16-mm-diameter rod made of a homogenous, isotropic material is observed to increase in length by 300 μm, and to decrease in diameter by 2.4 μm when subjected to an axial 12-kN load. Determine the modulus of elasticity and Poisson's ratio of the material.

The cross-sectional area of the rod is

$$A = \pi r^2 = \pi(8 \times 10^{-3}\,\text{m})^2 = 201 \times 10^{-6}\,\text{m}^2$$

Choosing the x axis along the axis of the rod (Fig. 2.37), we write

$$\sigma_x = \frac{P}{A} = \frac{12 \times 10^3\,\text{N}}{201 \times 10^{-6}\,\text{m}^2} = 59.7\,\text{MPa}$$

$$\epsilon_x = \frac{\delta_x}{L} = \frac{300\ \mu\text{m}}{500\,\text{mm}} = 600 \times 10^{-6}$$

$$\epsilon_y = \frac{\delta_y}{d} = \frac{-2.4\ \mu\text{m}}{16\,\text{mm}} = -150 \times 10^{-6}$$

From Hooke's law, $\sigma_x = E\epsilon_x$, we obtain

$$E = \frac{\sigma_x}{\epsilon_x} = \frac{59.7\,\text{MPa}}{600 \times 10^{-6}} = 99.5\,\text{GPa}$$

and, from Eq. (2.26),

$$\nu = -\frac{\epsilon_y}{\epsilon_x} = -\frac{-150 \times 10^{-6}}{600 \times 10^{-6}} = 0.25$$

MULTIAXIAL LOADING; GENERALIZED HOOKE'S LAW

All the examples considered so far in this chapter have dealt with slender members subjected to axial loads, i.e., to forces directed along a single axis. Choosing this axis as the x axis, and denoting by P the internal force at a given location, the corresponding stress components were found to be $\sigma_x = P/A$, $\sigma_y = 0$, and $\sigma_z = 0$.

Let us now consider structural elements subjected to loads acting in the directions of the three coordinate axes and producing normal stresses σ_x, σ_y, and σ_z which are all different from zero (Fig. 2.38). This condition is referred to as a *multiaxial loading*. Note that this is not the general stress condition described in Sec. 1.12, since no shearing stresses are included among the stresses shown in Fig. 2.38.

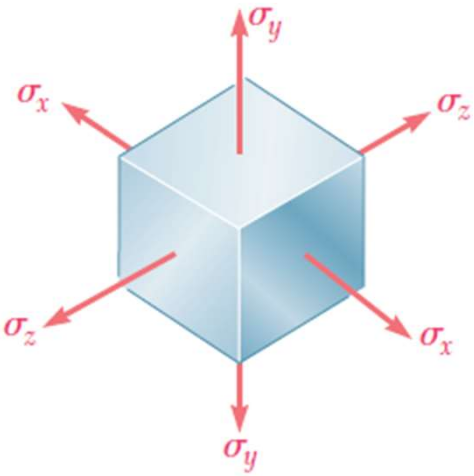
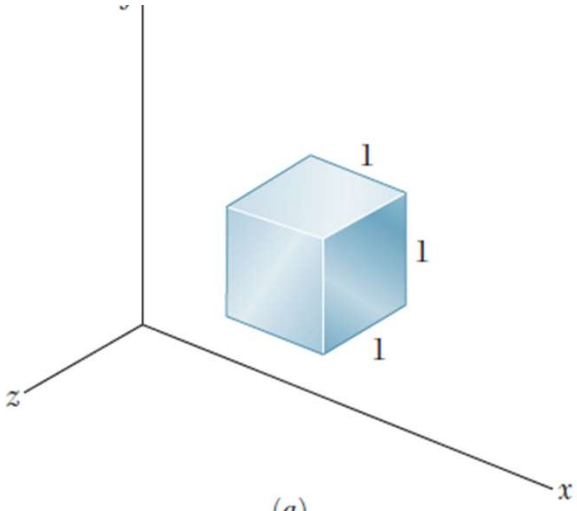
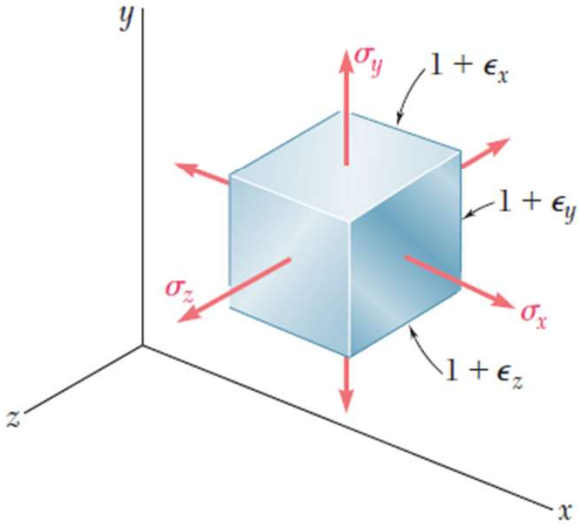


Fig. 2.38 Stress state for multiaxial loading.

Consider an element of an isotropic material in the shape of a cube (Fig. 2.39*a*). We can assume the side of the cube to be equal to unity, since it is always possible to select the side of the cube as a unit of length. Under the given multiaxial loading, the element will deform into a *rectangular parallelepiped* of sides equal, respectively, to $1 + \epsilon_x$, $1 + \epsilon_y$, and $1 + \epsilon_z$, where ϵ_x, ϵ_y, and ϵ_z denote the values of the normal strain in the directions of the three coordinate axes (Fig. 2.39*b*). You should note that, as a result of the deformations of the other elements of the material, the element under consideration could also undergo a translation, but we are concerned here only with the *actual deformation* of the element, and not with any possible superimposed rigid-body displacement.



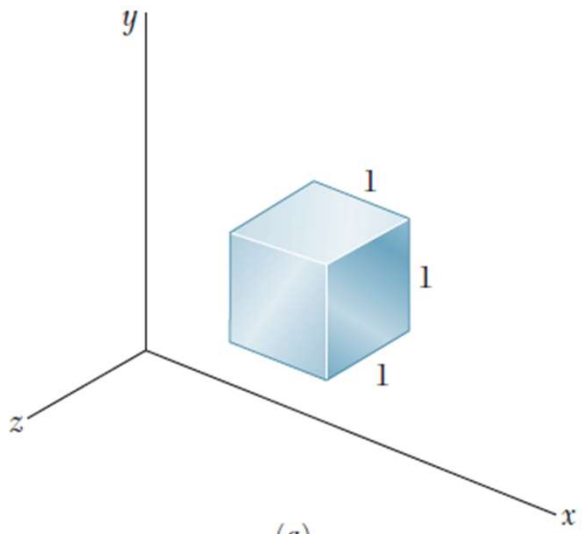
(*a*)



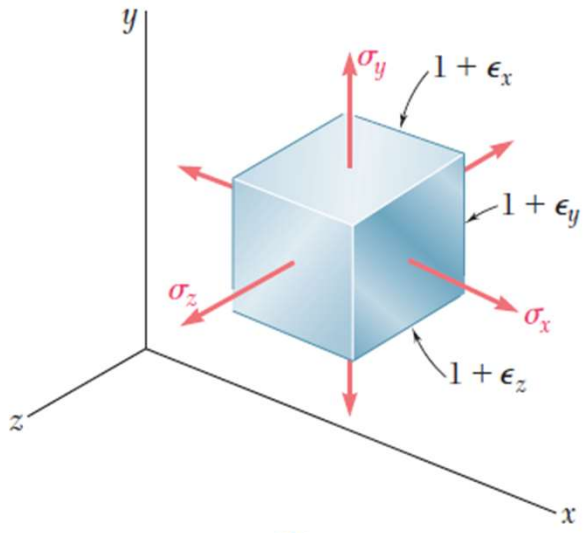
In order to express the strain components ϵ_x, ϵ_y, ϵ_z in terms of the stress components σ_x, σ_y, σ_z, we will consider separately the effect of each stress component and combine the results obtained. The approach we propose here will be used repeatedly in this text, and is based on the *principle of superposition*. This principle states that the effect of a given combined loading on a structure can be obtained by *determining separately the effects of the various loads and combining the results obtained*, provided that the following conditions are satisfied:

1. Each effect is linearly related to the load that produces it.
2. The deformation resulting from any given load is small and does not affect the conditions of application of the other loads.

In the case of a multiaxial loading, the first condition will be satisfied if the stresses do not exceed the proportional limit of the material, and the second condition will also be satisfied if the stress on any given face does not cause deformations of the other faces that are large enough to affect the computation of the stresses on those faces.



(*a*)



Considering first the effect of the stress component σ_x, we recall from Sec. 2.11 that σ_x causes a strain equal to σ_x/E in the x direction, and strains equal to $-\nu\sigma_x/E$ in each of the y and z directions. Similarly, the stress component σ_y, if applied separately, will cause a strain σ_y/E in the y direction and strains $-\nu\sigma_y/E$ in the other two directions. Finally, the stress component σ_z causes a strain σ_z/E in the z direction and strains $-\nu\sigma_z/E$ in the x and y directions. Combining the results obtained, we conclude that the components of strain corresponding to the given multiaxial loading are

$$\epsilon_x = +\frac{\sigma_x}{E} - \frac{\nu\sigma_y}{E} - \frac{\nu\sigma_z}{E}$$
$$\epsilon_y = -\frac{\nu\sigma_x}{E} + \frac{\sigma_y}{E} - \frac{\nu\sigma_z}{E} \qquad (2.28)$$
$$\epsilon_z = -\frac{\nu\sigma_x}{E} - \frac{\nu\sigma_y}{E} + \frac{\sigma_z}{E}$$

EXAMPLE 2.08

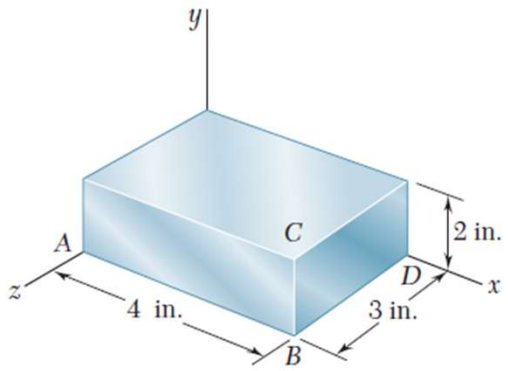


Fig. 2.40

The steel block shown (Fig. 2.40) is subjected to a uniform pressure on all its faces. Knowing that the change in length of edge AB is -1.2×10^{-3} in., determine (a) the change in length of the other two edges, (b) the pressure p applied to the faces of the block. Assume $E = 29 \times 10^6$ psi and $\nu = 0.29$.

(a) Change in Length of Other Edges. Substituting $\sigma_x = \sigma_y = \sigma_z = -p$ into the relations (2.28), we find that the three strain components have the common value

$$\epsilon_x = \epsilon_y = \epsilon_z = -\frac{p}{E}(1 - 2\nu) \tag{2.29}$$

Since

$$\epsilon_x = \delta_x/AB = (-1.2 \times 10^{-3}\text{ in.})/(4\text{ in.})$$
$$= -300 \times 10^{-6}\text{ in./in.}$$

we obtain

$$\epsilon_y = \epsilon_z = \epsilon_x = -300 \times 10^{-6}\text{ in./in.}$$

from which it follows that

$$\delta_y = \epsilon_y(BC) = (-300 \times 10^{-6})(2\text{ in.}) = -600 \times 10^{-6}\text{ in.}$$
$$\delta_z = \epsilon_z(BD) = (-300 \times 10^{-6})(3\text{ in.}) = -900 \times 10^{-6}\text{ in.}$$

(b) Pressure. Solving Eq. (2.29) for p, we write

$$p = -\frac{E\epsilon_x}{1 - 2\nu} = -\frac{(29 \times 10^6\text{ psi})(-300 \times 10^{-6})}{1 - 0.58}$$
$$p = 20.7\text{ ksi}$$

SHEARING STRAIN

When we derived in Sec. 2.12 the relations (2.28) between normal stresses and normal strains in a homogeneous isotropic material, we assumed that no shearing stresses were involved. In the more general stress situation represented in Fig. 2.41, shearing stresses τ_{xy}, τ_{yz}, and τ_{zx} will be present (as well, of course, as the corresponding shearing stresses τ_{yx}, τ_{zy}, and τ_{xz}). These stresses have no direct effect on the normal strains and, as long as all the deformations involved remain small, they will not affect the derivation nor the validity of the relations (2.28). The shearing stresses, however, will tend to deform a cubic element of material into an *oblique* parallelepiped.

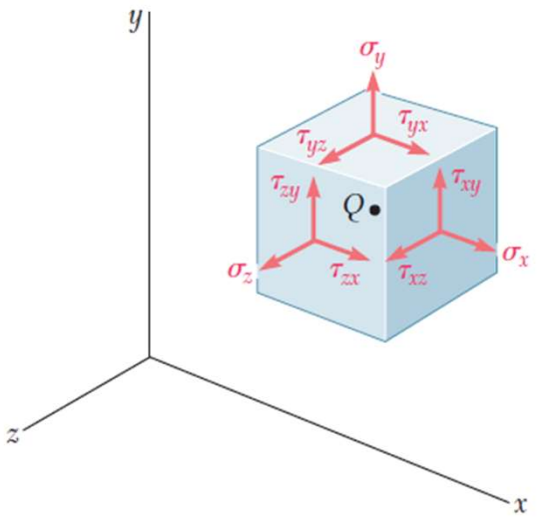


Fig. 2.41 General state of stress.

Consider first a cubic element of side one (Fig. 2.42) subjected to no other stresses than the shearing stresses τ_{xy} and τ_{yx} applied to faces of the element respectively perpendicular to the x and y axes. (We recall from Sec. 1.12 that $\tau_{xy} = \tau_{yx}$.) The element is observed to deform into a rhomboid of sides equal to one (Fig. 2.43). Two of the angles formed by the four faces under stress are reduced from $\frac{\pi}{2}$ to $\frac{\pi}{2} - \gamma_{xy}$, while the other two are increased from $\frac{\pi}{2}$ to $\frac{\pi}{2} + \gamma_{xy}$. The small angle γ_{xy} (expressed in radians) defines the *shearing strain* corresponding to the x and y directions. When the deformation involves a *reduction* of the angle formed by the two faces oriented respectively toward the positive x and y axes (as shown in Fig. 2.43), the shearing strain γ_{xy} is said to be *positive;* otherwise, it is said to be negative.

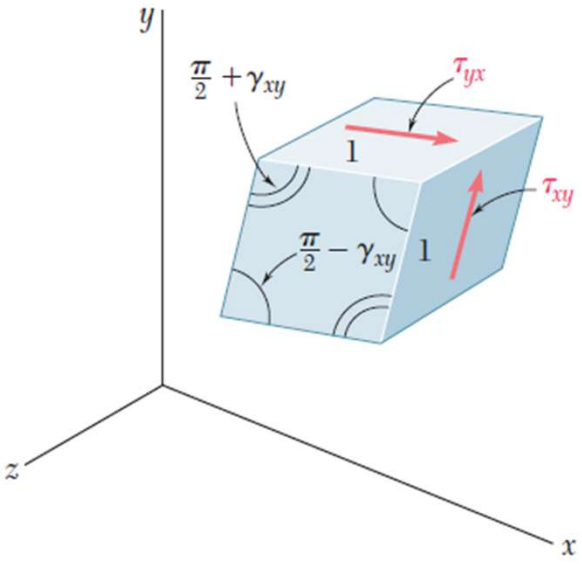
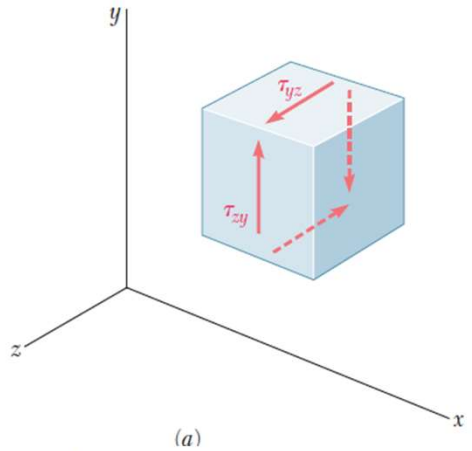


Fig. 2.43 Deformation of cubic element due to shearing stresses.

$$\tau_{xy} = G\gamma_{xy}$$



(a)

This relation is known as *Hooke's law for shearing stress and strain*, and the constant G is called the *modulus of rigidity* or *shear modulus* of the material. Since the strain γ_{xy} was defined as an angle in radians, it is dimensionless, and the modulus G is expressed in the same units as τ_{xy}, that is, in pascals or in psi. The modulus of rigidity G of any given material is less than one-half, but more than one-third of the modulus of elasticity E of that material.†

$$\tau_{yz} = G\gamma_{yz} \qquad \tau_{zx} = G\gamma_{zx}$$

$$\epsilon_x = +\frac{\sigma_x}{E} - \frac{\nu\sigma_y}{E} - \frac{\nu\sigma_z}{E}$$

$$\epsilon_y = -\frac{\nu\sigma_x}{E} + \frac{\sigma_y}{E} - \frac{\nu\sigma_z}{E}$$

$$\epsilon_z = -\frac{\nu\sigma_x}{E} - \frac{\nu\sigma y}{E} + \frac{\sigma_z}{E}$$

$$\gamma_{xy} = \frac{\tau_{xy}}{G} \qquad \gamma_{yz} = \frac{\tau_{yz}}{G} \qquad \gamma_{zx} = \frac{\tau_{zx}}{G}$$

$$G = \frac{E}{2(1+\nu)}$$

EXAMPLE 2.10

A rectangular block of a material with a modulus of rigidity $G = 90$ ksi is bonded to two rigid horizontal plates. The lower plate is fixed, while the upper plate is subjected to a horizontal force **P** (Fig. 2.47). Knowing that the upper plate moves through 0.04 in. under the action of the force, determine (*a*) the average shearing strain in the material, (*b*) the force **P** exerted on the upper plate.

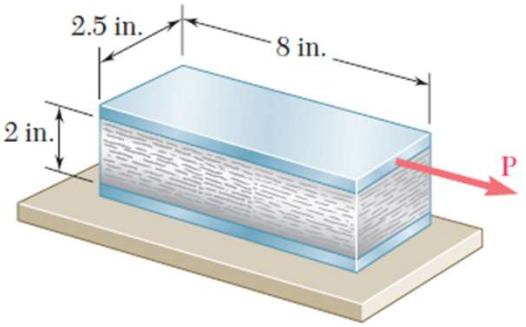


Fig. 2.47

(a) Shearing Strain. We select coordinate axes centered at the midpoint C of edge AB and directed as shown (Fig. 2.48). According to its definition, the shearing strain γ_{xy} is equal to the angle formed by the vertical and the line CF joining the midpoints of edges AB and DE. Noting that this is a very small angle and recalling that it should be expressed in radians, we write

$$\gamma_{xy} \approx \tan \gamma_{xy} = \frac{0.04 \text{ in.}}{2 \text{ in.}} \qquad \gamma_{xy} = 0.020 \text{ rad}$$

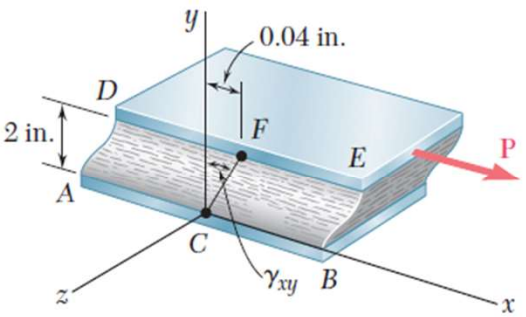


Fig. 2.48

(b) Force Exerted on Upper Plate. We first determine the shearing stress τ_{xy} in the material. Using Hooke's law for shearing stress and strain, we have

$$\tau_{xy} = G\gamma_{xy} = (90 \times 10^3 \text{ psi})(0.020 \text{ rad}) = 1800 \text{ psi}$$

The force exerted on the upper plate is thus

$$P = \tau_{xy} A = (1800 \text{ psi})(8 \text{ in.})(2.5 \text{ in.}) = 36.0 \times 10^3 \text{ lb}$$

$$P = 36.0 \text{ kips}$$