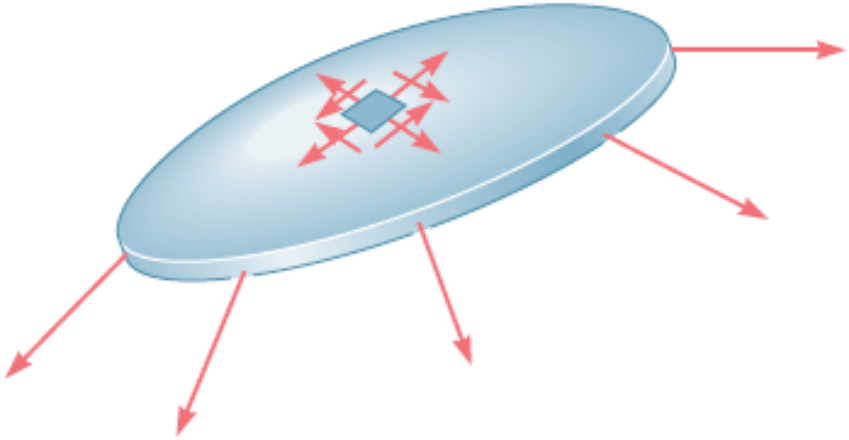
EWCE-211
Stress in thin wall pressure vessel and Beam Deflection Analysis

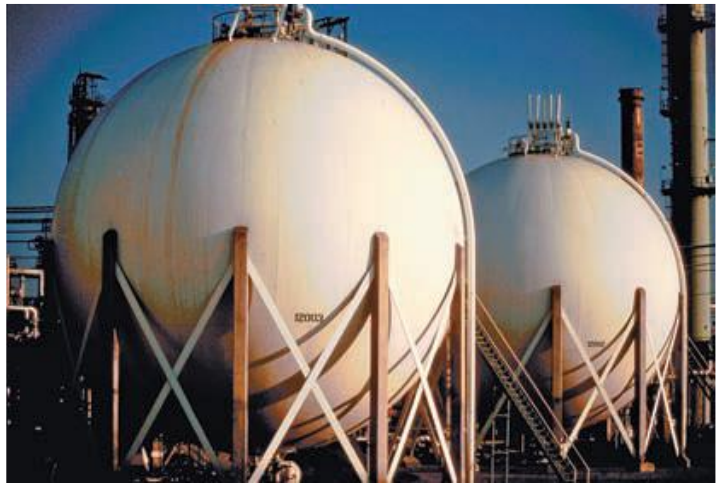
Dr. Md Saiful Islam

STRESSES IN THIN-WALLED PRESSURE VESSELS

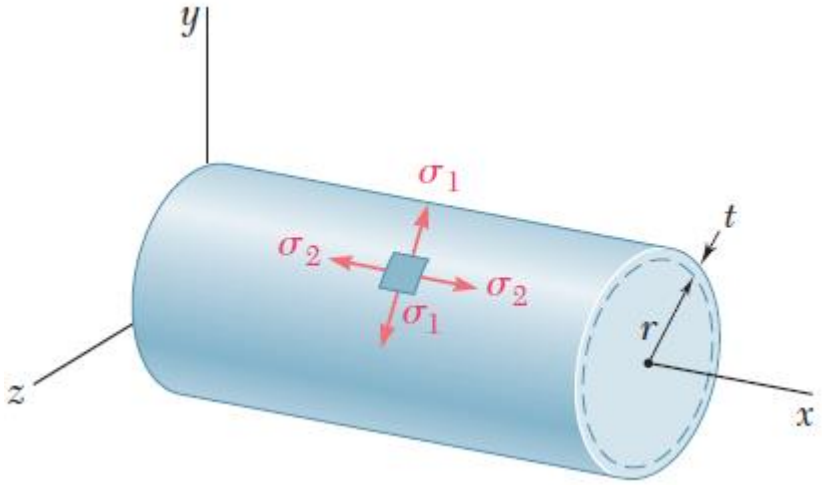
- Thin-walled pressure vessels provide an important application of the analysis of plane stress. Since their walls offer little resistance to bending, it can be assumed that the internal forces exerted on a given portion of wall are tangent to the surface of the vessel (Fig).

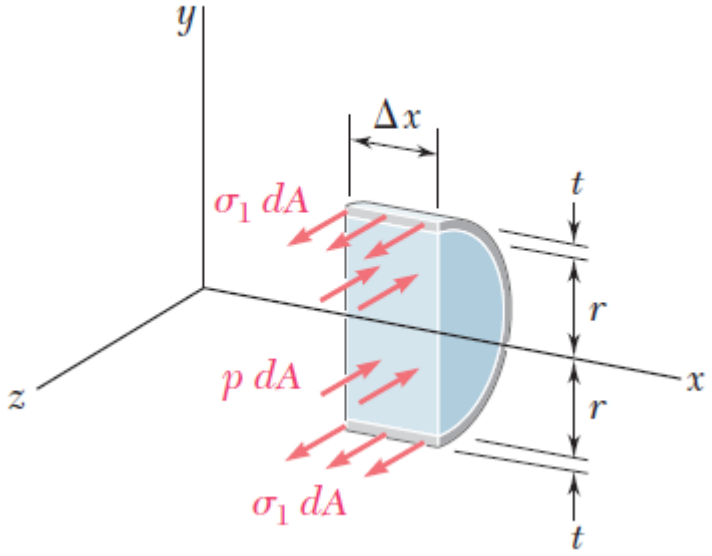




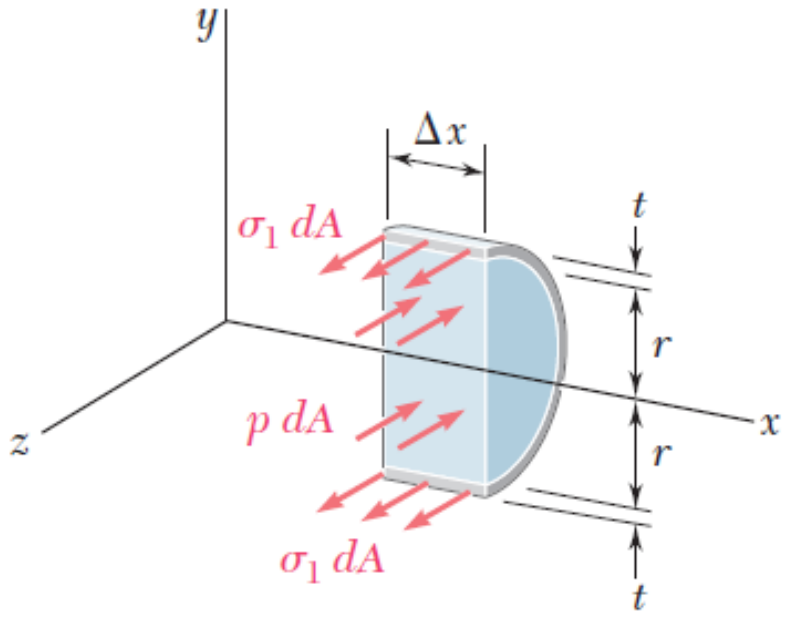


- Consider a cylindrical vessel of inner radius r and wall thickness t containing a fluid under pressure (Fig.). We propose to determine the stresses exerted on a small element of wall with sides respectively parallel and perpendicular to the axis of the cylinder.
- Because of the axisymmetry of the vessel and its contents, it is clear that no shearing stress is exerted on the element. The normal stresses σ_1 and σ_2 shown in Fig. are therefore principal stresses. The stress σ_1 is known as the hoop stress, because it is the type of stress found in hoops used to hold together the various slats of a wooden barrel, and the stress σ_2 is called the longitudinal stress.





- In order to determine the hoop stress s1, we detach a portion of the vessel and its contents bounded by the *xy* plane and by two planes parallel to the *yz* plane at a distance D*x* from each other (Fig.). The forces parallel to the *z* axis acting on the free body defined in this fashion consist of the elementary internal forces $\sigma_1 \, dA$ on the wall sections, and of the elementary pressure forces *p dA* exerted on the portion of fluid included in the free body. Note that *p* denotes the *gage pressure* of the fluid, i.e., the excess of the inside pressure over the outside atmospheric pressure. The resultant of the internal forces $\sigma_1 \, dA$ is equal to the product of σ_1 and of the cross-sectional area $2t \, \Delta x$ of the wall, while the resultant of the pressure forces *p dA* is equal to the product of *p* and of the area 2*r* D*x*. Writing the equilibrium equation Σ*Fz* = 0, we have

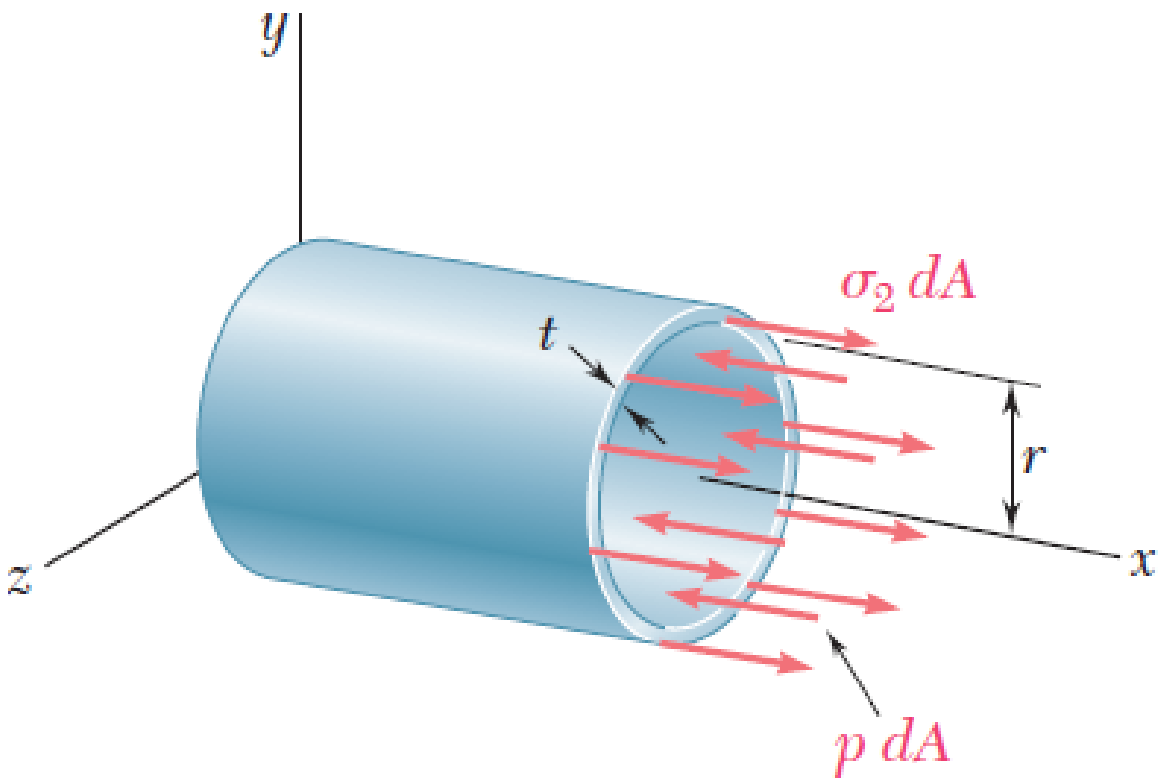


$$\Sigma F_z = 0: \qquad \sigma_1(2t\ \Delta x) - p(2r\ \Delta x) = 0$$

and, solving for the hoop stress σ_1,

$$\sigma_1 = \frac{pr}{t}$$

To determine the longitudinal stress σ_2, we now pass a section perpendicular to the x axis and consider the free body consisting of the portion of the vessel and its contents located to the left of the section

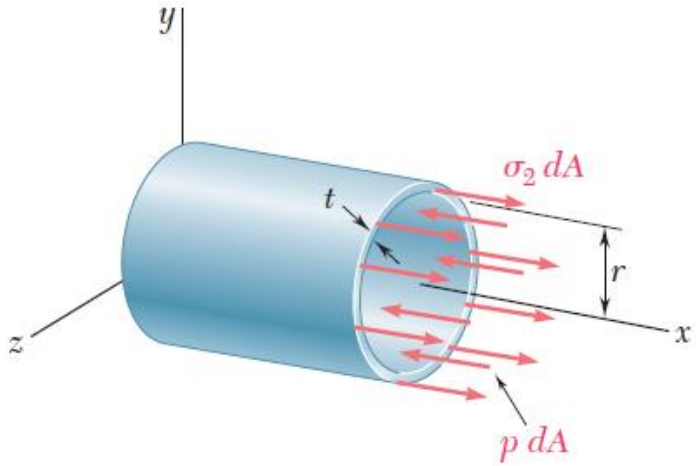


The forces acting on this free body are the elementary internal forces $\sigma_2 dA$ on the wall section and the elementary pressure forces $p\,dA$ exerted on the portion of fluid included in the free body. Noting that the area of the fluid section is πr^2 and that the area of the wall section can be obtained by multiplying the circumference $2\pi r$ of the cylinder by its wall thickness t, we write the equilibrium equation:†

$$\Sigma F_x = 0: \qquad \sigma_2(2\pi rt) - p(\pi r^2) = 0$$

and, solving for the longitudinal stress σ_2,

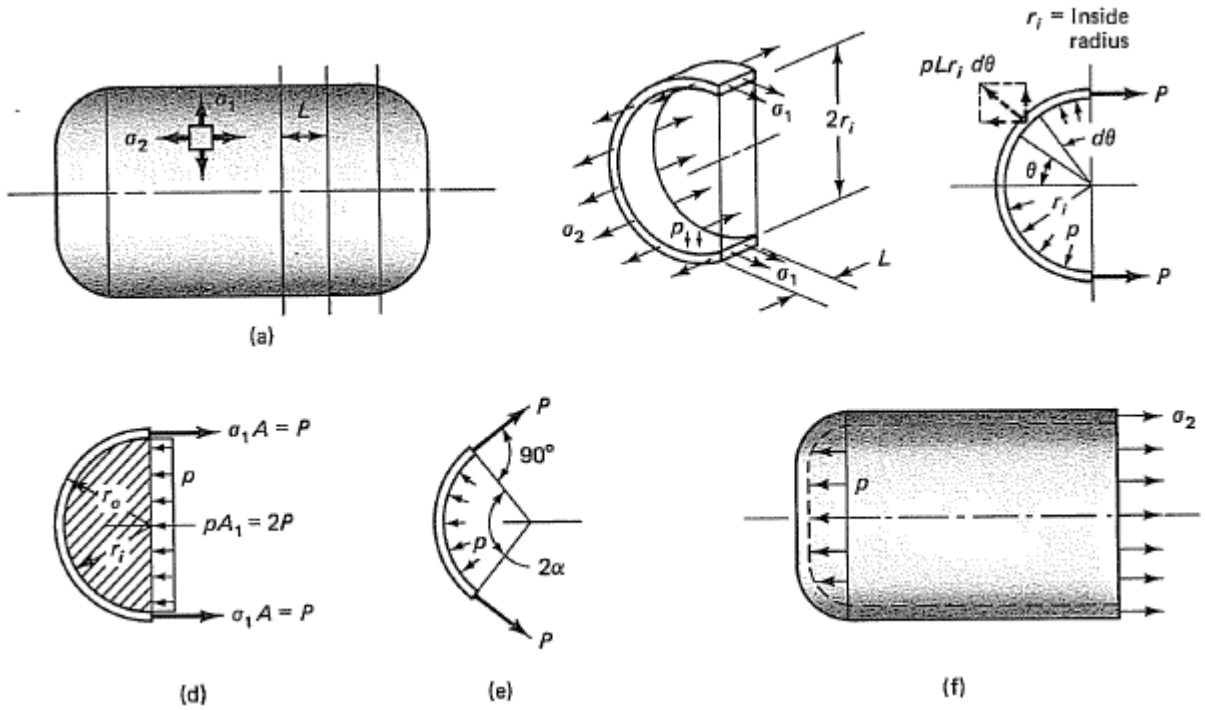
$$\sigma_2 = \frac{pr}{2t}$$



It can be observed from the previous two Eqs. that the hoop stress σ_1 is twice as large as the longitudinal stress σ_2

$$\sigma_1 = 2\sigma_2$$

Consider a closed cylindrical steel pressure vessel, as shown in Fig. The radius of the cylinder is 1000 mm and its wall thickness is 10 mm. (a) Determine the hoop and the longitudinal stresses in the cylindrical wall caused by an internal pressure of 0.80 MPa. (b) Calculate the change in diameter of the cylinder caused by pressurization. Let $E = 200$ GPa, and $\nu = 0.25$. Assume that $r_i \approx r_o \approx r$.



Solution

The stresses follow by direct application of Eqs.

$$\sigma_1 = \frac{pr}{t} = \frac{0.8 \times 1}{10 \times 10^{-3}} = 80 \text{ MPa}$$

and

$$\sigma_2 = \frac{pr}{2t} = \frac{0.8 \times 1}{2 \times 10 \times 10^{-3}} = 40 \text{ MPa}$$

The stress perpendicular to the cylinder wall, $\sigma_3 = p = 0.80$ MPa, on the inside decreases to zero on the outside. Being small, it can be neglected. Hence, on setting $\sigma_x = \sigma_1$, $\sigma_y = \sigma_2$, and $\sigma_z = 0$ in the first expression in Eq. 3-14, one obtains the hoop strain ε_1:

$$\varepsilon_1 = \frac{\sigma_1}{E} - \nu\frac{\sigma_2}{E} = \frac{80}{200 \times 10^3} - \frac{40}{4 \times 200 \times 10^3} = 0.35 \times 10^{-3} \text{ mm/mm}$$

On pressurizing the cylinder, the radius r increases by an amount Δ. For this condition, the hoop strain ε_1 can be found by calculating the difference in the strained and the unstrained hoop circumferences and dividing this quantity by the initial hoop length. Therefore,

$$\varepsilon_1 = \frac{2\pi(r + \Delta) - 2\pi r}{2\pi r} = \frac{\Delta}{r}$$

By recasting this expression and substituting the numerical value for ε_1 found earlier,

$$\Delta = \varepsilon_1 r = 0.35 \times 10^{-3} \times 10^3 = 0.35 \text{ mm}$$

We now consider a spherical vessel of inner radius r and wall thickness t, containing a fluid under a gage pressure p. For reasons of symmetry, the stresses exerted on the four faces of a small element of wall must be equal (Fig. 7.51). We have

$$\sigma_1 = \sigma_2 \tag{7.35}$$

To determine the value of the stress, we pass a section through the center C of the vessel and consider the free body consisting of the portion of the vessel and its contents located to the left of the section (Fig. 7.52). The equation of equilibrium for this free body is the same as for the free body of Fig. 7.49. We thus conclude that, for a spherical vessel,

$$\sigma_1 = \sigma_2 = \frac{pr}{2t} \tag{7.36}$$

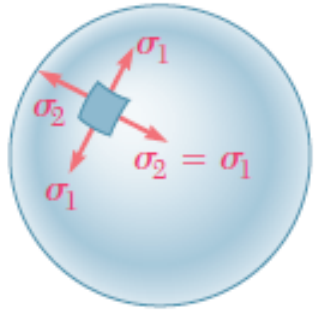


Fig. 7.51 Pressurized spherical vessel.

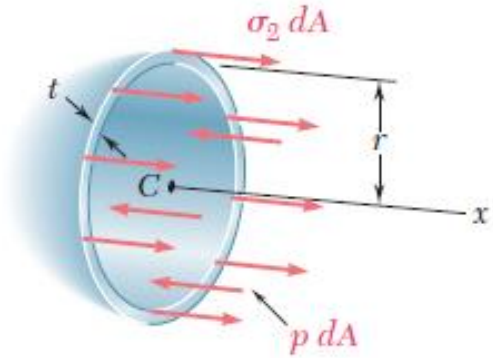


Fig. 7.52 Free body to determine wall stress.

Consider a steel spherical pressure vessel of radius 1000 mm having a wall thickness of 10 mm. (a) Determine the maximum membrane stresses caused by an internal pressure of 0.80 MPa. (b) Calculate the change in diameter in the sphere caused by pressurization. Let $E = 200$ GPa, and $\nu = 0.25$. Assume that $r_i \approx r_o \approx r$.

Solution

The maximum membrane normal stresses follow directly from Eq.

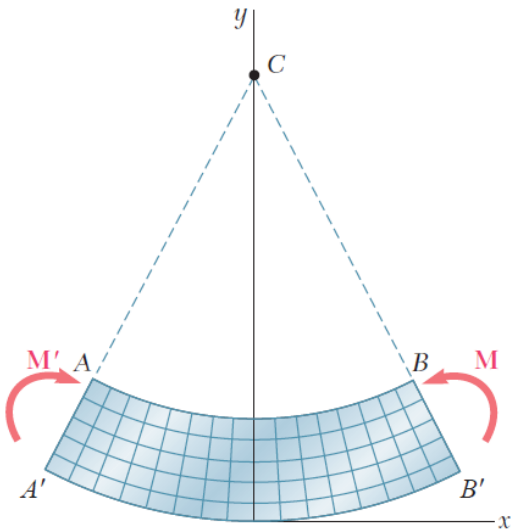
$$\sigma_1 = \sigma_2 = \frac{pr}{2t} = \frac{0.80 \times 1}{2 \times 10 \times 10^{-3}} = 40 \text{ MPa}$$

The same procedure as in the previous example can be used for finding the expansion of the sphere due to pressurization. Hence, if Δ is the increase in the radius r due to this cause, $\Delta = \varepsilon_1 r$, where ε_1 is the membrane strain on the great circle. However, from the first expression in Eq. 3-14, one has

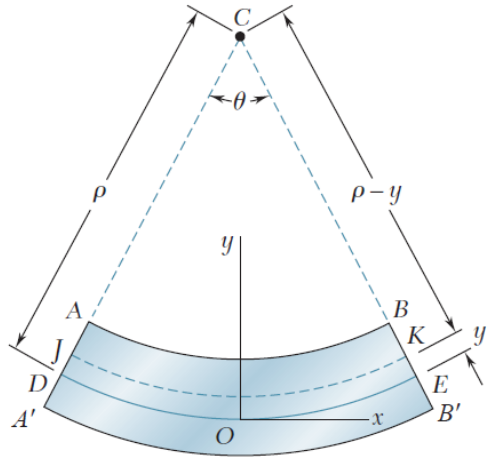
$$\varepsilon_1 = \frac{\sigma_1}{E} - \nu\frac{\sigma_2}{E} = \frac{40}{200 \times 10^3} - \frac{40}{4 \times 200 \times 10^3} = 0.15 \times 10^{-3} \text{ mm/mm}$$

$$\Delta = \varepsilon_1 r = 0.15 \times 10^{-3} \times 10^3 = 0.15 \text{ mm}$$

Beam Deflection Analysis



(*a*) Longitudinal, vertical section (plane of symmetry)



(*a*) Longitudinal, vertical section (plane of symmetry)

Denoting by ρ the radius of arc DE (Fig. 4.10*a*), by θ the central angle corresponding to DE, and observing that the length of DE is equal to the length L of the undeformed member, we write

$$L = \rho\theta \tag{4.4}$$

Considering now the arc JK located at a distance y above the neutral surface, we note that its length L' is

$$L' = (\rho - y)\theta \tag{4.5}$$

Since the original length of arc JK was equal to L, the deformation of JK is

$$\delta = L' - L \tag{4.6}$$

or, if we substitute from (4.4) and (4.5) into (4.6),

$$\delta = (\rho - y)\theta - \rho\theta = -y\theta \tag{4.7}$$

The longitudinal strain ϵ_x in the elements of JK is obtained by dividing δ by the original length L of JK. We write

$$\epsilon_x = \frac{\delta}{L} = \frac{-y\theta}{\rho\theta}$$

or

$$\epsilon_x = -\frac{y}{\rho} \tag{4.8}$$

The strain ϵ_x reaches its maximum absolute value when y itself is largest. Denoting by c the largest distance from the neutral surface (which corresponds to either the upper or the lower surface of the member), and by ϵ_m the *maximum absolute value* of the strain, we have

$$\epsilon_m = \frac{c}{\rho} \tag{4.9}$$

$$\sigma_m = \frac{Mc}{I}$$

$$\frac{1}{\rho} = \frac{\epsilon_m}{c} \qquad \frac{1}{\rho} = \frac{\sigma_m}{Ec} = \frac{1}{Ec}\frac{Mc}{I}$$

$$\frac{1}{\rho} = \frac{M}{EI}$$

We first recall from elementary calculus that the curvature of a plane curve at a point $Q(x,y)$ of the curve can be expressed as

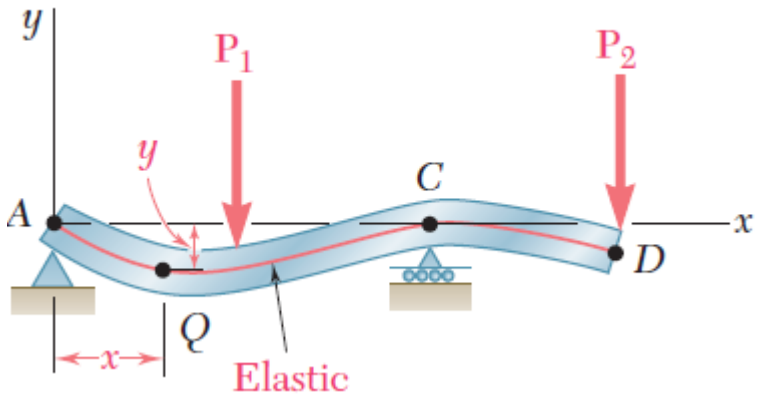
$$\frac{1}{\rho} = \frac{\dfrac{d^2y}{dx^2}}{\left[1 + \left(\dfrac{dy}{dx}\right)^2\right]^{3/2}}$$

$$\kappa = \frac{|f''(x)|}{\left(1 + [f'(x)]^2\right)^{\frac{3}{2}}}$$

You have seen this form of curvature equation in calculus

where dy/dx and d^2y/dx^2 are the first and second derivatives of the function $y(x)$ represented by that curve. But, in the case of the elastic curve of a beam, the slope dy/dx is very small, and its square is negligible compared to unity. We write, therefore,

$$\frac{1}{\rho} = \frac{d^2y}{dx^2} \tag{9.3}$$



Substituting for $1/\rho$ from (9.3) into (9.1), we have

$$\frac{d^2y}{dx^2} = \frac{M(x)}{EI}$$

The equation obtained is a second-order linear differential equation; it is the governing differential equation for the elastic curve.

The product EI is known as the *flexural rigidity* and, if it varies along the beam, as in the case of a beam of varying depth, we must express it as a function of x before proceeding to integrate Eq. (9.4). However, in the case of a prismatic beam, which is the case considered here, the flexural rigidity is constant. We may thus multiply both members of Eq. (8.4) by EI and integrate in x. We write

$$EI\frac{dy}{dx} = \int_0^x M(x)\,dx + C_1 \tag{9.5}$$

where C_1 is a constant of integration. Denoting by $\theta(x)$ the angle, measured in radians, that the tangent to the elastic curve at Q forms with the horizontal (Fig. 9.7), and recalling that this angle is very small, we have

$$\frac{dy}{dx} = \tan\theta \simeq \theta(x)$$

Thus, we write Eq. (9.5) in the alternative form

$$EI\,\theta(x) = \int_0^x M(x)\,dx + C_1 \tag{9.5'}$$

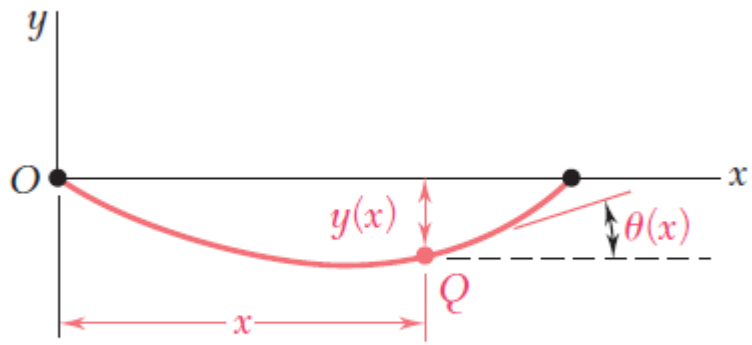


Fig. 9.7 Slope $\theta(x)$ of tangent to the elastic curve.

Integrating both members of Eq. (9.5) in x, we have

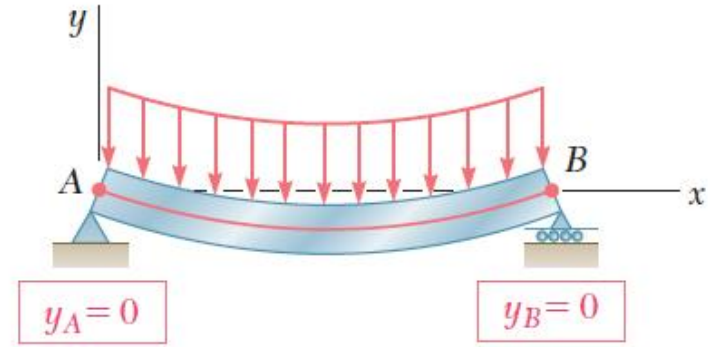
$$EI\,y = \int_0^x \left[\int_0^x M(x)\,dx + C_1 \right] dx + C_2$$

$$EI\,y = \int_0^x dx \int_0^x M(x)\,dx + C_1 x + C_2$$

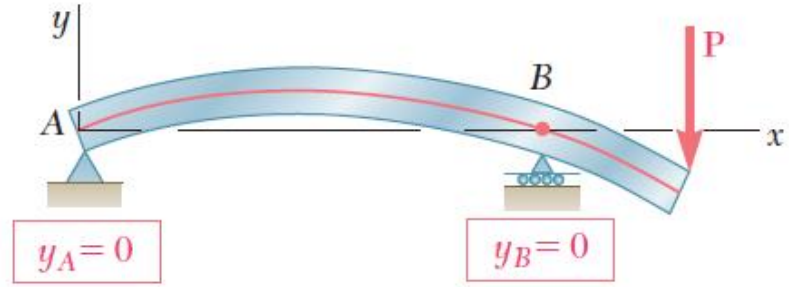
where C_2 is a second constant, and where the first term in the right-hand member represents the function of x obtained by integrating twice in x the bending moment $M(x)$. If it were not for the fact that the constants C_1 and C_2 are as yet undetermined, Eq. (9.6) would define the deflection of the beam at any given point Q, and Eq. (9.5) or (9.5′) would similarly define the slope of the beam at Q.

The constants C_1 and C_2 are determined from the *boundary conditions* or, more precisely, from the conditions imposed on the beam by its supports. Limiting our analysis in this section to *statically determinate beams*, i.e., to beams supported in such a way that the reactions at the supports can be obtained by the methods of statics, we note that only three types of beams need to be considered here (Fig. 9.8): (*a*) the *simply supported beam*, (*b*) the *overhanging beam*, and (*c*) the *cantilever beam*.

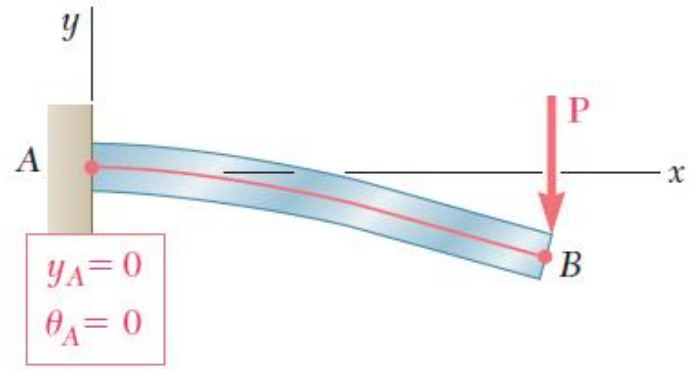
In the first two cases, the supports consist of a pin and bracket at A and of a roller at B, and require that the deflection be zero at each of these points. Letting first $x = x_A$, $y = y_A = 0$ in Eq. (9.6), and then $x = x_B$, $y = y_B = 0$ in the same equation, we obtain two equations that can be solved for C_1 and C_2. In the case of the cantilever beam (Fig. 9.8*c*), we note that both the deflection and the slope at A must be zero. Letting $x = x_A$, $y = y_A = 0$ in Eq. (9.6), and $x = x_A$, $\theta = \theta_A = 0$ in Eq. (9.5′), we obtain again two equations that can be solved for C_1 and C_2.



(*a*) Simply supported beam



(*b*) Overhanging beam



(*c*) Cantilever beam

Fig. 9.8 Boundary conditions for statically determinate beams.

The cantilever beam AB is of uniform cross section and carries a load **P** at its free end A (Fig. 9.9). Determine the equation of the elastic curve and the deflection and slope at A.

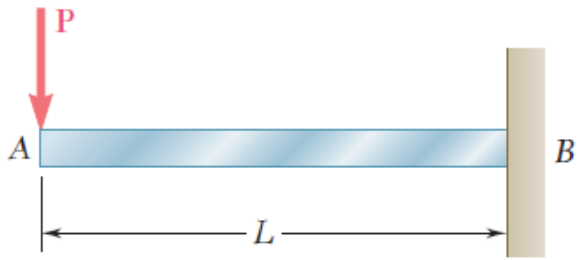


Fig. 9.9

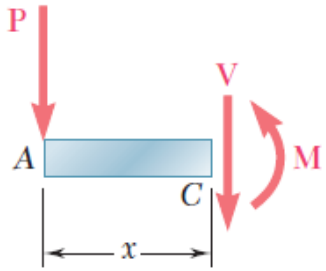


Fig. 9.10

Using the free-body diagram of the portion AC of the beam (Fig. 9.10), where C is located at a distance x from end A, we find

$$M = -Px \tag{9.7}$$

Substituting for M into Eq. (9.4) and multiplying both members by the constant EI, we write

$$EI\,\frac{d^2y}{dx^2} = -Px$$

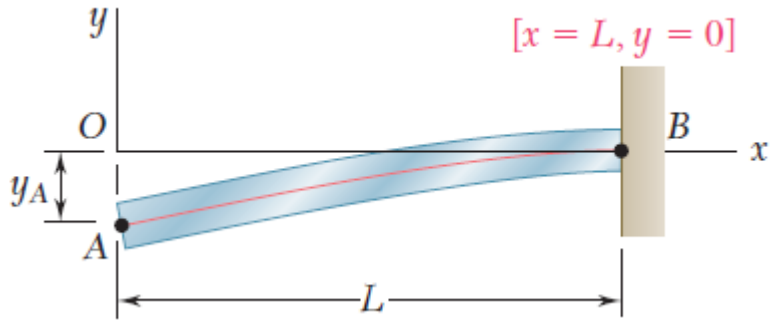
Integrating in x, we obtain

$$EI\,\frac{dy}{dx} = -\tfrac{1}{2}Px^2 + C_1 \tag{9.8}$$

We now observe that at the fixed end B we have $x = L$ and $\theta = dy/dx = 0$ (Fig. 9.11). Substituting these values into (9.8) and solving for C_1, we have

$$C_1 = \tfrac{1}{2}PL^2$$

$[x = L, \theta = 0]$
$[x = L, y = 0]$



We now observe that at the fixed end B we have $x = L$ and $\theta = dy/dx = 0$ (Fig. 9.11). Substituting these values into (9.8) and solving for C_1, we have

$$C_1 = \tfrac{1}{2}PL^2$$

which we carry back into (9.8):

$$EI\,\frac{dy}{dx} = -\tfrac{1}{2}Px^2 + \tfrac{1}{2}PL^2 \tag{9.9}$$

Integrating both members of Eq. (9.9), we write

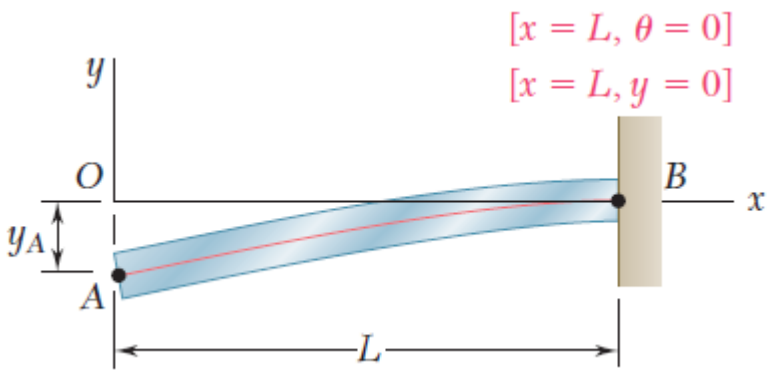
$$EI\,y = -\tfrac{1}{6}Px^3 + \tfrac{1}{2}PL^2x + C_2 \tag{9.10}$$

But, at B we have $x = L$, $y = 0$. Substituting into (9.10), we have

$$0 = -\tfrac{1}{6}PL^3 + \tfrac{1}{2}PL^3 + C_2$$
$$C_2 = -\tfrac{1}{3}PL^3$$

Carrying the value of C_2 back into Eq. (9.10), we obtain the equation of the elastic curve:

$$EI\,y = -\tfrac{1}{6}Px^3 + \tfrac{1}{2}PL^2x - \tfrac{1}{3}PL^3$$



Carrying the value of C_2 back into Eq. (9.10), we obtain the equation of the elastic curve:

$$EI\,y = -\tfrac{1}{6}Px^3 + \tfrac{1}{2}PL^2x - \tfrac{1}{3}PL^3$$

or

$$y = \frac{P}{6EI}(-x^3 + 3L^2x - 2L^3) \tag{9.11}$$

The deflection and slope at A are obtained by letting $x = 0$ in Eqs. (9.11) and (9.9). We find

$$y_A = -\frac{PL^3}{3EI} \qquad \text{and} \qquad \theta_A = \left(\frac{dy}{dx}\right)_A = \frac{PL^2}{2EI}$$

The simply supported prismatic beam AB carries a uniformly distributed load w per unit length (Fig. 9.12). Determine the equation of the elastic curve and the maximum deflection of the beam.

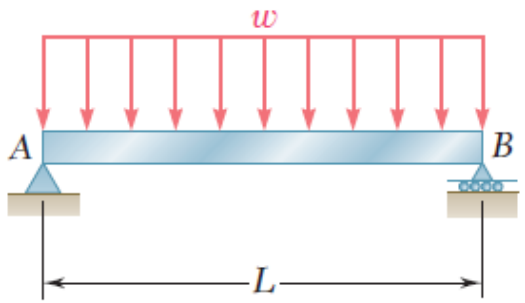


Fig. 9.12

Drawing the free-body diagram of the portion AD of the beam (Fig. 9.13) and taking moments about D, we find that

$$M = \tfrac{1}{2}wLx - \tfrac{1}{2}wx^2 \tag{9.12}$$

Substituting for M into Eq. (9.4) and multiplying both members of this equation by the constant EI, we write

$$EI\frac{d^2y}{dx^2} = -\frac{1}{2}wx^2 + \frac{1}{2}wLx \tag{9.13}$$

Integrating twice in x, we have

$$EI\frac{dy}{dx} = -\frac{1}{6}wx^3 + \frac{1}{4}wLx^2 + C_1 \tag{9.14}$$

$$EI\,y = -\frac{1}{24}wx^4 + \frac{1}{12}wLx^3 + C_1x + C_2 \tag{9.15}$$

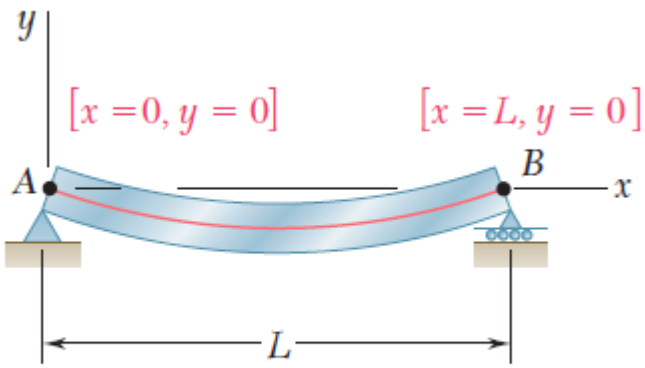


Fig. 9.14

Observing that $y = 0$ at both ends of the beam (Fig. 9.14), we first let $x = 0$ and $y = 0$ in Eq. (9.15) and obtain $C_2 = 0$. We then make $x = L$ and $y = 0$ in the same equation and write

$$0 = -\tfrac{1}{24}wL^4 + \tfrac{1}{12}wL^4 + C_1L$$
$$C_1 = -\tfrac{1}{24}wL^3$$

Carrying the values of C_1 and C_2 back into Eq. (9.15), we obtain the equation of the elastic curve:

$$EI\,y = -\tfrac{1}{24}wx^4 + \tfrac{1}{12}wLx^3 - \tfrac{1}{24}wL^3x$$

or

$$y = \frac{w}{24EI}(-x^4 + 2Lx^3 - L^3x) \tag{9.16}$$

Substituting into Eq. (9.14) the value obtained for C_1, we check that the slope of the beam is zero for $x = L/2$ and that the elastic curve has a minimum at the midpoint C of the beam (Fig. 9.15). Letting $x = L/2$ in Eq. (9.16), we have

$$y_C = \frac{w}{24EI}\left(-\frac{L^4}{16} + 2L\frac{L^3}{8} - L^3\frac{L}{2}\right) = -\frac{5wL^4}{384EI}$$

The maximum deflection or, more precisely, the maximum absolute value of the deflection, is thus

$$|y|_{\max} = \frac{5wL^4}{384EI}$$

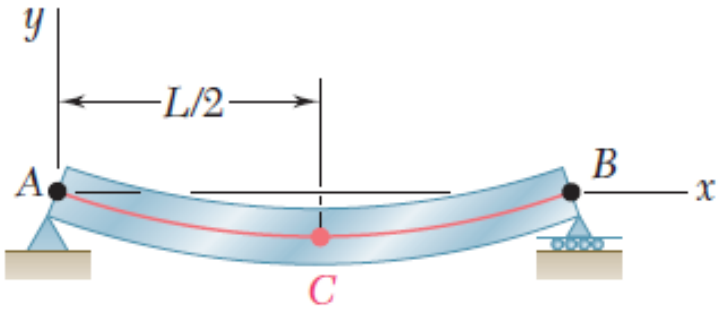


Fig. 9.15

*9.4 DIRECT DETERMINATION OF THE ELASTIC CURVE FROM THE LOAD DISTRIBUTION

We saw in Sec. 9.3 that the equation of the elastic curve can be obtained by integrating twice the differential equation

$$\frac{d^2y}{dx^2} = \frac{M(x)}{EI} \tag{9.4}$$

where $M(x)$ is the bending moment in the beam. We now recall from Sec. 5.3 that, when a beam supports a distributed load $w(x)$, we have $dM/dx = V$ and $dV/dx = -w$ at any point of the beam. Differentiating both members of Eq. (9.4) with respect to x and assuming EI to be constant, we have therefore

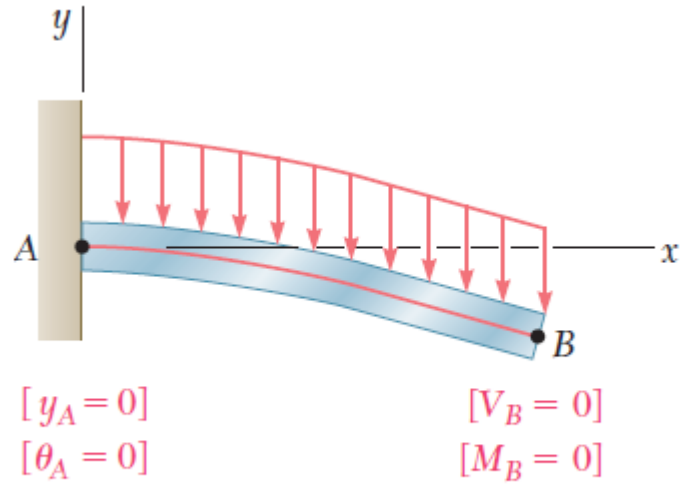
$$\frac{d^3y}{dx^3} = \frac{1}{EI}\frac{dM}{dx} = \frac{V(x)}{EI} \tag{9.31}$$

and, differentiating again,

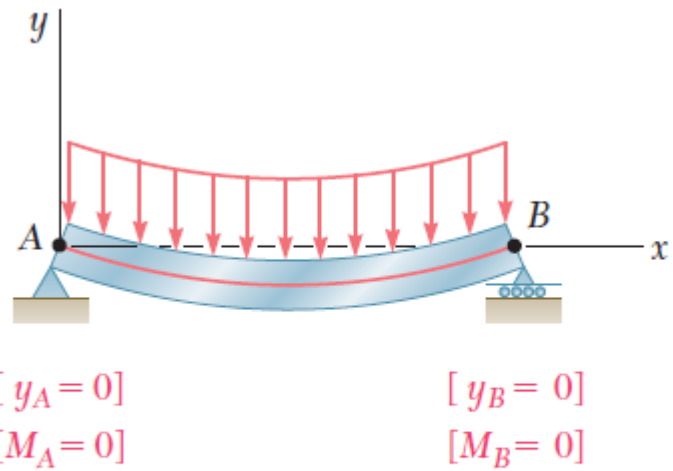
$$\frac{d^4y}{dx^4} = \frac{1}{EI}\frac{dV}{dx} = -\frac{w(x)}{EI}$$

We conclude that, when a prismatic beam supports a distributed load $w(x)$, its elastic curve is governed by the fourth-order linear differential equation

$$\frac{d^4y}{dx^4} = -\frac{w(x)}{EI} \tag{9.32}$$



(*a*) Cantilever beam



(*b*) Simply supported beam

Fig. 9.20 Boundary conditions.

Multiplying both members of Eq. (9.32) by the constant EI and integrating four times, we write

$$EI\frac{d^4y}{dx^4} = -w(x)$$

$$EI\frac{d^3y}{dx^3} = V(x) = -\int w(x)\,dx + C_1$$

$$EI\frac{d^2y}{dx^2} = M(x) = -\int dx \int w(x)\,dx + C_1x + C_2 \tag{9.33}$$

$$EI\frac{dy}{dx} = EI\,\theta(x) = -\int dx \int dx \int w(x)\,dx + \frac{1}{2}C_1x^2 + C_2x + C_3$$

$$EIy(x) = -\int dx \int dx \int dx \int w(x)\,dx + \frac{1}{6}C_1x^3 + \frac{1}{2}C_2x^2 + C_3x + C_4$$

The four constants of integration can be determined from the boundary conditions. These conditions include (*a*) the conditions imposed on the deflection or slope of the beam by its supports (cf. Sec. 9.3), and (*b*) the condition that V and M be zero at the free end of a cantilever beam, or that M be zero at both ends of a simply supported beam (cf. Sec. 5.3). This has been illustrated in Fig. 9.20.

The simply supported prismatic beam AB carries a uniformly distributed load w per unit length (Fig. 9.21). Determine the equation of the elastic curve and the maximum deflection of the beam. (This is the same beam and loading as in Example 9.02.)

Since w = constant, the first three of Eqs. (9.33) yield

$$EI\frac{d^4y}{dx^4} = -w$$

$$EI\frac{d^3y}{dx^3} = V(x) = -wx + C_1$$

$$EI\frac{d^2y}{dx^2} = M(x) = -\frac{1}{2}wx^2 + C_1x + C_2 \tag{9.34}$$

Noting that the boundary conditions require that $M = 0$ at both ends of the beam (Fig. 9.22), we first let $x = 0$ and $M = 0$ in Eq. (9.34) and obtain $C_2 = 0$. We then make $x = L$ and $M = 0$ in the same equation and obtain $C_1 = \frac{1}{2}wL$.

Carrying the values of C_1 and C_2 back into Eq. (9.34), and integrating twice, we write

$$EI\frac{d^2y}{dx^2} = -\frac{1}{2}wx^2 + \frac{1}{2}wLx$$

$$EI\frac{dy}{dx} = -\frac{1}{6}wx^3 + \frac{1}{4}wLx^2 + C_3$$

$$EI\,y = -\frac{1}{24}wx^4 + \frac{1}{12}wLx^3 + C_3x + C_4 \tag{9.35}$$

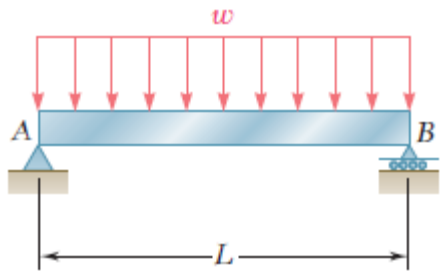


Fig. 9.21

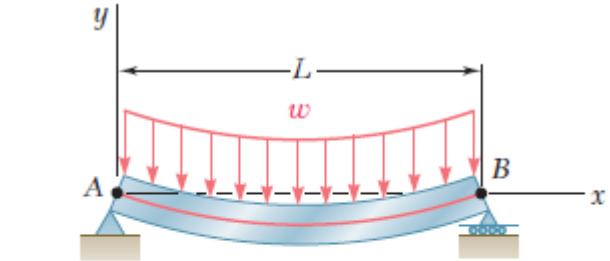


Fig. 9.22

But the boundary conditions also require that $y = 0$ at both ends of the beam. Letting $x = 0$ and $y = 0$ in Eq. (9.35), we obtain $C_4 = 0$; letting $x = L$ and $y = 0$ in the same equation, we write

$$0 = -\tfrac{1}{24}wL^4 + \tfrac{1}{12}wL^4 + C_3L$$

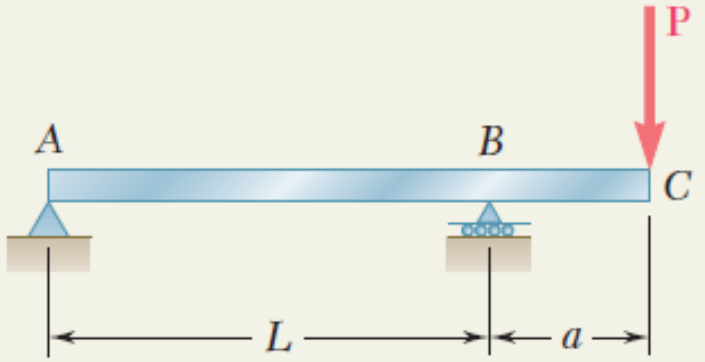
$$C_3 = -\tfrac{1}{24}wL^3$$

Carrying the values of C_3 and C_4 back into Eq. (9.35) and dividing both members by EI, we obtain the equation of the elastic curve:

$$y = \frac{w}{24EI}(-x^4 + 2Lx^3 - L^3x) \tag{9.36}$$

The value of the maximum deflection is obtained by making $x = L/2$ in Eq. (9.36). We have

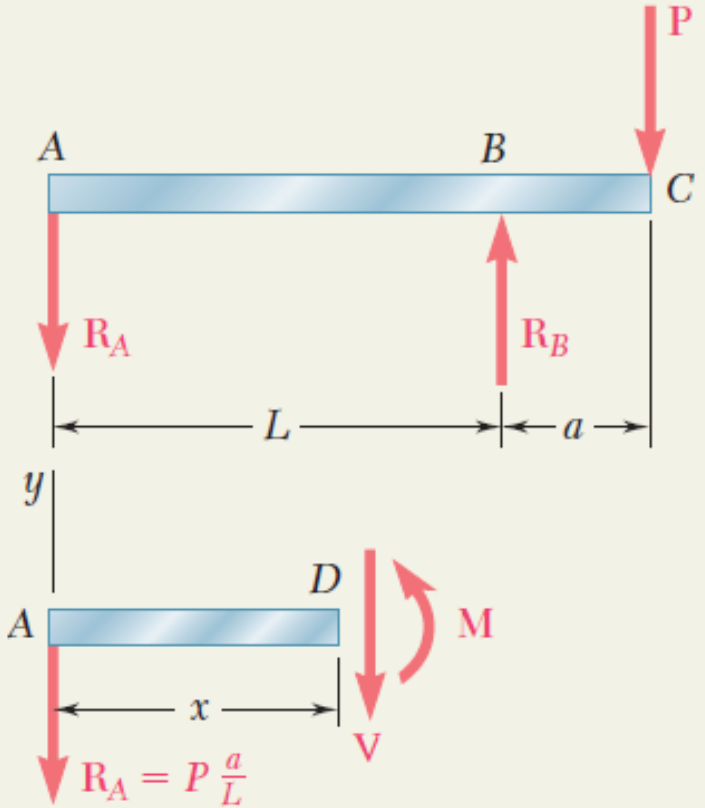
$$|y|_{\max} = \frac{5wL^4}{384EI}$$



The overhanging steel beam ABC carries a concentrated load **P** at end C. For portion AB of the beam, (a) derive the equation of the elastic curve, (b) determine the maximum deflection, (c) evaluate $y_{\max}$ for the following data:

W14 × 68	$I = 722 \text{ in}^4$	$E = 29 \times 10^6 \text{ psi}$
$P = 50 \text{ kips}$	$L = 15 \text{ ft} = 180 \text{ in.}$	$a = 4 \text{ ft} = 48 \text{ in.}$

SOLUTION



Free-Body Diagrams. Reactions: $\mathbf{R}_A = Pa/L \downarrow \quad \mathbf{R}_B = P(1 + a/L) \uparrow$
Using the free-body diagram of the portion of beam AD of length x, we find

$$M = -P\frac{a}{L}x \qquad (0 < x < L)$$

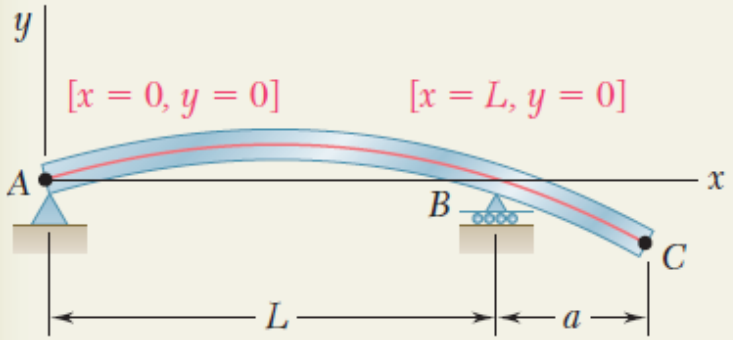
Differential Equation of the Elastic Curve. We use Eq. (9.4) and write

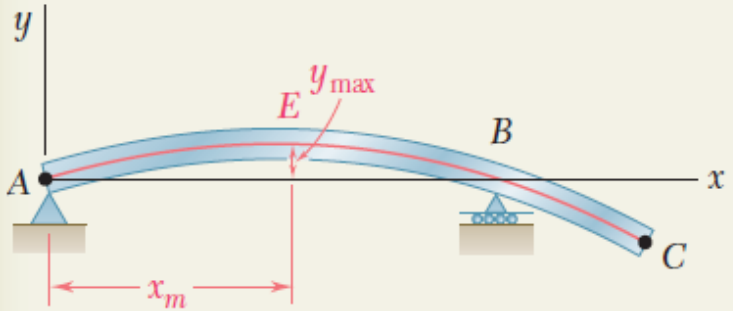
$$EI\frac{d^2y}{dx^2} = -P\frac{a}{L}x$$

Noting that the flexural rigidity EI is constant, we integrate twice and find

$$EI\frac{dy}{dx} = -\frac{1}{2}P\frac{a}{L}x^2 + C_1 \tag{1}$$

$$EI\,y = -\frac{1}{6}P\frac{a}{L}x^3 + C_1x + C_2 \tag{2}$$





Determination of Constants. For the boundary conditions shown, we have

$[x = 0, y = 0]$: From Eq. (2), we find $C_2 = 0$

$[x = L, y = 0]$: Again using Eq. (2), we write

$$EI(0) = -\frac{1}{6}P\frac{a}{L}L^3 + C_1L \qquad C_1 = +\frac{1}{6}PaL$$

a. Equation of the Elastic Curve. Substituting for C_1 and C_2 into Eqs. (1) and (2), we have

$$EI\frac{dy}{dx} = -\frac{1}{2}P\frac{a}{L}x^2 + \frac{1}{6}PaL \qquad \frac{dy}{dx} = \frac{PaL}{6EI}\left[1 - 3\left(\frac{x}{L}\right)^2\right] \tag{3}$$

$$EI\,y = -\frac{1}{6}P\frac{a}{L}x^3 + \frac{1}{6}PaLx \qquad y = \frac{PaL^2}{6EI}\left[\frac{x}{L} - \left(\frac{x}{L}\right)^3\right] \tag{4}$$

b. Maximum Deflection in Portion *AB*. The maximum deflection y_{max} occurs at point E where the slope of the elastic curve is zero. Setting $dy/dx = 0$ in Eq. (3), we determine the abscissa x_m of point E:

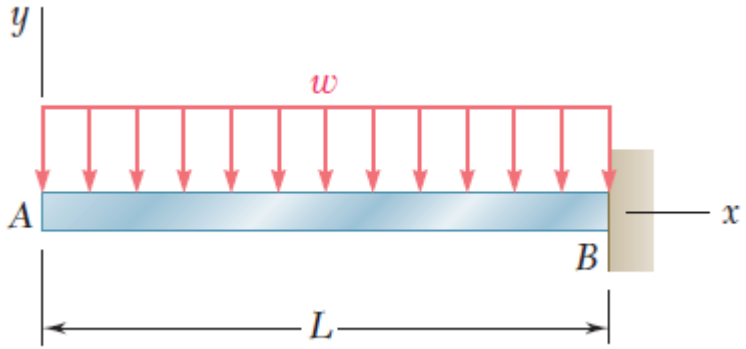
$$0 = \frac{PaL}{6EI}\left[1 - 3\left(\frac{x_m}{L}\right)^2\right] \qquad x_m = \frac{L}{\sqrt{3}} = 0.577L$$

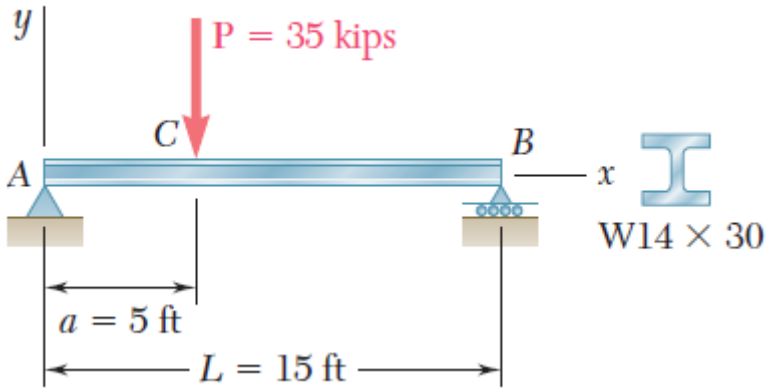
We substitute $x_m/L = 0.577$ into Eq. (4) and have

$$y_{max} = \frac{PaL^2}{6EI}[(0.577) - (0.577)^3] \qquad y_{max} = 0.0642\frac{PaL^2}{EI}$$

c. Evaluation of y_{max}. For the data given, the value of y_{max} is

$$y_{max} = 0.0642\frac{(50\text{ kips})(48\text{ in.})(180\text{ in.})^2}{(29 \times 10^6\text{ psi})(722\text{ in}^4)} \qquad y_{max} = 0.238\text{ in.}$$





at $x = L\,; y = 0\ and\ \frac{dy}{dx} =$ 0