Concept of Torsion

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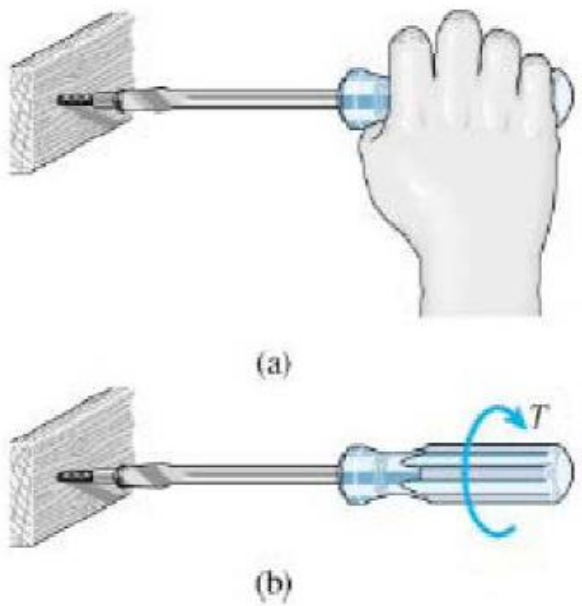
Torsion

Torsion is the twisting of a straight bar when it is loaded by twisting moments or torques that tend to produce rotation about the longitudinal axes of the bar.

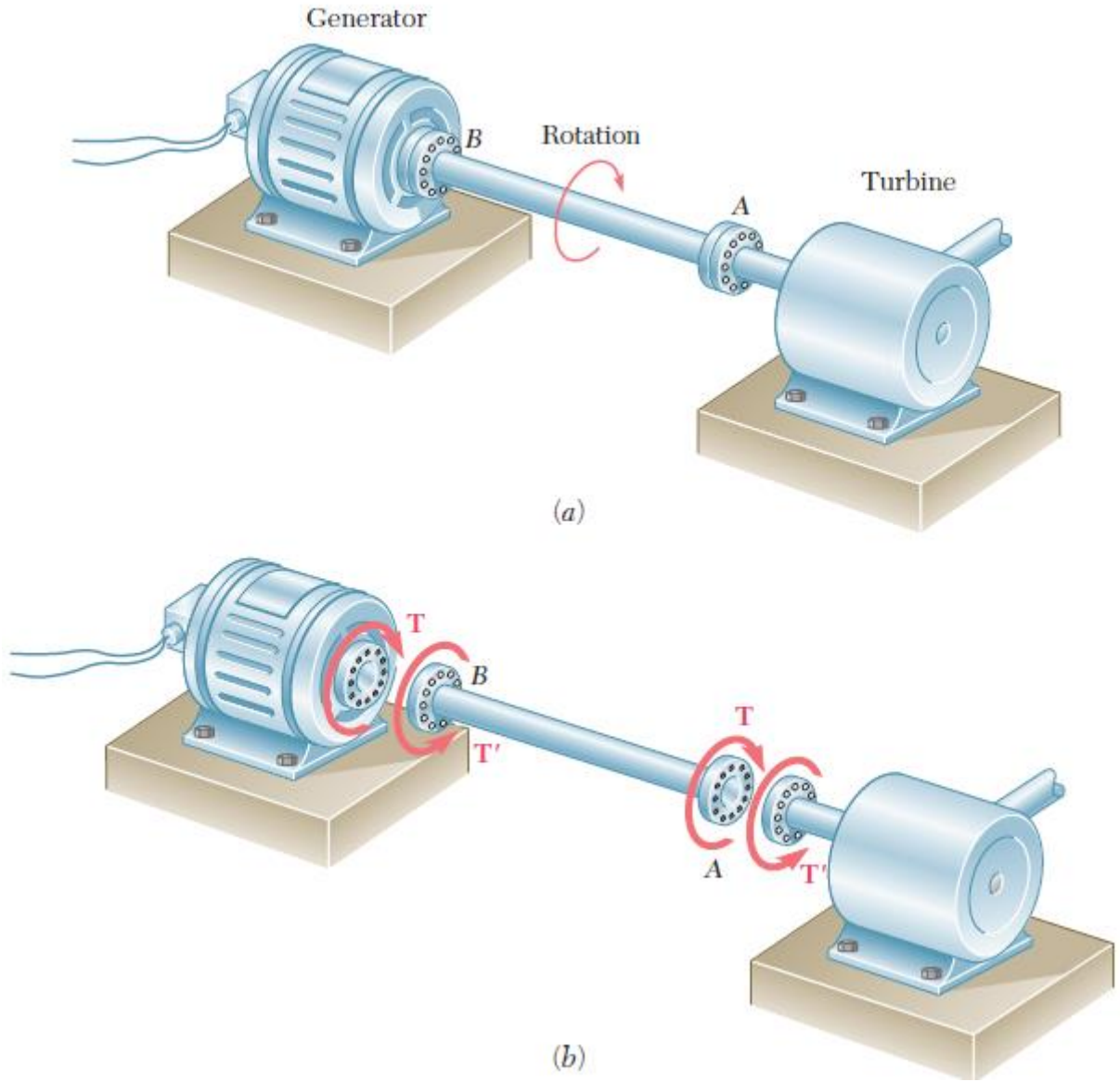
For instance, when we turn a screw driver to produce torsion our hand applies torque 'T' to the handle and twists the shank of the screw driver.

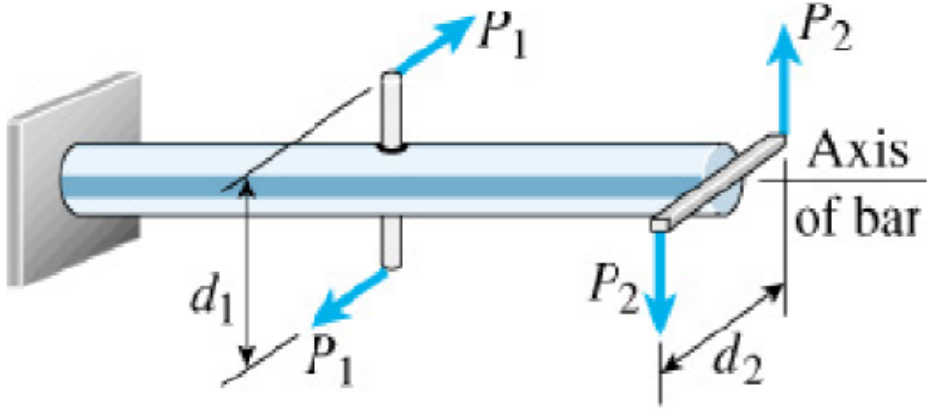
Sign convention – Right Hand Rule

Shafts are structural members with length significantly greater than the largest cross-sectional dimension used in transmitting torque from one plane to another.



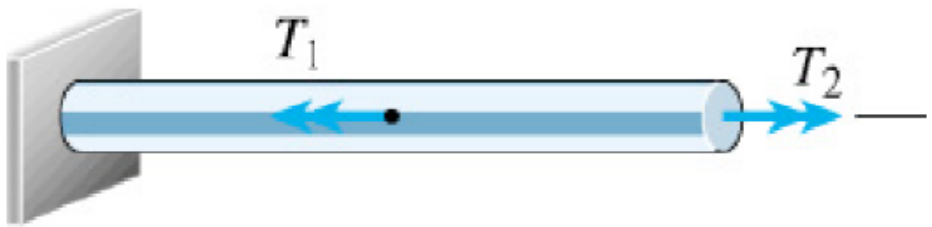
- Turbine exerts torque T on the shaft
- Shaft transmits the torque to the generator
- Generator creates an equal and opposite torque T'
- Interested in stresses and strains of circular shafts subjected to twisting couples or *torques*



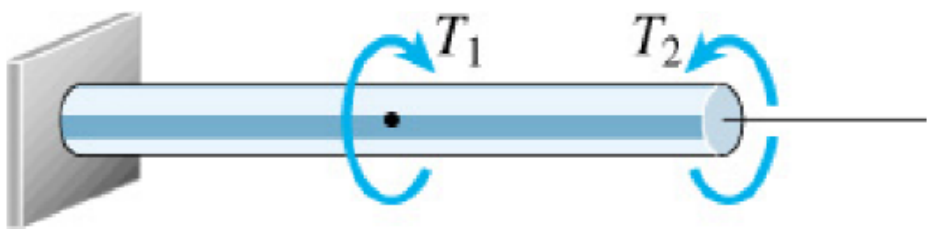


$T_1 = P_1 d_1$ $\quad$ $T_2 = P_2 d_2$

(a)



(b)

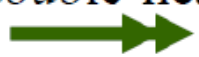


Torsion: twisting of a straight bar when loaded by moments (or torques)
Moments or torques tend to produce rotation about the longitudinal axis of the bar (twisting).

Couple: the pair of forces that tends to twist the bar about its longitudinal axis.

Moment of a couple:
T (Nm)= Force (N) x Arm (m) or
T (lb in.)= Force (lb) x Arm (in.)

Representation of the moment of a couple: a vector with a double-headed arrow



Then we can write the moments as

$T_1 = P_1 d_1$ $\quad$ $T_2 = P_2 d_2$

THE STRESSES IN A SHAFT

Considering a shaft AB subjected at A and B to equal and opposite torques $\mathbf{T}$ and $\mathbf{T}'$, we pass a section perpendicular to the axis of the shaft through some arbitrary point C (Fig. 3.3). The free-body diagram of the portion BC of the shaft must include the elementary shearing forces $d\mathbf{F}$, perpendicular to the radius of the shaft, that portion AC exerts on BC as the shaft is twisted (Fig. 3.4*a*). But the conditions of equilibrium for BC require that the system of these elementary forces be equivalent to an internal torque $\mathbf{T}$, equal and opposite to $\mathbf{T}'$ (Fig. 3.4*b*). Denoting by ρ the perpendicular distance from the force $d\mathbf{F}$ to the axis of the shaft, and expressing that the sum of the moments of the shearing forces $d\mathbf{F}$ about the axis of the shaft is equal in magnitude to the torque $\mathbf{T}$, we write

$$\int \rho \, dF = T$$

or, since $dF = \tau \, dA$, where τ is the shearing stress on the element of area dA,

$$\int \rho (\tau \, dA) = T \tag{3.1}$$

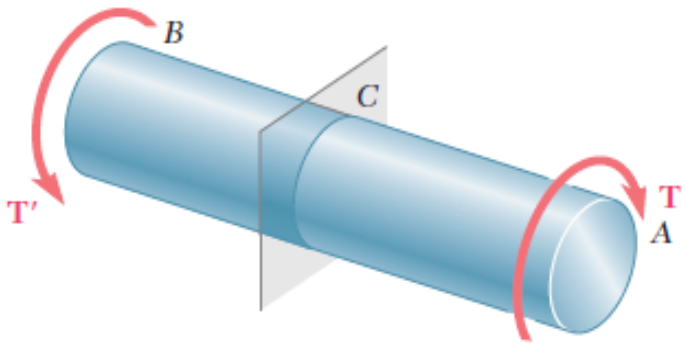


Fig. 3.3 Shaft subject to torques.

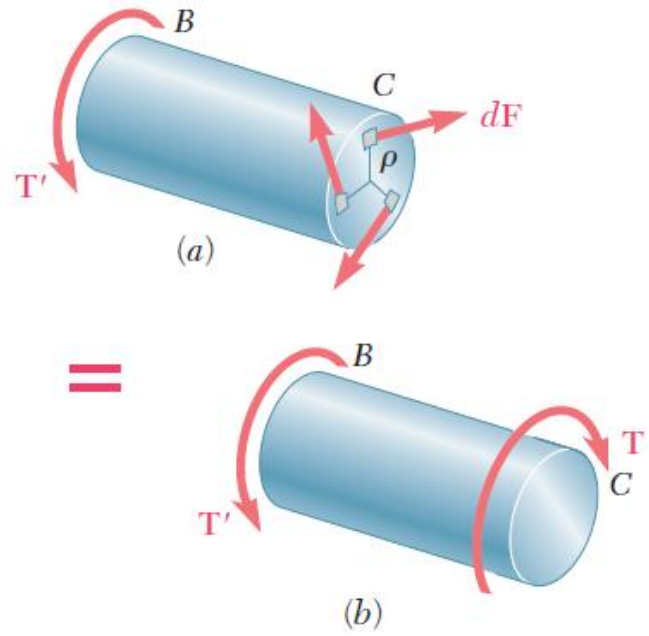


Fig. 3.4

or, since $dF = \tau dA$, where τ is the shearing stress on the element of area dA,

$$\int \rho(\tau dA) = T \tag{3.1}$$

While the relation obtained expresses an important condition that must be satisfied by the shearing stresses in any given cross section of the shaft, it does *not* tell us how these stresses are distributed in the cross section. We thus observe, as we already did in Sec. 1.5, that the actual distribution of stresses under a given load is *statically indeterminate*, i.e., this distribution *cannot be determined by the methods of statics*. However, having assumed in Sec. 1.5 that the normal stresses produced by an axial centric load were uniformly distributed, we found later (Sec. 2.17) that this assumption was justified, except in the neighborhood of concentrated loads. A similar assumption with respect to the distribution of shearing stresses in an elastic shaft *would be wrong*. We must withhold any judgment regarding the distribution of stresses in a shaft until we have analyzed the *deformations* that are produced in the shaft. This will be done in the next section.

One more observation should be made at this point. As was indicated in Sec. 1.12, shear cannot take place in one plane only. Consider the very small element of shaft shown in Fig. 3.5. We know that the torque applied to the shaft produces shearing stresses τ on

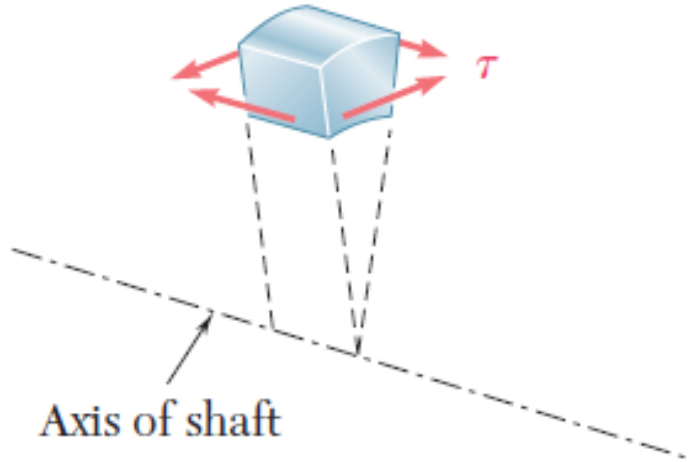


Fig. 3.5 Element in shaft.

the faces perpendicular to the axis of the shaft. But the conditions of equilibrium discussed in Sec. 1.12 require the existence of equal stresses on the faces formed by the two planes containing the axis of the shaft. That such shearing stresses actually occur in torsion can be demonstrated by considering a "shaft" made of separate slats pinned at both ends to disks as shown in Fig. 3.6*a*. If markings have been painted on two adjoining slats, it is observed that the slats slide with respect to each other when equal and opposite torques are applied to the ends of the "shaft" (Fig. 3.6*b*). While sliding will not actually take place in a shaft made of a homogeneous and cohesive material, the tendency for sliding will exist, showing that stresses occur on longitudinal planes as well as on planes perpendicular to the axis of the shaft.†

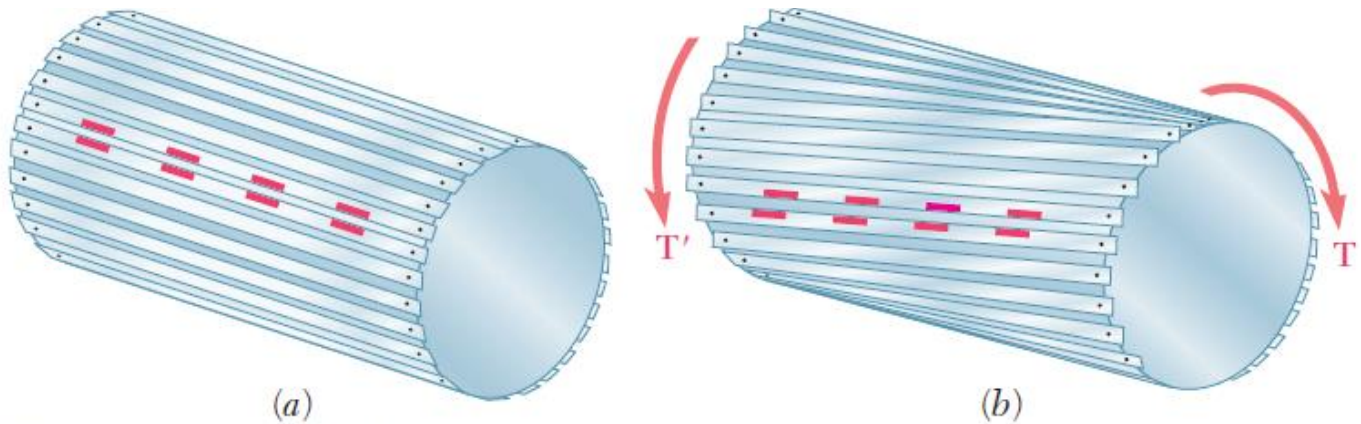
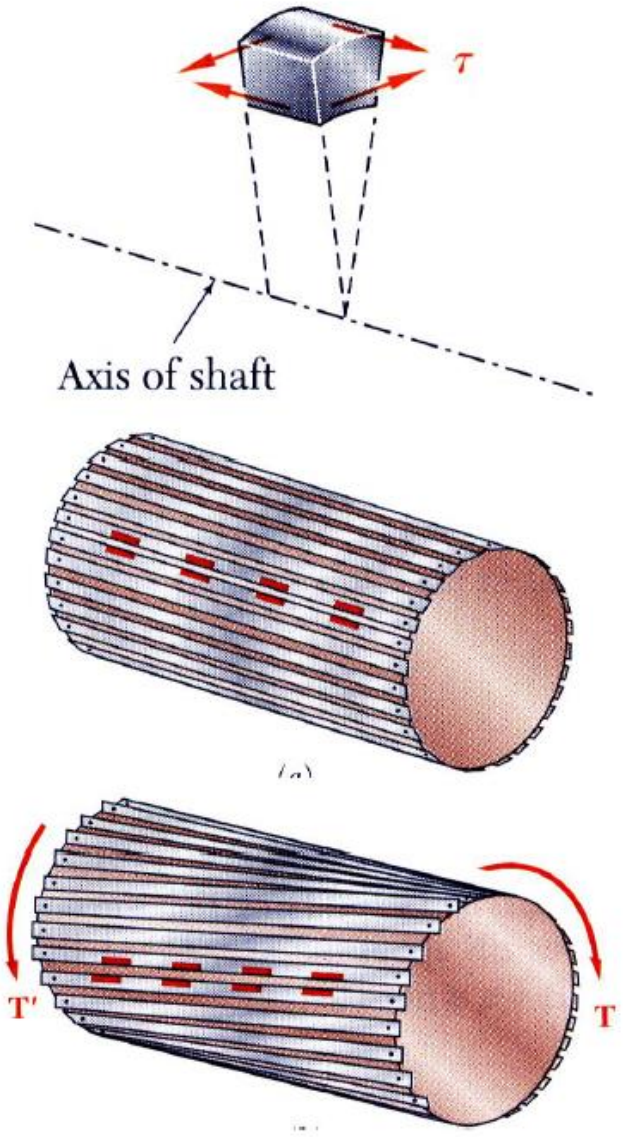


Fig. 3.6 Model of shaft.

- In summery



- Torque applied to shaft produces shearing stresses on the faces perpendicular to the axis.
- Conditions of equilibrium require the existence of equal stresses on the faces of the two planes containing the axis of the shaft
- The existence of the axial shear components is demonstrated by considering a shaft made up of axial slats.

  The slats slide with respect to each other when equal and opposite torques are applied to the ends of the shaft.

DEFORMATIONS IN A CIRCULAR SHAFT

Consider a circular shaft that is attached to a fixed support at one end (Fig. 3.7*a*). If a torque **T** is applied to the other end, the shaft will twist, with its free end rotating through an angle ϕ called *the angle of twist* (Fig. 3.7*b*). Observation shows that, within a certain range of values of T, the angle of twist ϕ is proportional to T. It also shows that ϕ is proportional to the length L of the shaft. In other words, the angle of twist for a shaft of the same material and same cross section, but twice as long, will be twice as large under the same torque **T.** One purpose of our analysis will be to find the specific relation existing among ϕ, L, and T; another purpose will be to determine the distribution of shearing stresses in the shaft, which we were unable to obtain in the preceding section on the basis of statics alone.

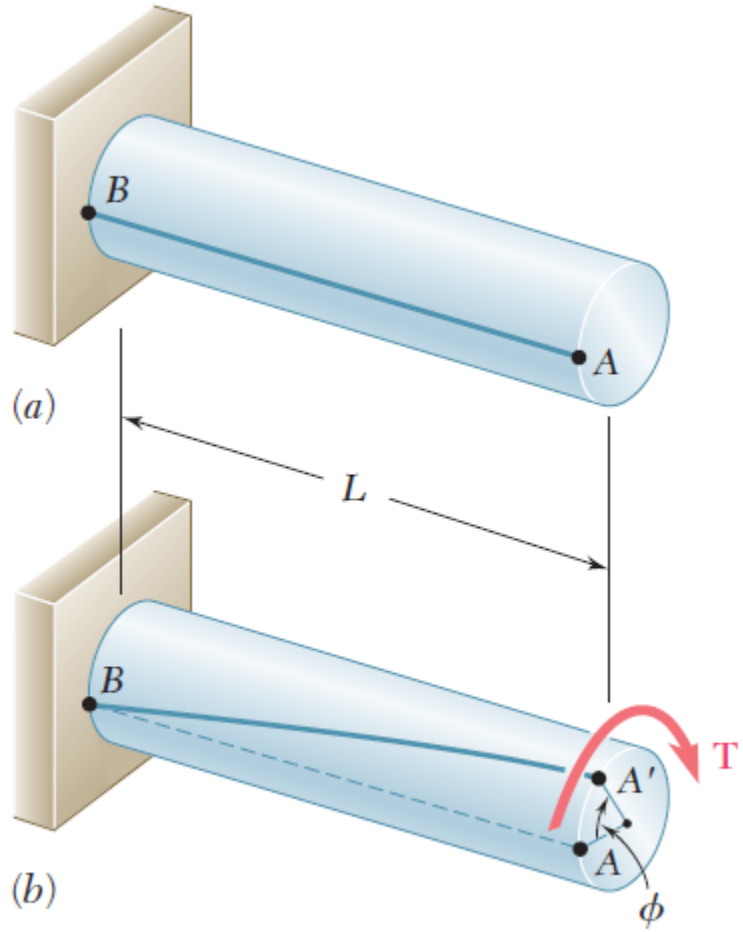


Fig. 3.7 Shaft with fixed support.

At this point, an important property of circular shafts should be noted: When a circular shaft is subjected to torsion, *every cross section remains plane and undistorted*. In other words, while the various cross sections along the shaft rotate through different amounts, each cross section rotates as a solid rigid slab. This is illustrated in Fig. 3.8*a*, which shows the deformations in a rubber model subjected to torsion. The property we are discussing is characteristic of circular shafts, whether solid or hollow; it is not enjoyed by members of noncircular cross section. For example, when a bar of square cross section is subjected to torsion, its various cross sections warp and do not remain plane (Fig. 3.8*b*).

The cross sections of a circular shaft remain plane and undistorted because a circular shaft is *axisymmetric*, i.e., its appearance remains the same when it is viewed from a fixed position and rotated about its axis through an arbitrary angle. (Square bars, on the other hand, retain the same appearance only if they are rotated through 90° or 180°.) As we will see presently, the axisymmetry of circular shafts may be used to prove theoretically that their cross sections remain plane and undistorted.

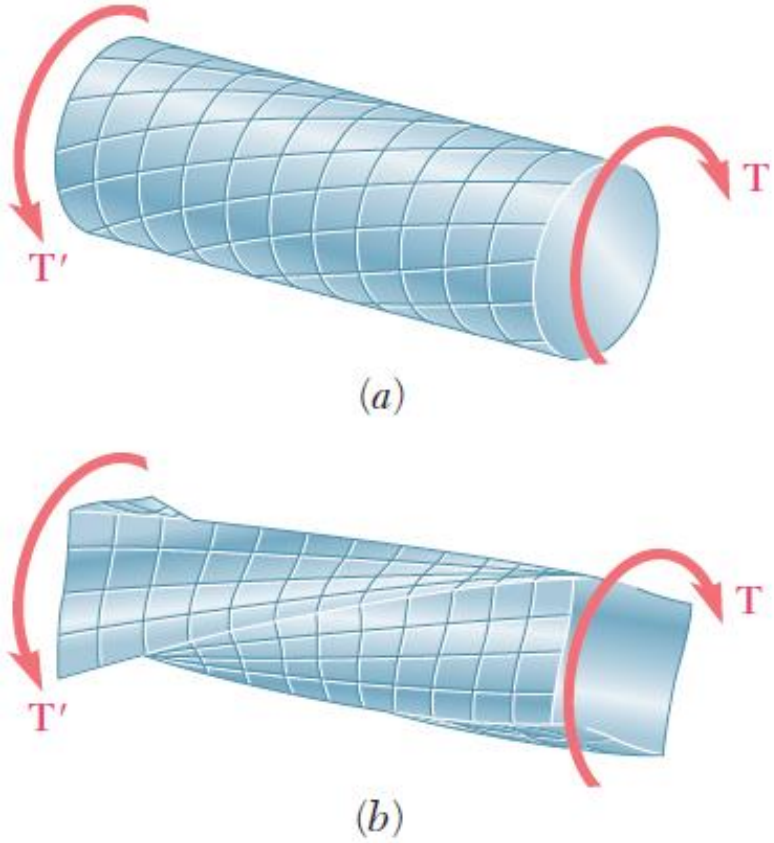


Fig. 3.8 Comparison of deformations in circular and square shafts.

Consider the points C and D located on the circumference of a given cross section of the shaft, and let C' and D' be the positions they will occupy after the shaft has been twisted (Fig. 3.9*a*). The axisymmetry of the shaft and of the loading requires that the rotation which would have brought D into D' should now bring C into C'. Thus C' and D' must lie on the circumference of a circle, and the arc $C'D'$ must be equal to the arc CD (Fig. 3.9*b*). We will now examine whether the circle on which C' and D' lie is different from the original circle. Let us assume that C' and D' do lie on a different circle and that the new circle is located to the left of the original circle, as shown in Fig. 3.9*b*. The same situation will prevail for any other cross section, since all the cross sections of the shaft are subjected to the same internal torque T, and an observer looking at the shaft from its end A will conclude that the loading causes any given circle drawn on the shaft to move *away*. But an observer located at B, to whom the given loading looks the same (a clockwise couple in the foreground and a counterclockwise couple in the background) will reach the opposite conclusion, i.e., that the circle moves *toward* him. This contradiction proves that our assumption is wrong and that C' and D' lie on the same circle as C and D. Thus, as the shaft is twisted, the original circle just rotates in its own plane. Since the same reasoning may be applied to any smaller, concentric circle located in the cross section under consideration, we conclude that the entire cross section remains plane (Fig. 3.10).

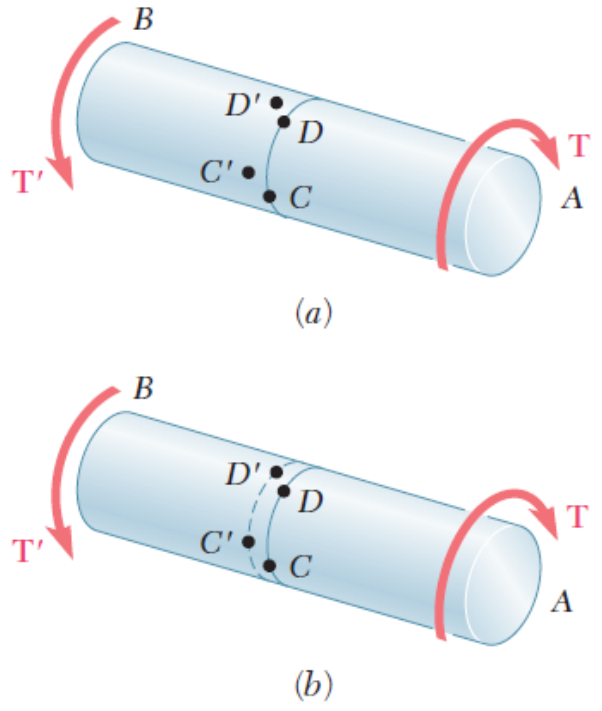


Fig. 3.9 Shaft subject to twisting.

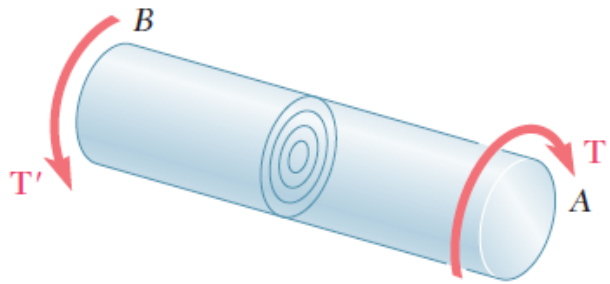


Fig. 3.10 Concentric circles.

The above argument does not preclude the possibility for the various concentric circles of Fig. 3.10 to rotate by different amounts when the shaft is twisted. But if that were so, a given diameter of the cross section would be distorted into a curve which might look as shown in Fig. 3.11*a*. An observer looking at this curve from *A* would conclude that the outer layers of the shaft get more twisted than the inner ones, while an observer looking from *B* would reach the opposite conclusion (Fig. 3.11*b*). This inconsistency leads us to conclude that any diameter of a given cross section remains straight (Fig. 3.11*c*) and, therefore, that any given cross section of a circular shaft remains plane and undistorted.

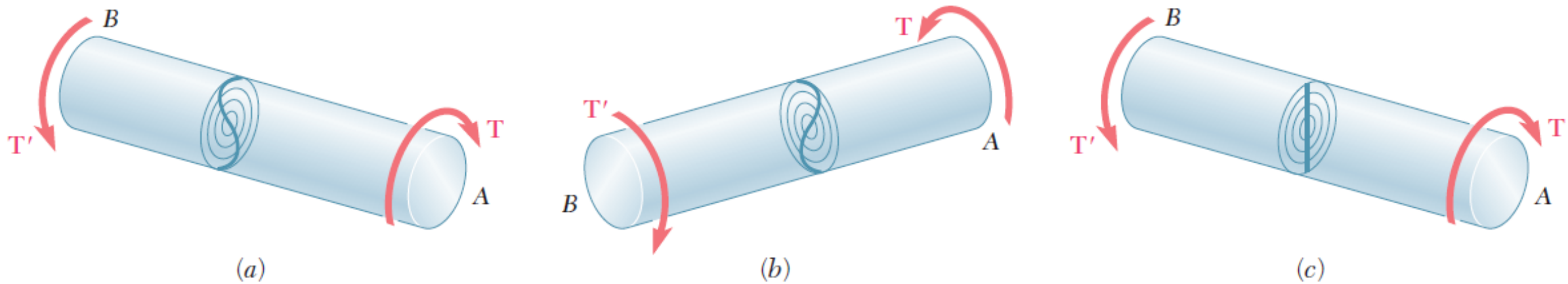


Fig. 3.11 Potential deformations of cross section.

Our discussion so far has ignored the mode of application of the twisting couples **T** and **T′**. If *all* sections of the shaft, from one end to the other, are to remain plane and undistorted, we must make sure that the couples are applied in such a way that the ends of the shaft themselves remain plane and undistorted. This may be accomplished by applying the couples **T** and **T′** to rigid plates, which are solidly attached to the ends of the shaft (Fig. 3.12a). We can then be sure that all sections will remain plane and undistorted when the loading is applied, and that the resulting deformations will occur in a uniform fashion throughout the entire length of the shaft. All of the equally spaced circles shown in Fig. 3.12*a* will rotate by the same amount relative to their neighbors, and each of the straight lines will be transformed into a curve (helix) intersecting the various circles at the same angle (Fig. 3.12*b*).

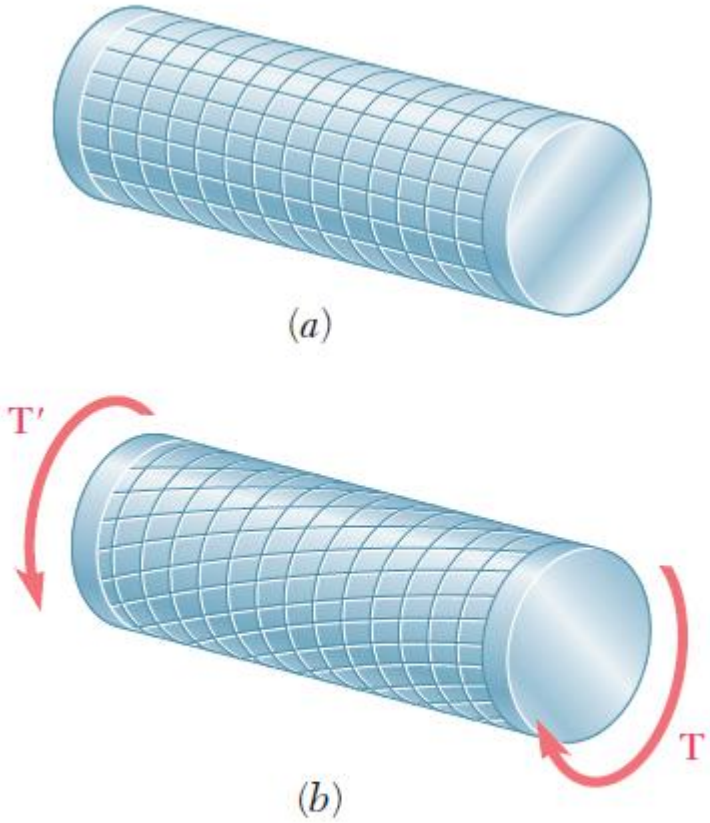


Fig. 3.12 Deformation of shaft subject to twisting couples.

We will now determine the distribution of *shearing strains* in a circular shaft of length L and radius c that has been twisted through an angle ϕ (Fig. 3.13*a*). Detaching from the shaft a cylinder of radius ρ, we consider the small square element formed by two adjacent circles and two adjacent straight lines traced on the surface of the cylinder before any load is applied (Fig. 3.13*b*). As the shaft is subjected to a torsional load, the element deforms into a rhombus (Fig. 3.13*c*). We now recall from Sec. 2.14 that the shearing strain γ in a given element is measured by the change in the angles formed by the sides of that element. Since the circles defining two of the sides of the element considered here remain unchanged, the shearing strain γ must be equal to the angle between lines AB and $A'B$. (We recall that γ should be expressed in radians.)

We observe from Fig. 3.13*c* that, for small values of γ, we can express the arc length AA' as $AA' = L\gamma$. But, on the other hand, we have $AA' = \rho\phi$. It follows that $L\gamma = \rho\phi$, or

$$\gamma = \frac{\rho\phi}{L} \tag{3.2}$$

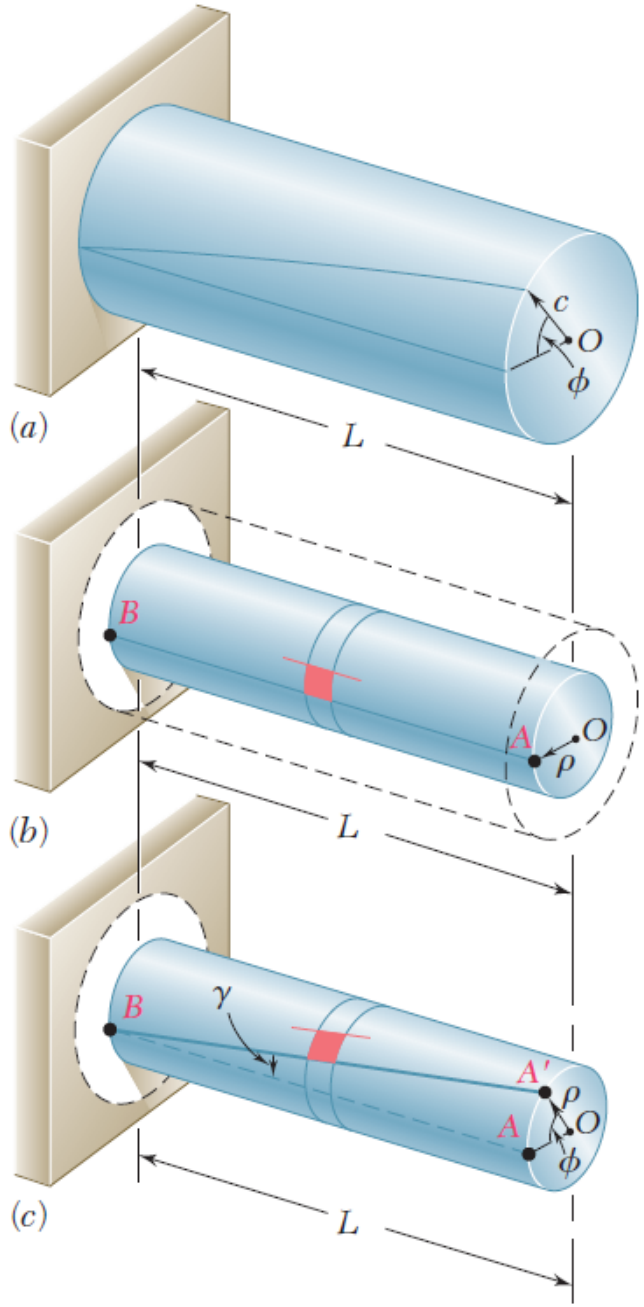


Fig. 3.13 Shearing strain.

$$\gamma = \frac{\rho\phi}{L}$$

where γ and ϕ are both expressed in radians. The equation obtained shows, as we could have anticipated, that the shearing strain γ at a given point of a shaft in torsion is proportional to the angle of twist ϕ. It also shows that γ is proportional to the distance ρ from the axis of the shaft to the point under consideration. Thus, *the shearing strain in a circular shaft varies linearly with the distance from the axis of the shaft.*

It follows from Eq. (3.2) that the shearing strain is maximum on the surface of the shaft, where $\rho = c$. We have

$$\gamma_{\max} = \frac{c\phi}{L} \tag{3.3}$$

Eliminating ϕ from Eqs. (3.2) and (3.3), we can express the shearing strain γ at a distance ρ from the axis of the shaft as

$$\gamma = \frac{\rho}{c}\gamma_{\max} \tag{3.4}$$

3.4 STRESSES IN THE ELASTIC RANGE

No particular stress-strain relationship has been assumed so far in our discussion of circular shafts in torsion. Let us now consider the case when the torque **T** is such that all shearing stresses in the shaft remain below the yield strength τ_Y. We know from Chap. 2 that, for all practical purposes, this means that the stresses in the shaft will remain below the proportional limit and below the elastic limit as well. Thus, Hooke's law will apply and there will be no permanent deformation.

Recalling Hooke's law for shearing stress and strain from Sec. 2.14, we write

$$\tau = G\gamma \tag{3.5}$$

where G is the modulus of rigidity or shear modulus of the material. Multiplying both members of Eq. (3.4) by G, we write

$$G\gamma = \frac{\rho}{c} G\gamma_{\max}$$

or, making use of Eq. (3.5),

$$\tau = \frac{\rho}{c} \tau_{\max} \tag{3.6}$$

The equation obtained shows that, as long as the yield strength (or proportional limit) is not exceeded in any part of a circular shaft, *the shearing stress in the shaft varies linearly with the distance* ρ *from the axis of the shaft*. Figure 3.14*a* shows the stress distribution in a solid circular shaft of radius c, and Fig. 3.14*b* in a hollow circular shaft of inner radius c_1 and outer radius c_2. From Eq. (3.6), we find that, in the latter case,

$$\tau_{\min} = \frac{c_1}{c_2}\tau_{\max} \tag{3.7}$$

We now recall from Sec. 3.2 that the sum of the moments of the elementary forces exerted on any cross section of the shaft must be equal to the magnitude T of the torque exerted on the shaft:

$$\int \rho(\tau\, dA) = T \tag{3.1}$$

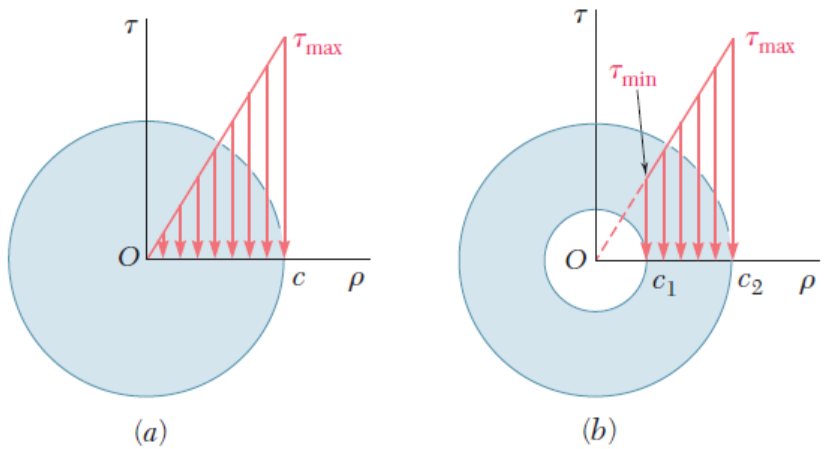


Fig. 3.14 Distribution of shearing stresses.

Substituting for τ from (3.6) into (3.1), we write

$$T = \int \rho\tau \, dA = \frac{\tau_{\max}}{c} \int \rho^2 \, dA$$

But the integral in the last member represents the polar moment of inertia J of the cross section with respect to its center O. We have therefore

$$T = \frac{\tau_{\max} J}{c} \tag{3.8}$$

or, solving for $\tau_{\max}$,

$$\tau_{\max} = \frac{Tc}{J} \tag{3.9}$$

Substituting for $\tau_{\max}$ from (3.9) into (3.6), we express the shearing stress at any distance ρ from the axis of the shaft as

$$\tau = \frac{T\rho}{J} \tag{3.10}$$

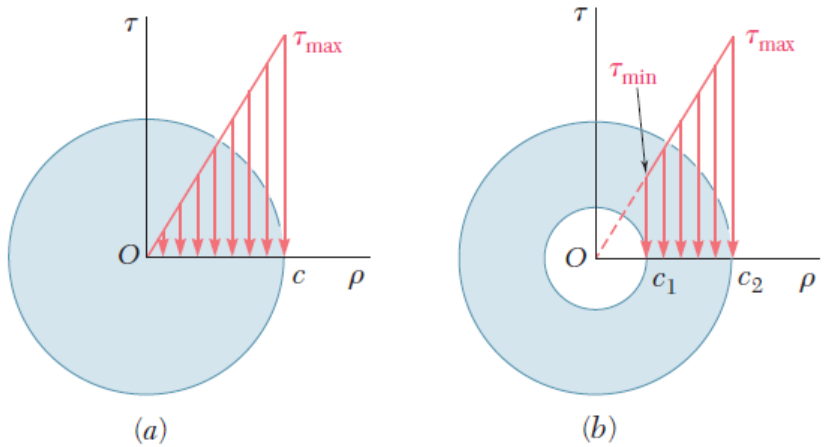


Fig. 3.14 Distribution of shearing stresses.

Equations (3.9) and (3.10) are known as the *elastic torsion formulas*. We recall from statics that the polar moment of inertia of a circle of radius c is $J = \frac{1}{2}\pi c^4$. In the case of a hollow circular shaft of inner radius c_1 and outer radius c_2, the polar moment of inertia is

$$J = \tfrac{1}{2}\pi c_2^4 - \tfrac{1}{2}\pi c_1^4 = \tfrac{1}{2}\pi(c_2^4 - c_1^4) \tag{3.11}$$

We note that, if SI metric units are used in Eq. (3.9) or (3.10), T will be expressed in N · m, c or ρ in meters, and J in m^4; we check that the resulting shearing stress will be expressed in N/m^2, that is, pascals (Pa). If U.S. customary units are used, T should be expressed in lb · in., c or ρ in inches, and J in in^4, with the resulting shearing stress expressed in psi.

EXAMPLE 3.01

A hollow cylindrical steel shaft is 1.5 m long and has inner and outer diameters respectively equal to 40 and 60 mm (Fig. 3.15). (*a*) What is the largest torque that can be applied to the shaft if the shearing stress is not to exceed 120 MPa? (*b*) What is the corresponding minimum value of the shearing stress in the shaft?

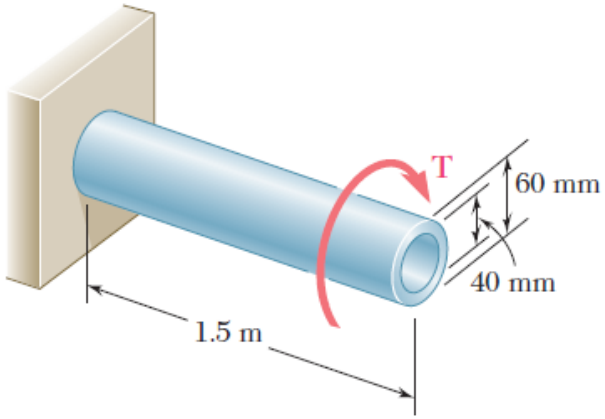
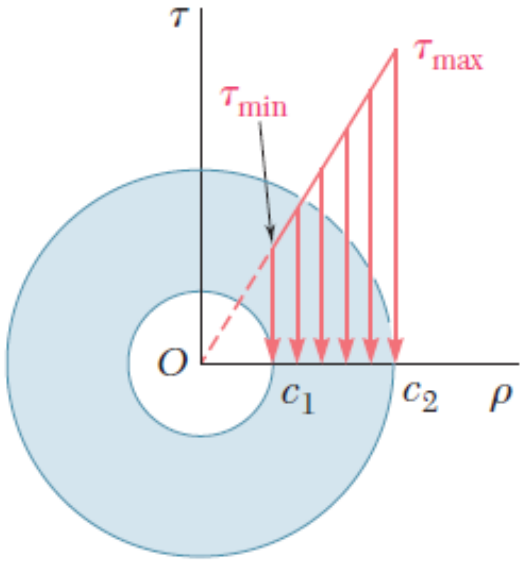


Fig. 3.15



(a) Largest Permissible Torque. The largest torque **T** that can be applied to the shaft is the torque for which $\tau_{max} = 120$ MPa. Since this value is less than the yield strength for steel, we can use Eq. (3.9). Solving this equation for T, we have

$$T = \frac{J\tau_{max}}{c} \tag{3.12}$$

Recalling that the polar moment of inertia J of the cross section is given by Eq. (3.11), where $c_1 = \frac{1}{2}(40\text{ mm}) = 0.02\text{ m}$ and $c_2 = \frac{1}{2}(60\text{ mm}) = 0.03\text{ m}$, we write

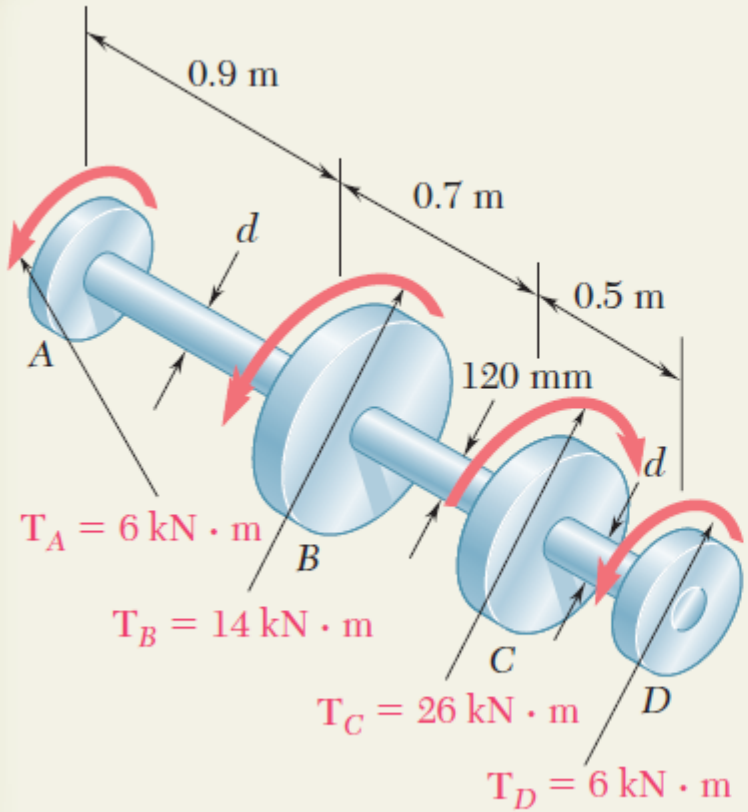
$$J = \tfrac{1}{2}\pi\,(c_2^4 - c_1^4) = \tfrac{1}{2}\pi(0.03^4 - 0.02^4) = 1.021 \times 10^{-6}\text{ m}^4$$

Substituting for J and τ_{max} into (3.12), and letting $c = c_2 = 0.03$ m, we have

$$T = \frac{J\tau_{max}}{c} = \frac{(1.021 \times 10^{-6}\text{ m}^4)(120 \times 10^{6}\text{ Pa})}{0.03\text{ m}} = 4.08\text{ kN} \cdot \text{m}$$

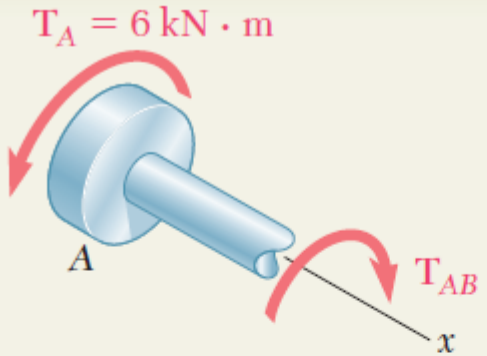
(b) Minimum Shearing Stress. The minimum value of the shearing stress occurs on the inner surface of the shaft. It is obtained from Eq. (3.7), which expresses that τ_{min} and τ_{max} are respectively proportional to c_1 and c_2:

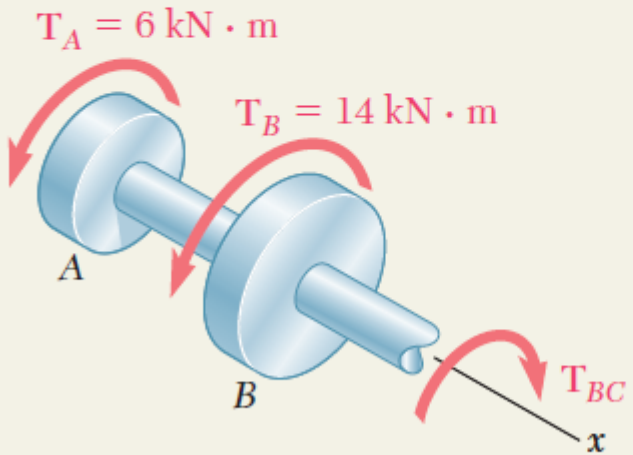
$$\tau_{min} = \frac{c_1}{c_2}\tau_{max} = \frac{0.02\text{ m}}{0.03\text{ m}}(120\text{ MPa}) = 80\text{ MPa}$$



SAMPLE PROBLEM 3.1

Shaft *BC* is hollow with inner and outer diameters of 90 mm and 120 mm, respectively. Shafts *AB* and *CD* are solid and of diameter *d*. For the loading shown, determine (*a*) the maximum and minimum shearing stress in shaft *BC*, (*b*) the required diameter *d* of shafts *AB* and *CD* if the allowable shearing stress in these shafts is 65 MPa.







SOLUTION

Equations of Statics. Denoting by $\mathbf{T}_{AB}$ the torque in shaft AB, we pass a section through shaft AB and, for the free body shown, we write

$$\Sigma M_x = 0: \qquad (6\ \text{kN} \cdot \text{m}) - T_{AB} = 0 \qquad T_{AB} = 6\ \text{kN} \cdot \text{m}$$

We now pass a section through shaft BC and, for the free body shown, we have

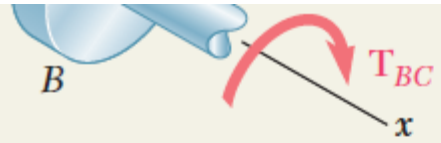
$$\Sigma M_x = 0: \qquad (6\ \text{kN} \cdot \text{m}) + (14\ \text{kN} \cdot \text{m}) - T_{BC} = 0 \qquad T_{BC} = 20\ \text{kN} \cdot \text{m}$$

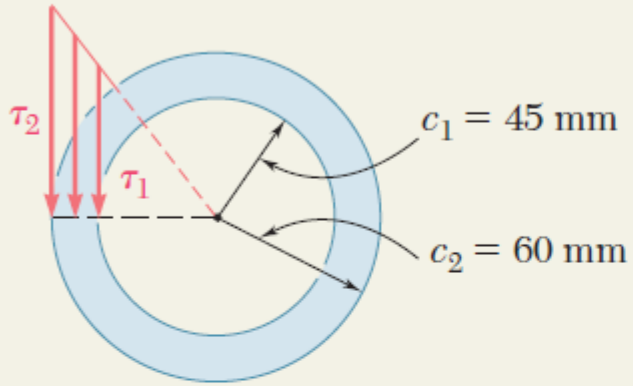
a. Shaft *BC*. For this hollow shaft we have

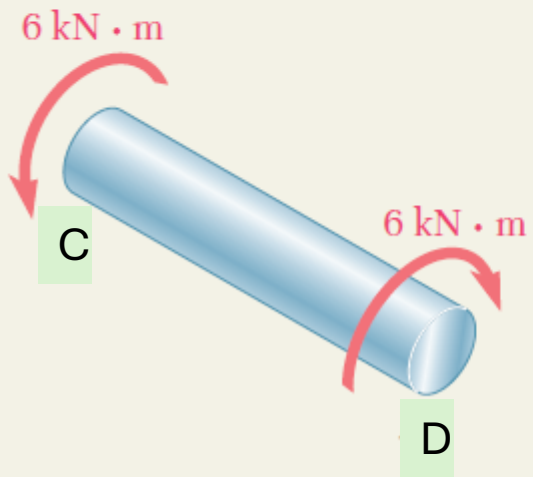
$$J = \frac{\pi}{2}(c_2^4 - c_1^4) = \frac{\pi}{2}[(0.060)^4 - (0.045)^4] = 13.92 \times 10^{-6}\ \text{m}^4$$

Maximum Shearing Stress. On the outer surface, we have

$$\tau_{\max} = \tau_2 = \frac{T_{BC}c_2}{J} = \frac{(20\ \text{kN} \cdot \text{m})(0.060\ \text{m})}{13.92 \times 10^{-6}\ \text{m}^4} \qquad \tau_{\max} = 86.2\ \text{MPa}$$







Maximum Shearing Stress. On the outer surface, we have

$$\tau_{\max} = \tau_2 = \frac{T_{BC}c_2}{J} = \frac{(20\text{ kN} \cdot \text{m})(0.060\text{ m})}{13.92 \times 10^{-6}\text{ m}^4} \qquad \tau_{\max} = 86.2\text{ MPa}$$

Minimum Shearing Stress. We write that the stresses are proportional to the distance from the axis of the shaft.

$$\frac{\tau_{\min}}{\tau_{\max}} = \frac{c_1}{c_2} \qquad \frac{\tau_{\min}}{86.2\text{ MPa}} = \frac{45\text{ mm}}{60\text{ mm}} \qquad \tau_{\min} = 64.7\text{ MPa}$$

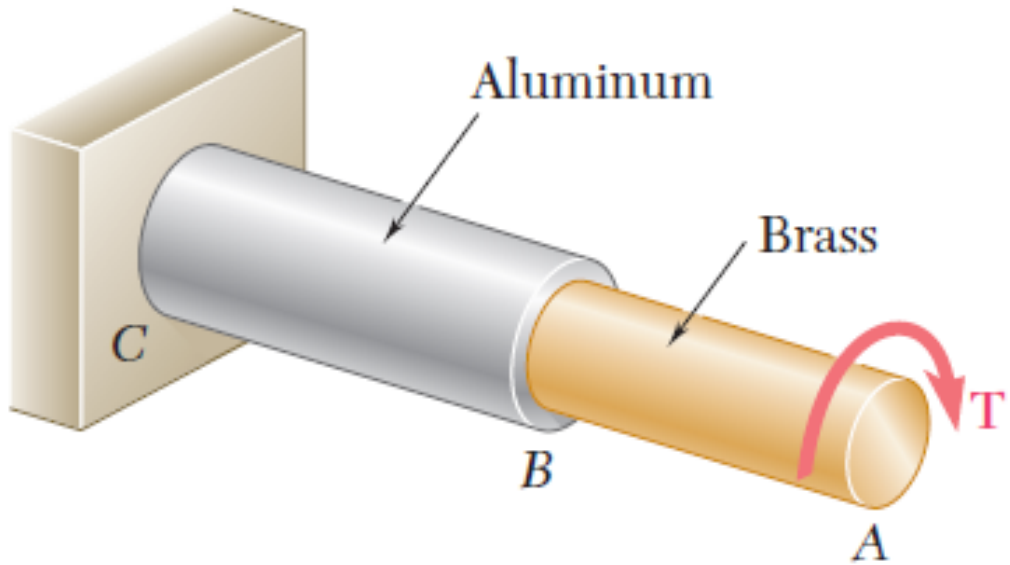
b. Shafts *AB* and *CD*. We note that in both of these shafts the magnitude of the torque is $T = 6\text{ kN} \cdot \text{m}$ and $\tau_{\text{all}} = 65\text{ MPa}$. Denoting by c the radius of the shafts, we write

$$\tau = \frac{Tc}{J} \qquad 65\text{ MPa} = \frac{(6\text{ kN} \cdot \text{m})c}{\frac{\pi}{2}c^4}$$

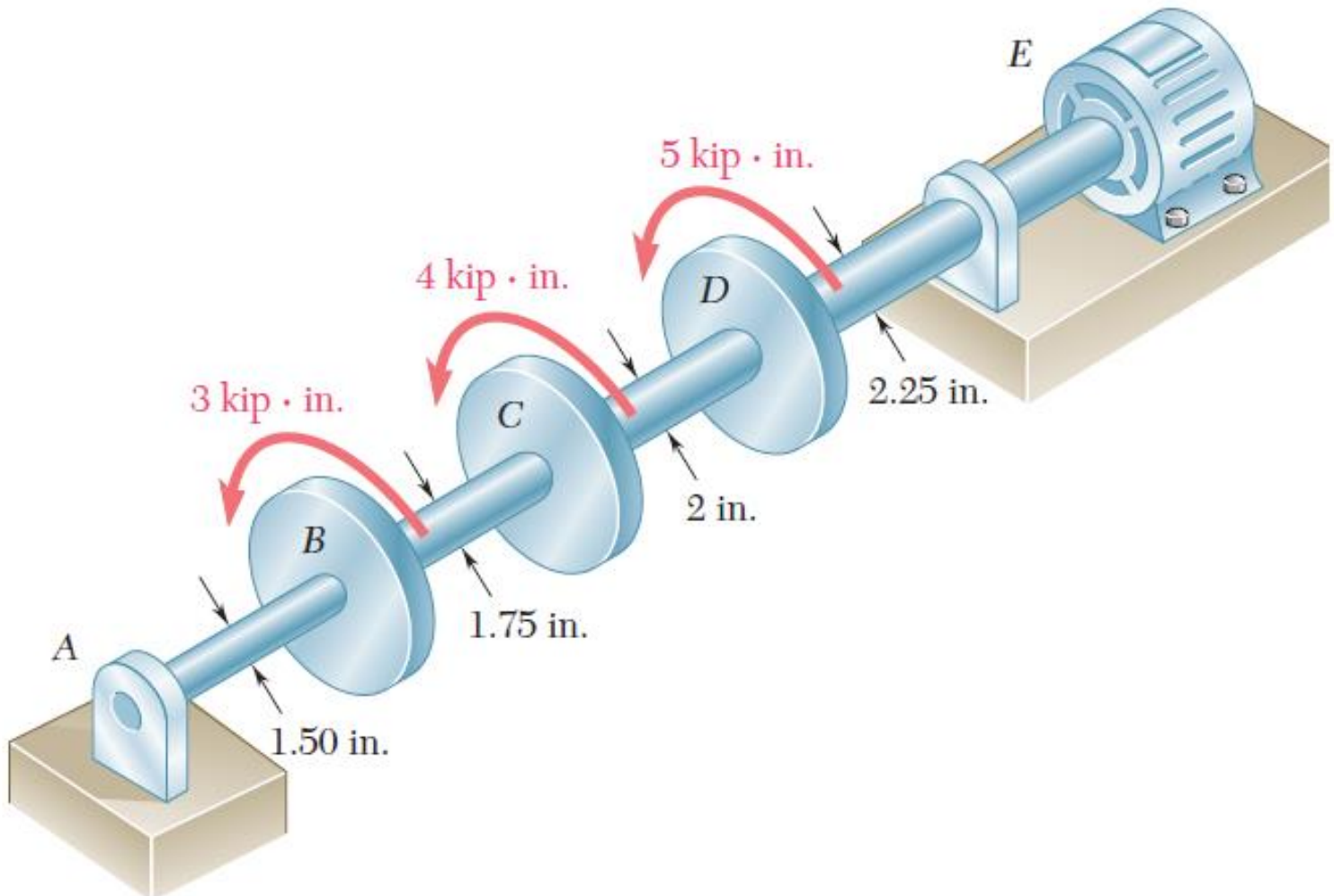
$$c^3 = 58.8 \times 10^{-6}\text{ m}^3 \qquad c = 38.9 \times 10^{-3}\text{ m}$$

$$d = 2c = 2(38.9\text{ mm}) \qquad d = 77.8\text{ mm}$$

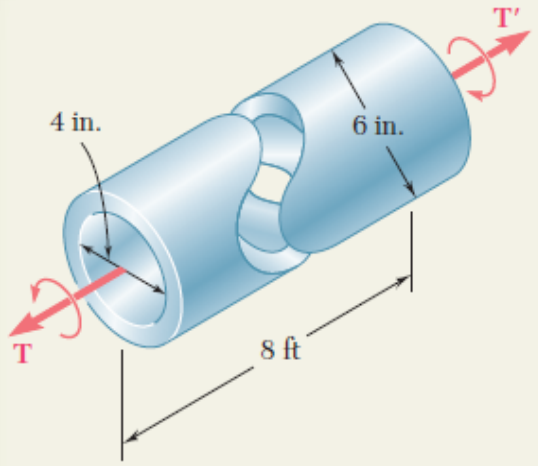
The allowable stress is 50 MPa in the brass rod AB and 25 MPa in the aluminum rod BC. Knowing that a torque of magnitude $T = 1250\ \text{N} \cdot \text{m}$ is applied at A, determine the required diameter of (a) rod AB, (b) rod BC.



Under normal operating conditions, the electric motor exerts a 12-kip · in. torque at E. Knowing that each shaft is solid, determine the maximum shearing stress in (a) shaft BC, (b) shaft CD, (c) shaft DE.

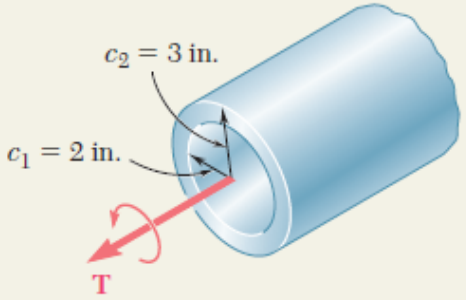


SAMPLE PROBLEM 3.2



The preliminary design of a large shaft connecting a motor to a generator calls for the use of a hollow shaft with inner and outer diameters of 4 in. and 6 in., respectively. Knowing that the allowable shearing stress is 12 ksi, determine the maximum torque that can be transmitted (*a*) by the shaft as designed, (*b*) by a solid shaft of the same weight, (*c*) by a hollow shaft of the same weight and of 8-in. outer diameter.

SOLUTION

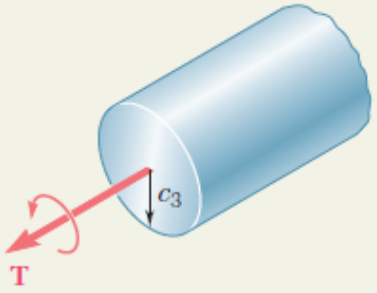


a. Hollow Shaft as Designed. For the hollow shaft we have

$$J = \frac{\pi}{2}(c_2^4 - c_1^4) = \frac{\pi}{2}[(3\text{ in.})^4 - (2\text{ in.})^4] = 102.1\text{ in}^4$$

Using Eq. (3.9), we write

$$\tau_{\max} = \frac{Tc_2}{J} \qquad 12\text{ ksi} = \frac{T(3\text{ in.})}{102.1\text{ in}^4} \qquad T = 408\text{ kip} \cdot \text{in.}$$

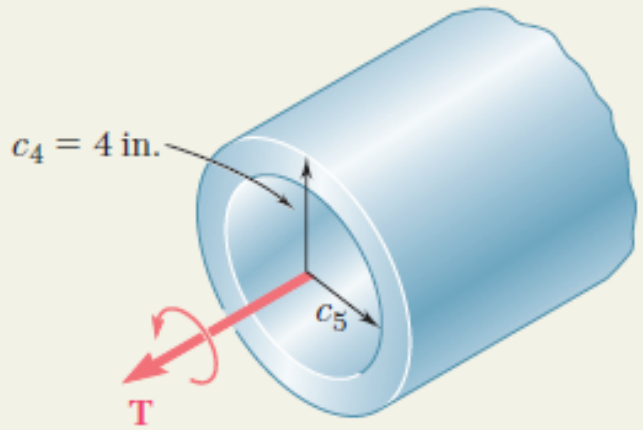


b. Solid Shaft of Equal Weight. For the shaft as designed and this solid shaft to have the same weight and length, their cross-sectional areas must be equal.

$$A_{(a)} = A_{(b)}$$
$$\pi[(3\text{ in.})^2 - (2\text{ in.})^2] = \pi c_3^2 \qquad c_3 = 2.24\text{ in.}$$

Since $\tau_{\text{all}} = 12$ ksi, we write

$$\tau_{\max} = \frac{Tc_3}{J} \qquad 12\text{ ksi} = \frac{T(2.24\text{ in.})}{\frac{\pi}{2}(2.24\text{ in.})^4} \qquad T = 211\text{ kip} \cdot \text{in.}$$



c. Hollow Shaft of 8-in. Diameter. For equal weight, the cross-sectional areas again must be equal. We determine the inside diameter of the shaft by writing

$$A_{(a)} = A_{(c)}$$

$$\pi[(3 \text{ in.})^2 - (2 \text{ in.})^2] = \pi[(4 \text{ in.})^2 - c_5^2] \qquad c_5 = 3.317 \text{ in.}$$

For $c_5 = 3.317$ in. and $c_4 = 4$ in.,

$$J = \frac{\pi}{2}[(4 \text{ in.})^4 - (3.317 \text{ in.})^4] = 212 \text{ in}^4$$

With $\tau_{all} = 12$ ksi and $c_4 = 4$ in.,

$$\tau_{max} = \frac{Tc_4}{J} \qquad 12 \text{ ksi} = \frac{T(4 \text{ in.})}{212 \text{ in}^4} \qquad T = 636 \text{ kip} \cdot \text{in.}$$

ANGLE OF TWIST IN THE ELASTIC RANGE

In this section, a relation will be derived between the angle of twist ϕ of a circular shaft and the torque **T** exerted on the shaft. The entire shaft will be assumed to remain elastic. Considering first the case of a shaft of length L and of uniform cross section of radius c subjected to a torque **T** at its free end we recall from Sec. 3.3 that the angle of twist ϕ and the maximum shearing strain γ_{max} are related as follows:

$$\gamma_{max} = \frac{c\phi}{L} \tag{3.3}$$

But, in the elastic range, the yield stress is not exceeded anywhere in the shaft, Hooke's law applies, and we have $\gamma_{max} = \tau_{max}/G$ or, recalling Eq. (3.9),

$$\gamma_{max} = \frac{\tau_{max}}{G} = \frac{Tc}{JG} \tag{3.15}$$

Equating the right-hand members of Eqs. (3.3) and (3.15), and solving for ϕ, we write

$$\phi = \frac{TL}{JG} \tag{3.16}$$

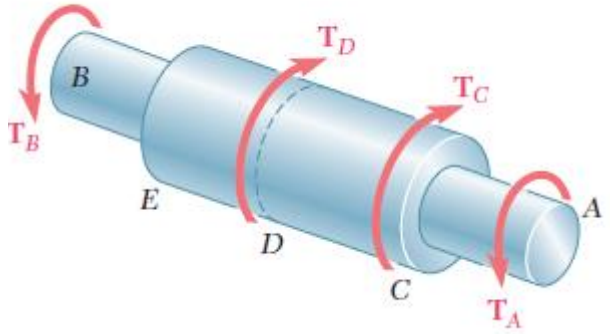
where ϕ is expressed in radians. The relation obtained shows that, within the elastic range, *the angle of twist ϕ is proportional to the torque T applied to the shaft*. This is in accordance with the experimental evidence cited at the beginning of Sec. 3.3.

Equation (3.16) provides us with a convenient method for determining the modulus of rigidity of a given material. A specimen of the material, in the form of a cylindrical rod of known diameter and length, is placed in a *torsion testing machine* (Photo 3.3). Torques of increasing magnitude T are applied to the specimen, and the corresponding values of the angle of twist ϕ in a length L of the specimen are recorded. As long as the yield stress of the material is not exceeded, the points obtained by plotting ϕ against T will fall on a straight line. The slope of this line represents the quantity JG/L, from which the modulus of rigidity G may be computed.



- If the torsional loading or shaft cross-section changes along the length, the angle of rotation is found as the sum of segment rotations

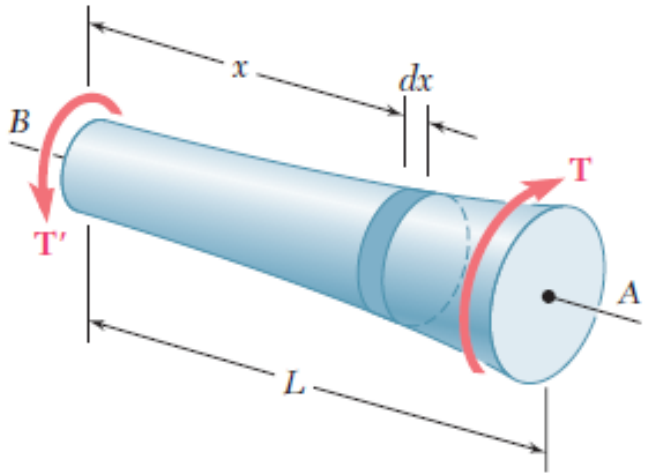
$$\phi = \sum_i \frac{T_i L_i}{J_i G_i}$$



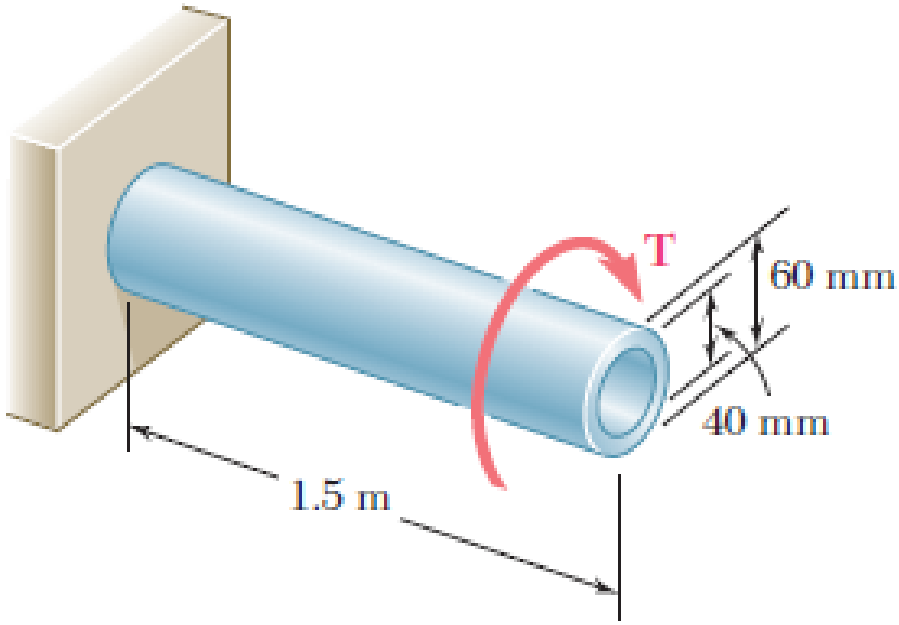
$$d\phi = \frac{T\,dx}{JG}$$

For Shaft with variable cross-section.

$$\phi = \int_0^L \frac{T\,dx}{JG}$$



What torque should be applied to the end of the shaft of Example 3.01 to produce a twist of 2°? Use the value $G = 77$ GPa for the modulus of rigidity of steel.



Solving Eq. (3.16) for T, we write

$$T = \frac{JG}{L}\phi$$

Substituting the given values

$$G = 77 \times 10^9 \text{ Pa} \qquad L = 1.5 \text{ m}$$

$$\phi = 2°\left(\frac{2\pi \text{ rad}}{360°}\right) = 34.9 \times 10^{-3} \text{ rad}$$

and recalling from Example 3.01 that, for the given cross section,

$$J = 1.021 \times 10^{-6} \text{ m}^4$$

we have

$$T = \frac{JG}{L}\phi = \frac{(1.021 \times 10^{-6} \text{ m}^4)(77 \times 10^9 \text{ Pa})}{1.5 \text{ m}}(34.9 \times 10^{-3} \text{ rad})$$

$$T = 1.829 \times 10^3 \text{ N} \cdot \text{m} = 1.829 \text{ kN} \cdot \text{m}$$

What angle of twist will create a shearing stress of 70 MPa on the inner surface of the hollow steel shaft of Examples 3.01 and 3.02?

The method of attack for solving this problem that first comes to mind is to use Eq. (3.10) to find the torque T corresponding to the given value of τ, and Eq. (3.16) to determine the angle of twist ϕ corresponding to the value of T just found.

A more direct solution, however, may be used. From Hooke's law, we first compute the shearing strain on the inner surface of the shaft:

$$\gamma_{\text{min}} = \frac{\tau_{\text{min}}}{G} = \frac{70 \times 10^6 \text{ Pa}}{77 \times 10^9 \text{ Pa}} = 909 \times 10^{-6}$$

Recalling Eq. (3.2), which was obtained by expressing the length of arc AA' in Fig. 3.13*c* in terms of both γ and ϕ, we have

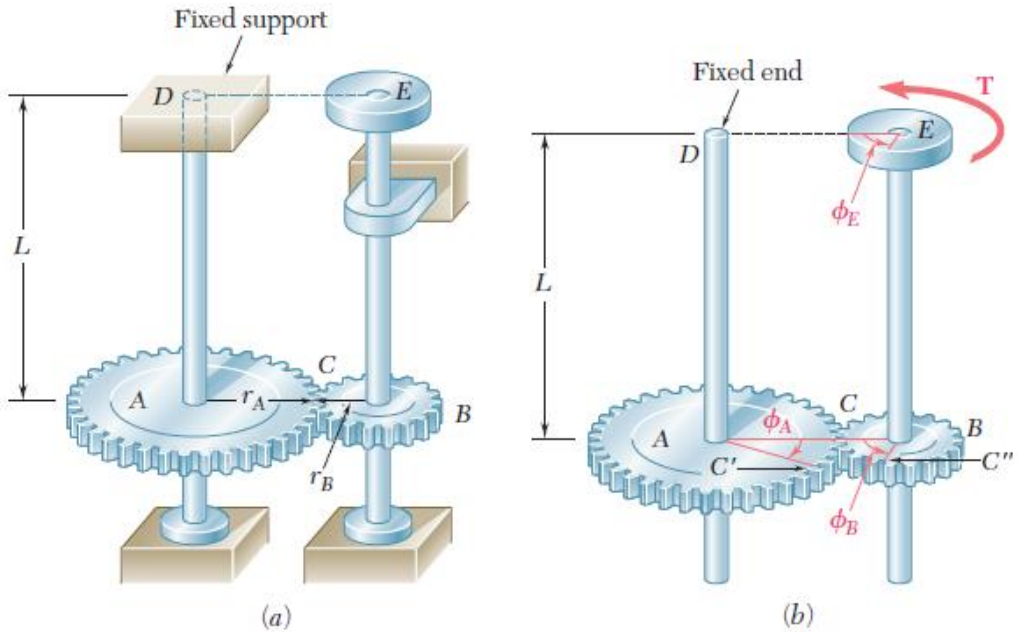
$$\phi = \frac{L\gamma_{\text{min}}}{c_1} = \frac{1500 \text{ mm}}{20 \text{ mm}}(909 \times 10^{-6}) = 68.2 \times 10^{-3} \text{ rad}$$

To obtain the angle of twist in degrees, we write

$$\phi = (68.2 \times 10^{-3} \text{ rad})\left(\frac{360^\circ}{2\pi \text{ rad}}\right) = 3.91^\circ$$

When both ends of a shaft rotate, however, the angle of twist of the shaft is equal to the angle through which one end of the shaft rotates *with respect to the other*. Consider, for instance, the assembly shown in Fig. 3.23*a*, consisting of two elastic shafts *AD* and *BE*, each of length *L*, radius *c*, and modulus of rigidity *G*, which are attached to gears meshed at *C*. If a torque **T** is applied at *E* (Fig. 3.23*b*), both shafts will be twisted. Since the end *D* of shaft *AD* is fixed, the angle of twist of *AD* is measured by the angle of rotation ϕ_A of end *A*. On the other hand, since both ends of shaft *BE* rotate, the angle of twist of *BE* is equal to the difference between the angles of rotation ϕ_B and ϕ_E, i.e., the angle of twist is equal to the angle through which end *E* rotates with respect to end *B*. Denoting this relative angle of rotation by $\phi_{E/B}$, we write

$$\phi_{E/B} = \phi_E - \phi_B = \frac{TL}{JG}$$



EXAMPLE 3.04

For the assembly of Fig. 3.23, knowing that $r_A = 2r_B$, determine the angle of rotation of end E of shaft BE when the torque $\mathbf{T}$ is applied at E.

We first determine the torque $\mathbf{T}_{AD}$ exerted on shaft AD. Observing that equal and opposite forces $\mathbf{F}$ and $\mathbf{F}'$ are applied on the two gears at C (Fig. 3.24), and recalling that $r_A = 2r_B$, we conclude that the torque exerted on shaft AD is twice as large as the torque exerted on shaft BE; thus, $T_{AD} = 2T$.

Since the end D of shaft AD is fixed, the angle of rotation ϕ_A of gear A is equal to the angle of twist of the shaft and is obtained by writing

$$\phi_A = \frac{T_{AD}L}{JG} = \frac{2TL}{JG}$$

Observing that the arcs CC' and CC'' in Fig. 3.26*b* must be equal, we write $r_A\phi_A = r_B\phi_B$ and obtain

$$\phi_B = (r_A/r_B)\phi_A = 2\phi_A$$

We have, therefore,

$$\phi_B = 2\phi_A = \frac{4TL}{JG}$$

Considering now shaft BE, we recall that the angle of twist of the shaft is equal to the angle $\phi_{E/B}$ through which end E rotates with respect to end B. We have

$$\phi_{E/B} = \frac{T_{BE}L}{JG} = \frac{TL}{JG}$$

The angle of rotation of end E is obtained by writing

$$\phi_E = \phi_B + \phi_{E/B}$$

$$= \frac{4TL}{JG} + \frac{TL}{JG} = \frac{5TL}{JG}$$

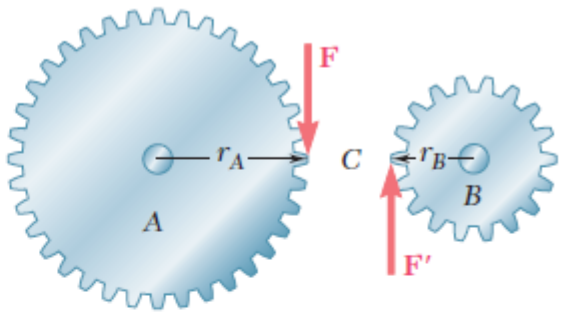


Fig. 3.24

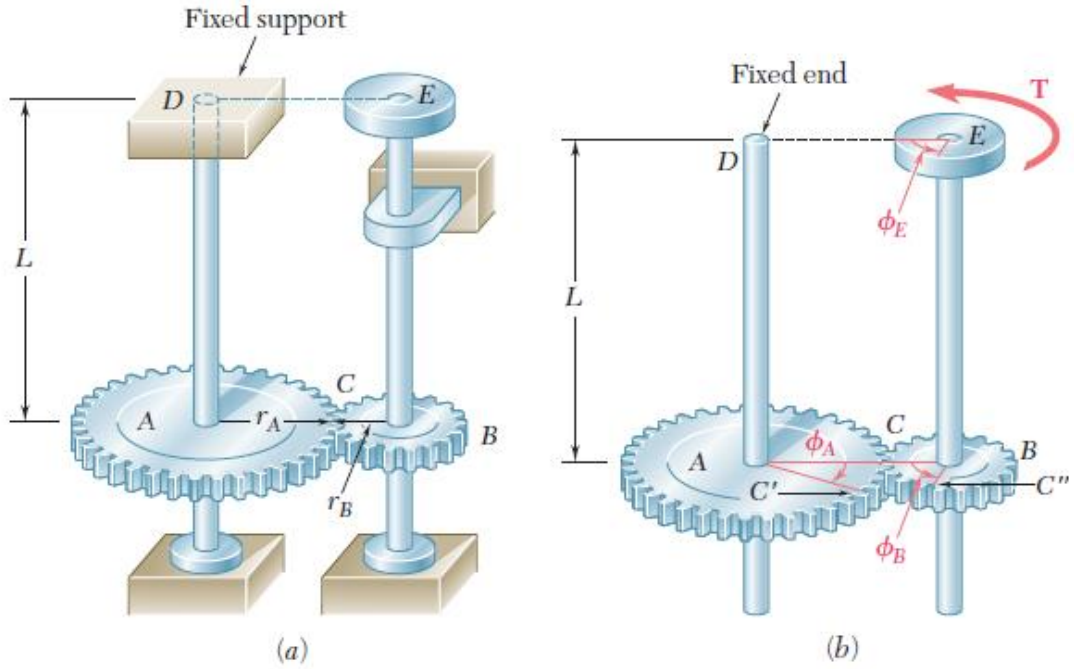


Fig. 3.23 Gear assembly.

EXAMPLE 3.05

A circular shaft AB consists of a 10-in.-long, $\frac{7}{8}$-in.-diameter steel cylinder, in which a 5-in.-long, $\frac{5}{8}$-in.-diameter cavity has been drilled from end B. The shaft is attached to fixed supports at both ends, and a 90 lb · ft torque is applied at its midsection (Fig. 3.25). Determine the torque exerted on the shaft by each of the supports.

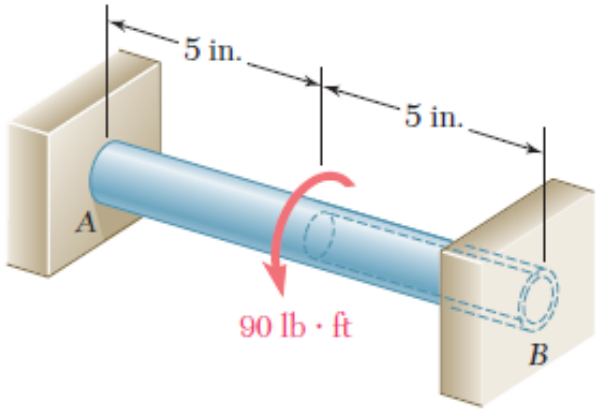


Fig. 3.25

Drawing the free-body diagram of the shaft and denoting by $\mathbf{T}_A$ and $\mathbf{T}_B$ the torques exerted by the supports (Fig. 3.26*a*), we obtain the equilibrium equation

$$T_A + T_B = 90 \text{ lb} \cdot \text{ft}$$

Since this equation is not sufficient to determine the two unknown torques $\mathbf{T}_A$ and $\mathbf{T}_B$, the shaft is statically indeterminate.

However, $\mathbf{T}_A$ and $\mathbf{T}_B$ can be determined if we observe that the total angle of twist of shaft AB must be zero, since both of its ends are restrained. Denoting by ϕ_1 and ϕ_2, respectively, the angles of twist of portions AC and CB, we write

$$\phi = \phi_1 + \phi_2 = 0$$

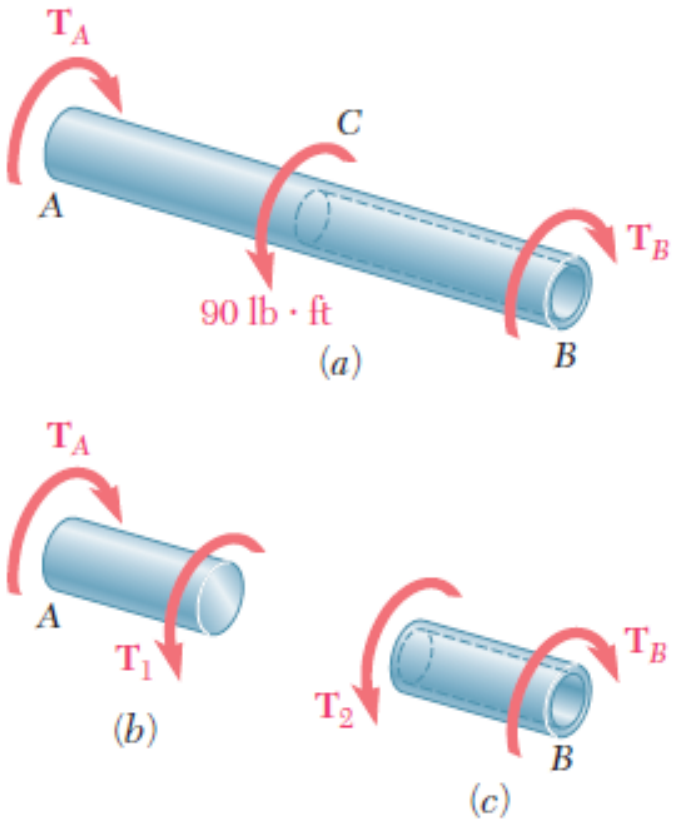


Fig. 3.26

From the free-body diagram of a small portion of shaft including end A (Fig. 3.26*b*), we note that the internal torque T_1 in AC is equal to T_A; from the free-body diagram of a small portion of shaft including end B (Fig. 3.26*c*), we note that the internal torque T_2 in CB is equal to T_B. Recalling Eq. (3.16) and observing that portions AC and CB of the shaft are twisted in opposite senses, we write

$$\phi = \phi_1 + \phi_2 = \frac{T_A L_1}{J_1 G} - \frac{T_B L_2}{J_2 G} = 0$$

Solving for T_B, we have

$$T_B = \frac{L_1 J_2}{L_2 J_1} T_A$$

Substituting the numerical data gives

$$L_1 = L_2 = 5 \text{ in.}$$
$$J_1 = \tfrac{1}{2}\pi(\tfrac{7}{16} \text{ in.})^4 = 57.6 \times 10^{-3} \text{ in}^4$$
$$J_2 = \tfrac{1}{2}\pi[(\tfrac{7}{16} \text{ in.})^4 - (\tfrac{5}{16} \text{ in.})^4] = 42.6 \times 10^{-3} \text{ in}^4$$

we obtain

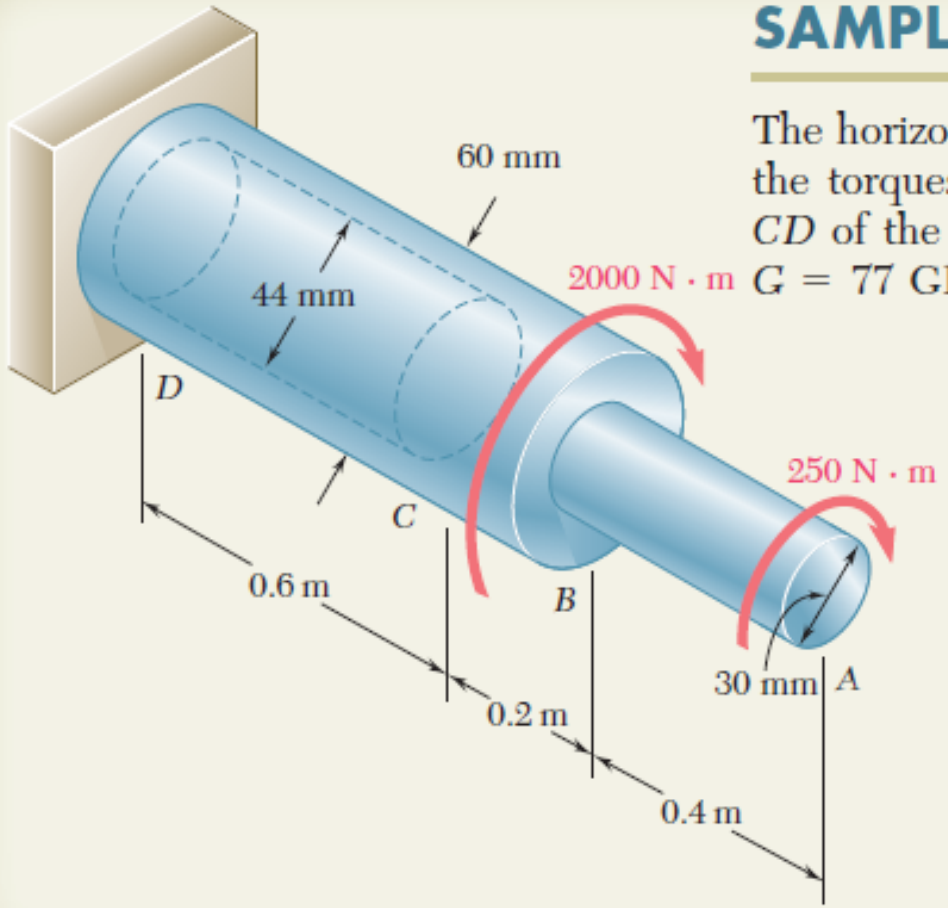
$$T_B = 0.740\ T_A$$

Substituting this expression into the original equilibrium equation, we write

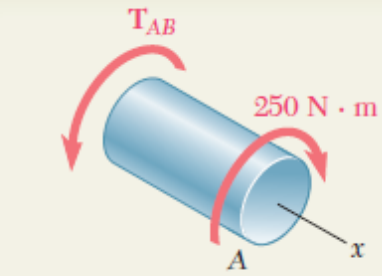
$$1.740\ T_A = 90 \text{ lb} \cdot \text{ft}$$

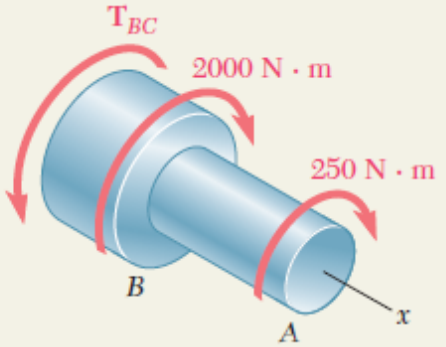
$$T_A = 51.7 \text{ lb} \cdot \text{ft} \qquad T_B = 38.3 \text{ lb} \cdot \text{ft}$$

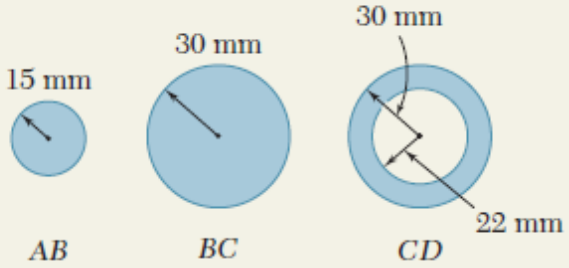
SAMPLE PROBLEM 3.3

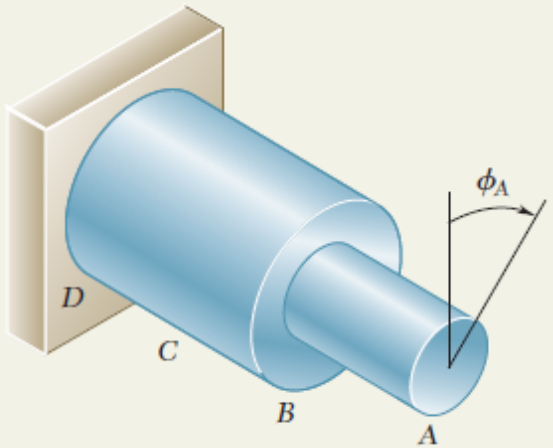


The horizontal shaft AD is attached to a fixed base at D and is subjected to the torques shown. A 44-mm-diameter hole has been drilled into portion CD of the shaft. Knowing that the entire shaft is made of steel for which $G = 77$ GPa, determine the angle of twist at end A.









SOLUTION

Since the shaft consists of three portions *AB*, *BC*, and *CD*, each of uniform cross section and each with a constant internal torque, Eq. (3.17) may be used.

Statics. Passing a section through the shaft between *A* and *B* and using the free body shown, we find

$$\Sigma M_x = 0: \qquad (250 \text{ N} \cdot \text{m}) - T_{AB} = 0 \qquad T_{AB} = 250 \text{ N} \cdot \text{m}$$

Passing now a section between *B* and *C*, we have

$$\Sigma M_x = 0: (250 \text{ N} \cdot \text{m}) + (2000 \text{ N} \cdot \text{m}) - T_{BC} = 0 \qquad T_{BC} = 2250 \text{ N} \cdot \text{m}$$

Since no torque is applied at *C*,

$$T_{CD} = T_{BC} = 2250 \text{ N} \cdot \text{m}$$

Polar Moments of Inertia

$$J_{AB} = \frac{\pi}{2} c^4 = \frac{\pi}{2} (0.015 \text{ m})^4 = 0.0795 \times 10^{-6} \text{ m}^4$$

$$J_{BC} = \frac{\pi}{2} c^4 = \frac{\pi}{2} (0.030 \text{ m})^4 = 1.272 \times 10^{-6} \text{ m}^4$$

$$J_{CD} = \frac{\pi}{2} (c_2^4 - c_1^4) = \frac{\pi}{2} [(0.030 \text{ m})^4 - (0.022 \text{ m})^4] = 0.904 \times 10^{-6} \text{ m}^4$$

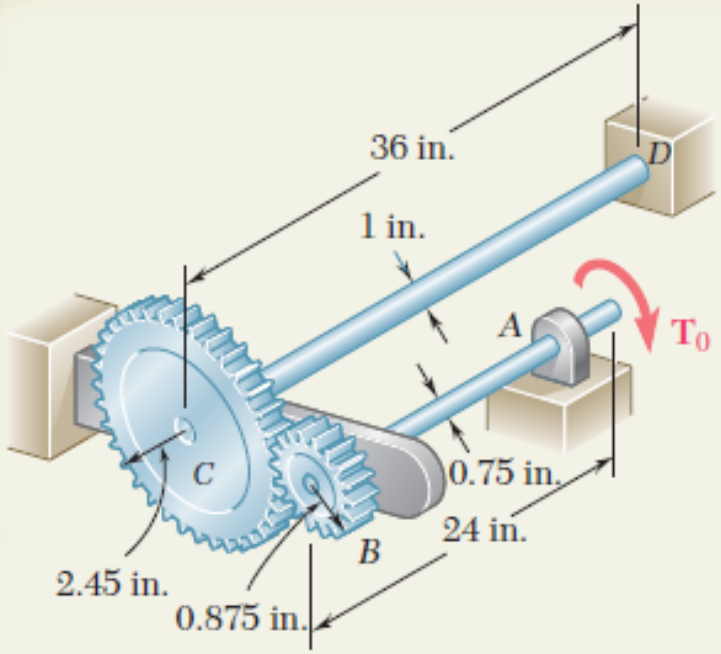
Angle of Twist. Using Eq. (3.17) and recalling that $G = 77$ GPa for the entire shaft, we have

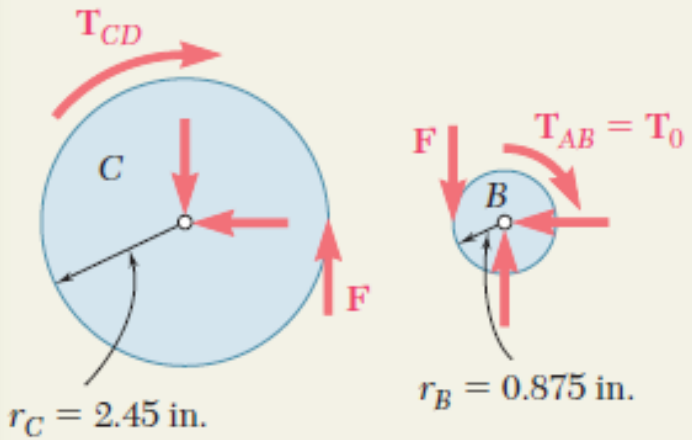
$$\phi_A = \sum_i \frac{T_i L_i}{J_i G} = \frac{1}{G} \left(\frac{T_{AB} L_{AB}}{J_{AB}} + \frac{T_{BC} L_{BC}}{J_{BC}} + \frac{T_{CD} L_{CD}}{J_{CD}} \right)$$

$$\phi_A = \frac{1}{77 \text{ GPa}} \left[\frac{(250 \text{ N} \cdot \text{m})(0.4 \text{ m})}{0.0795 \times 10^{-6} \text{ m}^4} + \frac{(2250)(0.2)}{1.272 \times 10^{-6}} + \frac{(2250)(0.6)}{0.904 \times 10^{-6}} \right]$$

$$= 0.01634 + 0.00459 + 0.01939 = 0.0403 \text{ rad}$$

$$\phi_A = (0.0403 \text{ rad}) \frac{360^\circ}{2\pi \text{ rad}} \qquad \phi_A = 2.31^\circ \blacktriangleleft$$





SAMPLE PROBLEM 3.4

Two solid steel shafts are connected by the gears shown. Knowing that for each shaft $G = 11.2 \times 10^6$ psi and that the allowable shearing stress is 8 ksi, determine (*a*) the largest torque $\mathbf{T}_0$ that may be applied to end *A* of shaft *AB*, (*b*) the corresponding angle through which end *A* of shaft *AB* rotates.

SOLUTION

Statics. Denoting by *F* the magnitude of the tangential force between gear teeth, we have

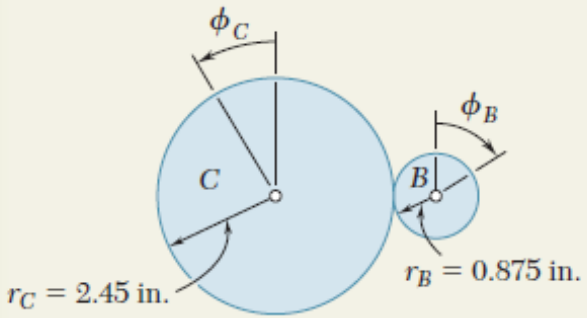
Gear B. $\Sigma M_B = 0$: $F(0.875 \text{ in.}) - T_0 = 0$

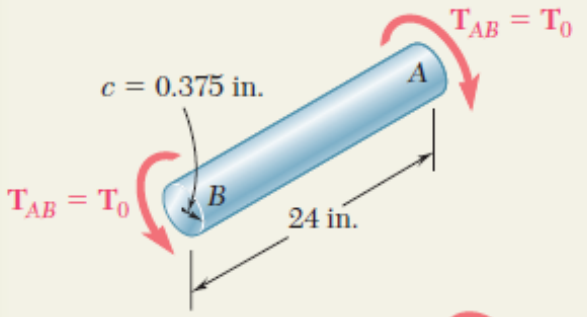
Gear C. $\Sigma M_C = 0$: $F(2.45 \text{ in.}) - T_{CD} = 0$

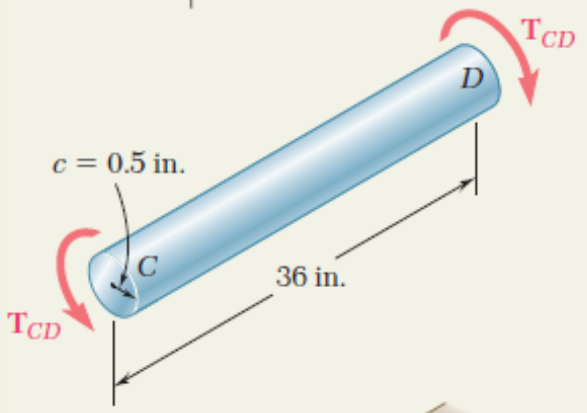
$$T_{CD} = 2.8T_0 \qquad (1)$$

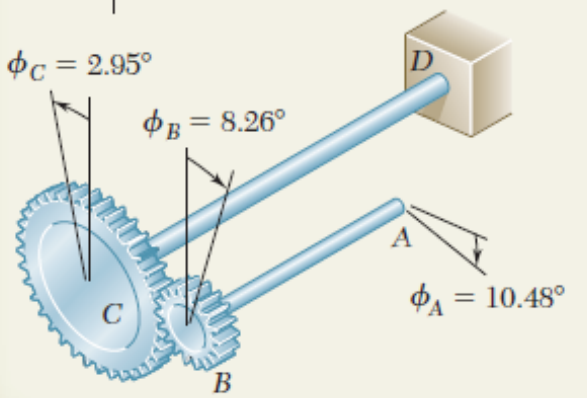
Kinematics. Noting that the peripheral motions of the gears are equal, we write

$$r_B\phi_B = r_C\phi_C \qquad \phi_B = \phi_C\frac{r_C}{r_B} = \phi_C\frac{2.45 \text{ in.}}{0.875 \text{ in.}} = 2.8\phi_C \qquad (2)$$









a. Torque T_0

Shaft AB. With $T_{AB} = T_0$ and $c = 0.375$ in., together with a maximum permissible shearing stress of 8000 psi, we write

$$\tau = \frac{T_{AB}c}{J} \qquad 8000 \text{ psi} = \frac{T_0(0.375 \text{ in.})}{\frac{1}{2}\pi(0.375 \text{ in.})^4} \qquad T_0 = 663 \text{ lb} \cdot \text{in.}$$

Shaft CD. From (1) we have $T_{CD} = 2.8T_0$. With $c = 0.5$ in. and $\tau_{all} = 8000$ psi, we write

$$\tau = \frac{T_{CD}c}{J} \qquad 8000 \text{ psi} = \frac{2.8T_0(0.5 \text{ in.})}{\frac{1}{2}\pi(0.5 \text{ in.})^4} \qquad T_0 = 561 \text{ lb} \cdot \text{in.}$$

Maximum Permissible Torque. We choose the smaller value obtained for T_0

$$T_0 = 561 \text{ lb} \cdot \text{in.}$$

b. Angle of Rotation at End A.

We first compute the angle of twist for each shaft.

Shaft AB. For $T_{AB} = T_0 = 561$ lb · in., we have

$$\phi_{A/B} = \frac{T_{AB}L}{JG} = \frac{(561 \text{ lb} \cdot \text{in.})(24 \text{ in.})}{\frac{1}{2}\pi(0.375 \text{ in.})^4(11.2 \times 10^6 \text{ psi})} = 0.0387 \text{ rad} = 2.22°$$

Shaft CD. $T_{CD} = 2.8T_0 = 2.8(561 \text{ lb} \cdot \text{in.})$

$$\phi_{C/D} = \frac{T_{CD}L}{JG} = \frac{2.8(561 \text{ lb} \cdot \text{in.})(36 \text{ in.})}{\frac{1}{2}\pi(0.5 \text{ in.})^4(11.2 \times 10^6 \text{ psi})} = 0.0514 \text{ rad} = 2.95°$$

Since end D of shaft CD is fixed, we have $\phi_C = \phi_{C/D} = 2.95°$. Using (2), we find the angle of rotation of gear B to be

$$\phi_B = 2.8\phi_C = 2.8(2.95°) = 8.26°$$

For end A of shaft AB, we have

$$\phi_A = \phi_B + \phi_{A/B} = 8.26° + 2.22° \qquad \phi_A = 10.48°$$

DESIGN OF TRANSMISSION SHAFTS

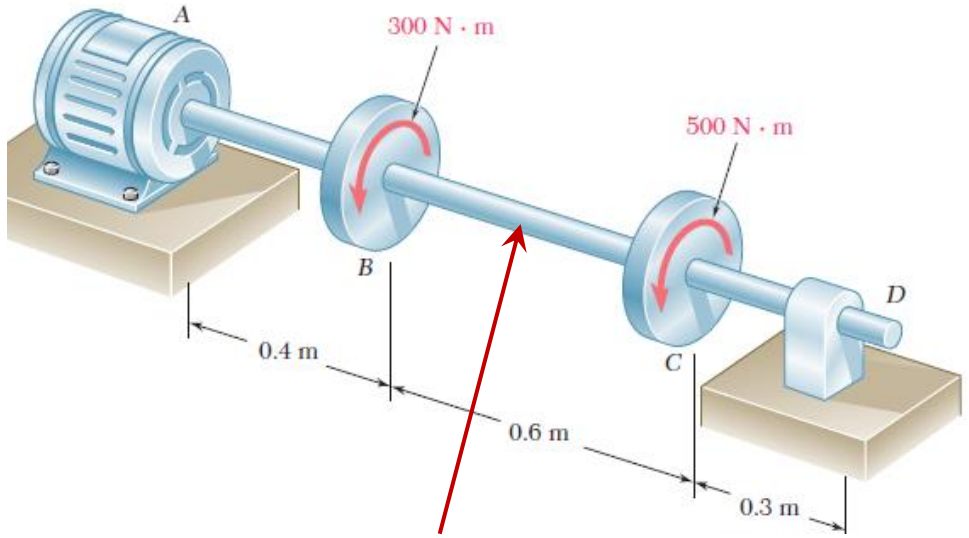
The principal specifications to be met in the design of a transmission shaft are the *power* to be transmitted and the *speed of rotation* of the shaft. The role of the designer is to select the material and the dimensions of the cross section of the shaft, so that the maximum shearing stress allowable in the material will not be exceeded when the shaft is transmitting the required power at the specified speed.

To determine the torque exerted on the shaft, we recall from elementary dynamics that the power P associated with the rotation of a rigid body subjected to a torque $\mathbf{T}$ is

$$P = T\omega \tag{3.19}$$

where ω is the angular velocity of the body expressed in radians per second. But $\omega = 2\pi f$, where f is the frequency of the rotation, i.e., the number of revolutions per second. The unit of frequency is thus $1\ s^{-1}$ and is called a *hertz* (Hz). Substituting for ω into Eq. (3.19), we write

$$P = 2\pi f T \tag{3.20}$$



Transmission Shaft

If SI units are used we verify that, with f expressed in Hz and T in N · m, the power will be expressed in N · m/s, that is, in *watts* (W). Solving Eq. (3.20) for T, we obtain the torque exerted on a shaft transmitting the power P at a frequency of rotation f,

$$T = \frac{P}{2\pi f} \tag{3.21}$$

where P, f, and T are expressed in the units indicated above.

After having determined the torque **T** that will be applied to the shaft and having selected the material to be used, the designer will carry the values of T and of the maximum allowable stress into the elastic torsion formula (3.9). Solving for J/c, we have

$$\frac{J}{c} = \frac{T}{\tau_{\max}} \tag{3.22}$$

and obtain in this way the minimum value allowable for the parameter J/c. We check that, if SI units are used, T will be expressed in N · m, $\tau_{\max}$ in Pa (or N/m^2), and J/c will be obtained in m^3. In the case of a solid circular shaft, $J = \frac{1}{2}\pi c^4$, and $J/c = \frac{1}{2}\pi c^3$; substituting this value for J/c into Eq. (3.22) and solving for c yields the minimum allowable value for the radius of the shaft. In the case of a hollow circular shaft, the critical parameter is J/c_2, where c_2 is the outer radius of the shaft; the value of this parameter may be computed from Eq. (3.11) of Sec. 3.4 to determine whether a given cross section will be acceptable.

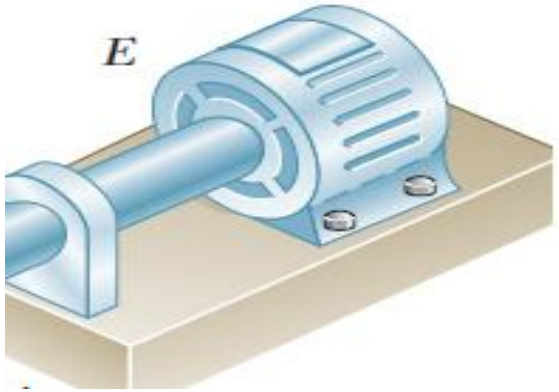
When U.S. customary units are used, the frequency is usually expressed in rpm and the power in horsepower (hp). It is then necessary, before applying formula (3.21), to convert the frequency into revolutions per second (i.e., hertzes) and the power into ft · lb/s or in · lb/s through the use of the following relations:

$$1 \text{ rpm} = \frac{1}{60} \text{ s}^{-1} = \frac{1}{60} \text{Hz}$$

$$1 \text{ hp} = 550 \text{ ft} \cdot \text{lb/s} = 6600 \text{ in} \cdot \text{lb/s}$$

EXAMPLE 3.06

What size of shaft should be used for the rotor of a 5-hp motor operating at 3600 rpm if the shearing stress is not to exceed 8500 psi in the shaft?



We first express the power of the motor in in · lb/s and its frequency in cycles per second (or hertzes).

$$P = (5\text{ hp})\left(\frac{6600\text{ in} \cdot \text{lb/s}}{1\text{ hp}}\right) = 33{,}000\text{ in} \cdot \text{lb/s}$$

$$f = (3600\text{ rpm})\frac{1\text{ Hz}}{60\text{ rpm}} = 60\text{ Hz} = 60\text{ s}^{-1}$$

The torque exerted on the shaft is given by Eq. (3.21):

$$T = \frac{P}{2\pi f} = \frac{33{,}000\text{ in} \cdot \text{lb/s}}{2\pi\,(60\text{ s}^{-1})} = 87.54\text{ lb} \cdot \text{in.}$$

Substituting for T and τ_{max} into Eq. (3.22), we write

$$\frac{J}{c} = \frac{T}{\tau_{max}} = \frac{87.54\text{ lb} \cdot \text{in.}}{8500\text{ psi}} = 10.30 \times 10^{-3}\text{ in}^3$$

But $J/c = \frac{1}{2}\pi c^3$ for a solid shaft. We have, therefore,

$$\tfrac{1}{2}\pi c^3 = 10.30 \times 10^{-3}\text{ in}^3$$

$$c = 0.1872\text{ in.}$$

$$d = 2c = 0.374\text{ in.}$$

A $\frac{3}{8}$-in. shaft should be used.

EXAMPLE 3.07

A shaft consisting of a steel tube of 50-mm outer diameter is to transmit 100 kW of power while rotating at a frequency of 20 Hz. Determine the tube thickness that should be used if the shearing stress is not to exceed 60 MPa.

The torque exerted on the shaft is given by Eq. (3.21):

$$T = \frac{P}{2\pi f} = \frac{100 \times 10^3 \text{ W}}{2\pi (20 \text{ Hz})} = 795.8 \text{ N} \cdot \text{m}$$

From Eq. (3.22) we conclude that the parameter J/c_2 must be at least equal to

$$\frac{J}{c_2} = \frac{T}{\tau_{\max}} = \frac{795.8 \text{ N} \cdot \text{m}}{60 \times 10^6 \text{ N/m}^2} = 13.26 \times 10^{-6} \text{ m}^3 \qquad (3.23)$$

But, from Eq. (3.10) we have

$$\frac{J}{c_2} = \frac{\pi}{2c_2}(c_2^4 - c_1^4) = \frac{\pi}{0.050}[(0.025)^4 - c_1^4] \qquad (3.24)$$

Equating the right-hand members of Eqs. (3.23) and (3.24), we obtain:

$$(0.025)^4 - c_1^4 = \frac{0.050}{\pi}(13.26 \times 10^{-6})$$

$$c_1^4 = 390.6 \times 10^{-9} - 211.0 \times 10^{-9} = 179.6 \times 10^{-9} \text{ m}^4$$

$$c_1 = 20.6 \times 10^{-3} \text{ m} = 20.6 \text{ mm}$$

The corresponding tube thickness is

$$c_2 - c_1 = 25 \text{ mm} - 20.6 \text{ mm} = 4.4 \text{ mm}$$

A tube thickness of 5 mm should be used.