

Photo 10.1 Laboratory test showing a buckled column.

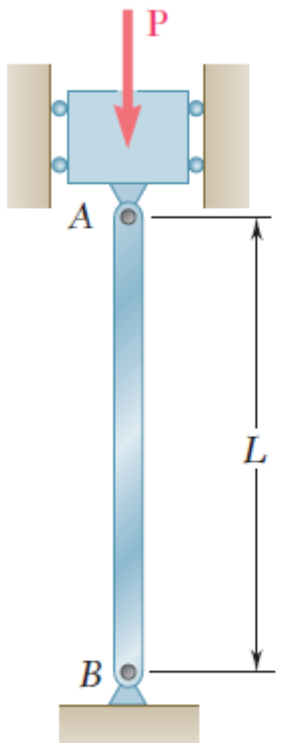
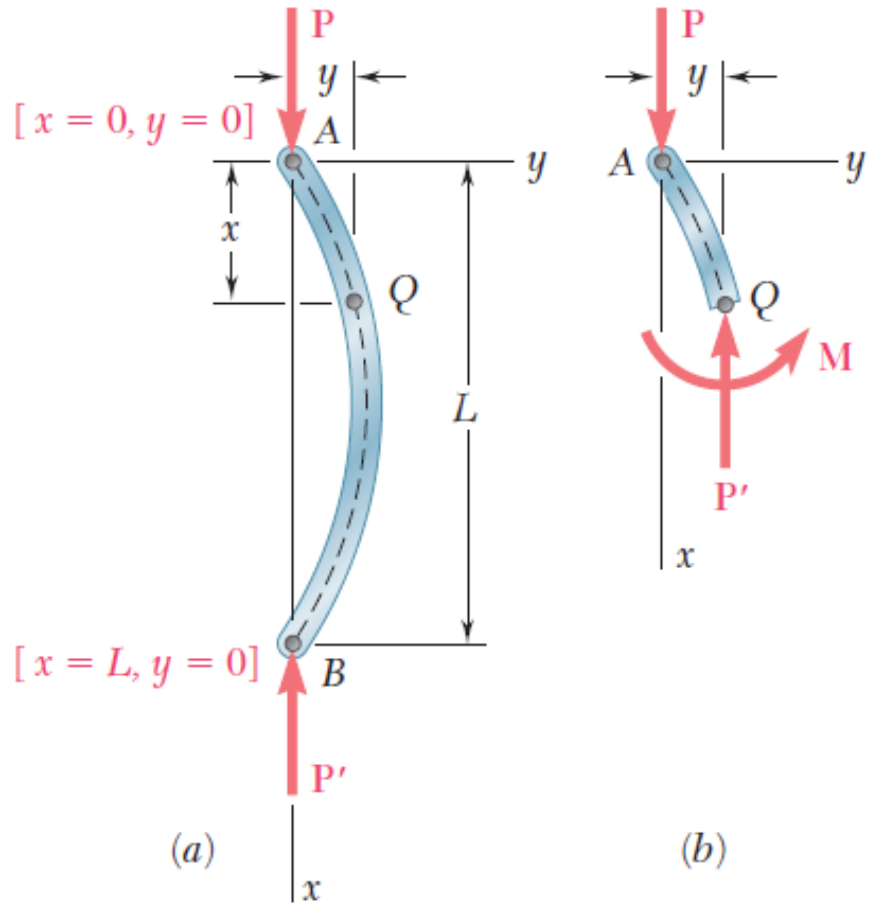


Fig. 10.1 Column *(repeated)*



Fig. 10.2 Buckled column *(repeated)*



$$\frac{d^2y}{dx^2} = \frac{M}{EI} = -\frac{P}{EI}y$$

$$\frac{d^2y}{dx^2} + \frac{P}{EI}y = 0$$

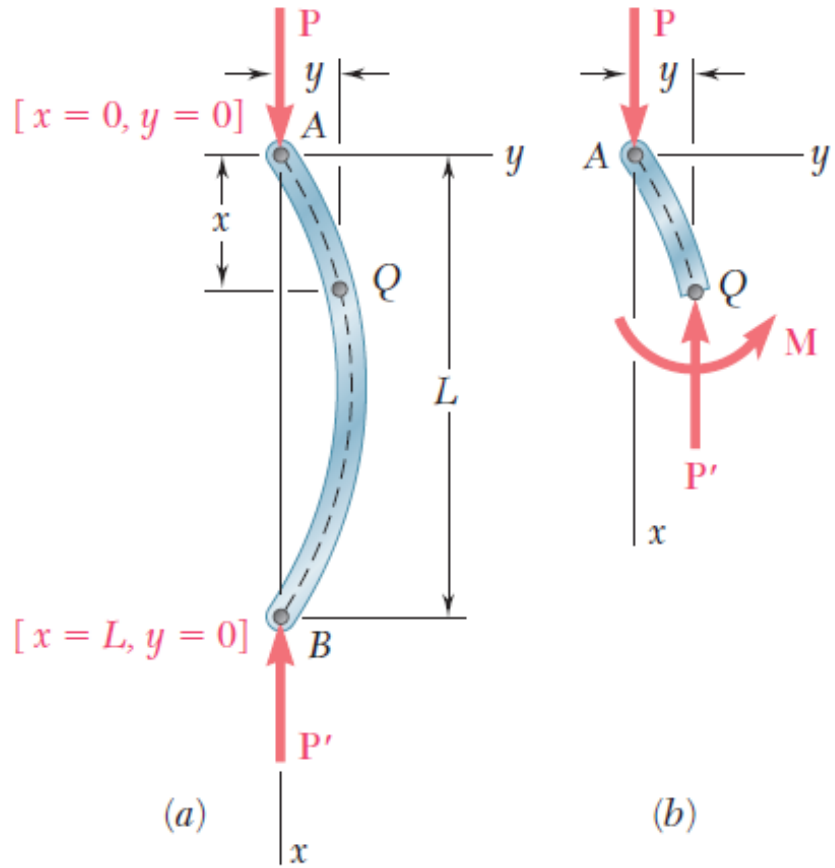
$$p^2 = \frac{P}{EI}$$

$$\frac{d^2y}{dx^2} + p^2y = 0$$

which is the same as that of the differential equation for simple harmonic motion except, of course, that the independent variable is now the distance x instead of the time t. The general solution of Eq. (10.7) is

$$y = A \sin px + B \cos px$$

as we easily check by computing d^2y/dx^2 and substituting for y and d^2y/dx^2 into Eq. (10.7).



Recalling the boundary conditions that must be satisfied at ends A and B of the column (Fig. 10.7*a*), we first make $x = 0$, $y = 0$ in Eq. (10.8) and find that $B = 0$. Substituting next $x = L$, $y = 0$, we obtain

$$A \sin pL = 0 \tag{10.9}$$

This equation is satisfied either if $A = 0$, or if $\sin pL = 0$. If the first of these conditions is satisfied, Eq. (10.8) reduces to $y = 0$ and the column is straight (Fig. 10.1). For the second condition to be satisfied, we must have $pL = n\pi$ or, substituting for p from (10.6) and solving for P,

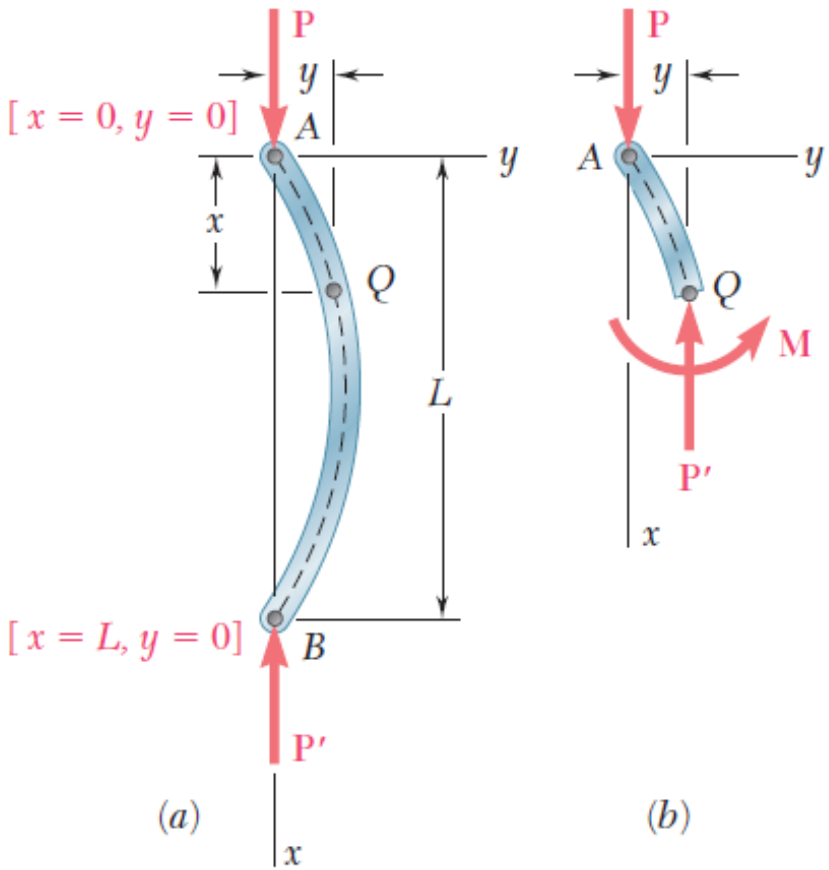
$$P = \frac{n^2\pi^2 EI}{L^2} \tag{10.10}$$

The smallest of the values of P defined by Eq. (10.10) is that corresponding to $n = 1$. We thus have

$$P_{cr} = \frac{\pi^2 EI}{L^2} \tag{10.11}$$

The expression obtained is known as *Euler's formula*, after the Swiss mathematician Leonhard Euler (1707–1783). Substituting this expression for P into Eq. (10.6) and the value obtained for p into Eq. (10.8), and recalling that $B = 0$, we write

$$y = A \sin \frac{\pi x}{L} \tag{10.12}$$



$$y = A \sin \frac{\pi x}{L} \qquad (10.12)$$

which is the equation of the elastic curve after the column has buckled (Fig. 10.2). We note that the value of the maximum deflection, $y_m = A$, is indeterminate. This is due to the fact that the differential equation (10.5) is a linearized approximation of the actual governing differential equation for the elastic curve.†

If $P < P_{cr}$, the condition $\sin pL = 0$ cannot be satisfied, and the solution given by Eq. (10.12) does not exist. We must then have $A = 0$, and the only possible configuration for the column is a straight one. Thus, for $P < P_{cr}$ the straight configuration of Fig. 10.1 is stable.

In the case of a column with a circular or square cross section, the moment of inertia I of the cross section is the same about any centroidal axis, and the column is as likely to buckle in one plane as another, except for the restraints that can be imposed by the end connections. For other shapes of cross section, the critical load should be computed by making $I = I_{min}$ in Eq. (10.11); if buckling occurs, it will take place in a plane perpendicular to the corresponding principal axis of inertia.

The value of the stress corresponding to the critical load is called the *critical stress* and is denoted by σ_{cr}. Recalling Eq. (10.11)

and setting $I = Ar^2$, where A is the cross-sectional area and r its radius of gyration, we have

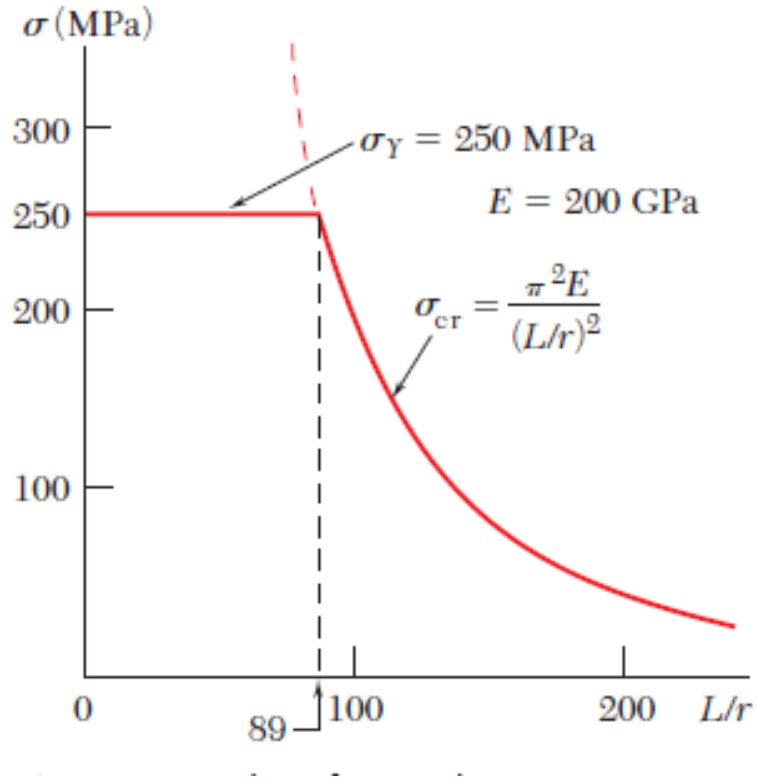
$$\sigma_{cr} = \frac{P_{cr}}{A} = \frac{\pi^2 E A r^2}{A L^2}$$

or

$$\sigma_{cr} = \frac{\pi^2 E}{(L/r)^2} \tag{10.13}$$

The quantity L/r is called the *slenderness ratio* of the column. It is clear, in view of the remark of the preceding paragraph, that the minimum value of the radius of gyration r should be used in computing the slenderness ratio and the critical stress in a column.

Equation (10.13) shows that the critical stress is proportional to the modulus of elasticity of the material, and inversely proportional to the square of the slenderness ratio of the column. The plot of σ_{cr} versus L/r is shown in Fig. 10.8 for structural steel, assuming $E = 200$ GPa and $\sigma_Y = 250$ MPa. We should keep in mind that no factor of safety has been used in plotting σ_{cr}. We also note that, if the value obtained for σ_{cr} from Eq. (10.13) or from the curve of Fig. 10.8 is larger than the yield strength σ_Y, this value is of no interest to us, since the column will yield in compression and cease to be elastic before it has a chance to buckle.



EXAMPLE 10.01

A 2-m-long pin-ended column of square cross section is to be made of wood. Assuming $E = 13$ GPa, $\sigma_{all} = 12$ MPa, and using a factor of safety of 2.5 in computing Euler's critical load for buckling, determine the size of the cross section if the column is to safely support (*a*) a 100-kN load, (*b*) a 200-kN load.

(a) For the 100-kN Load. Using the given factor of safety, we make

$$P_{cr} = 2.5(100\text{ kN}) = 250\text{ kN} \qquad L = 2\text{ m} \qquad E = 13\text{ GPa}$$

in Euler's formula (10.11) and solve for I. We have

$$I = \frac{P_{cr}L^2}{\pi^2 E} = \frac{(250 \times 10^3\text{ N})(2\text{ m})^2}{\pi^2(13 \times 10^9\text{ Pa})} = 7.794 \times 10^{-6}\text{ m}^4$$

Recalling that, for a square of side a, we have $I = a^4/12$, we write

$$\frac{a^4}{12} = 7.794 \times 10^{-6}\text{ m}^4 \qquad a = 98.3\text{ mm} \approx 100\text{ mm}$$

We check the value of the normal stress in the column:

$$\sigma = \frac{P}{A} = \frac{100\text{ kN}}{(0.100\text{ m})^2} = 10\text{ MPa}$$

Since σ is smaller than the allowable stress, a 100×100-mm cross section is acceptable.

(b) For the 200-kN Load. Solving again Eq. (10.11) for I, but making now $P_{cr} = 2.5(200) = 500$ kN, we have

$$I = 15.588 \times 10^{-6}\ \text{m}^4$$

$$\frac{a^4}{12} = 15.588 \times 10^{-6} \qquad a = 116.95\ \text{mm}$$

The value of the normal stress is

$$\sigma = \frac{P}{A} = \frac{200\ \text{kN}}{(0.11695\ \text{m})^2} = 14.62\ \text{MPa}$$

Since this value is larger than the allowable stress, the dimension obtained is not acceptable, and we must select the cross section on the basis of its resistance to compression. We write

$$A = \frac{P}{\sigma_{all}} = \frac{200\ \text{kN}}{12\ \text{MPa}} = 16.67 \times 10^{-3}\ \text{m}^2$$

$$a^2 = 16.67 \times 10^{-3}\ \text{m}^2 \qquad a = 129.1\ \text{mm}$$

A 130 × 130-mm cross section is acceptable.

EXTENSION OF EULER'S FORMULA TO COLUMNS WITH OTHER END CONDITIONS

$$P_{cr} = \frac{\pi^2 EI}{L^2}$$



$$P_{cr} = \frac{\pi^2 EI}{L_e^2} \qquad (10.11')$$

The critical stress is found in a similar way from the formula

$$\sigma_{cr} = \frac{\pi^2 E}{(L_e/r)^2} \qquad (10.13')$$

The quantity L_e/r is referred to as the *effective slenderness ratio* of the column and, in the case considered here, is equal to $2L/r$.

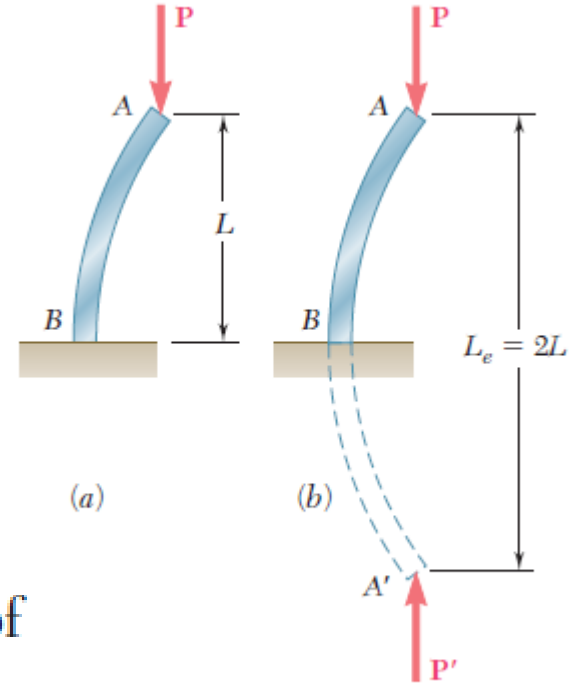
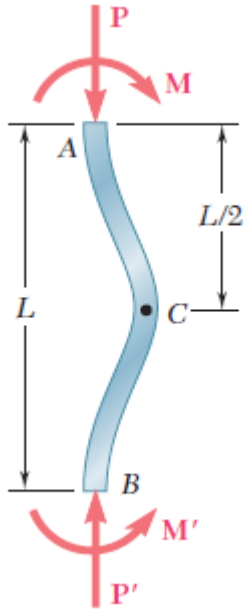


Fig. 10.9 Column with free end.

Consider next a column with two fixed ends A and B supporting a load **P** (Fig. 10.10). The symmetry of the supports and of the loading about a horizontal axis through the midpoint C requires that the shear at C and the horizontal components of the reactions at A and B be zero (Fig. 10.11). It follows that the restraints imposed upon the upper half AC of the column by the support at A and by the lower half CB are identical (Fig. 10.12). Portion AC must thus be symmetric about its midpoint D, and this point must be a point of inflection, where the bending moment is zero. A similar reasoning shows that the bending moment at the midpoint E of the lower half of the column must also be zero (Fig. 10.13*a*). Since the bending moment at the ends of a pin-ended column is zero, it follows that the portion DE of the column of Fig. 10.13*a* must behave as a pin-ended column (Fig. 10.13*b*). We thus conclude that the effective length of a column with two fixed ends is $L_e = L/2$.



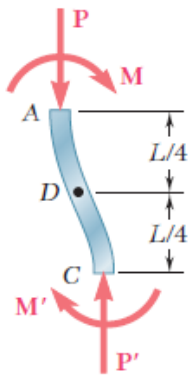


Fig. 10.12

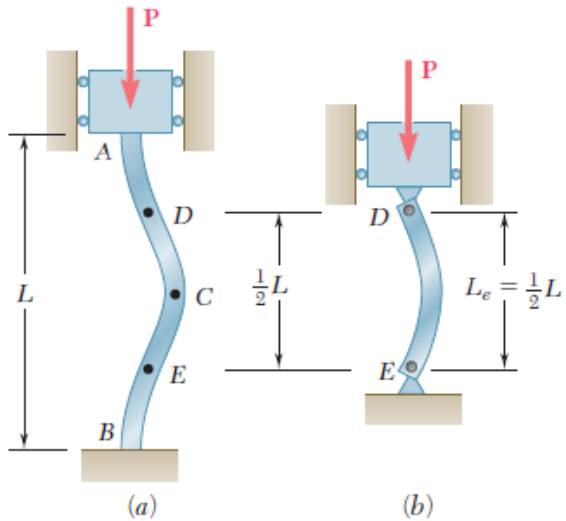
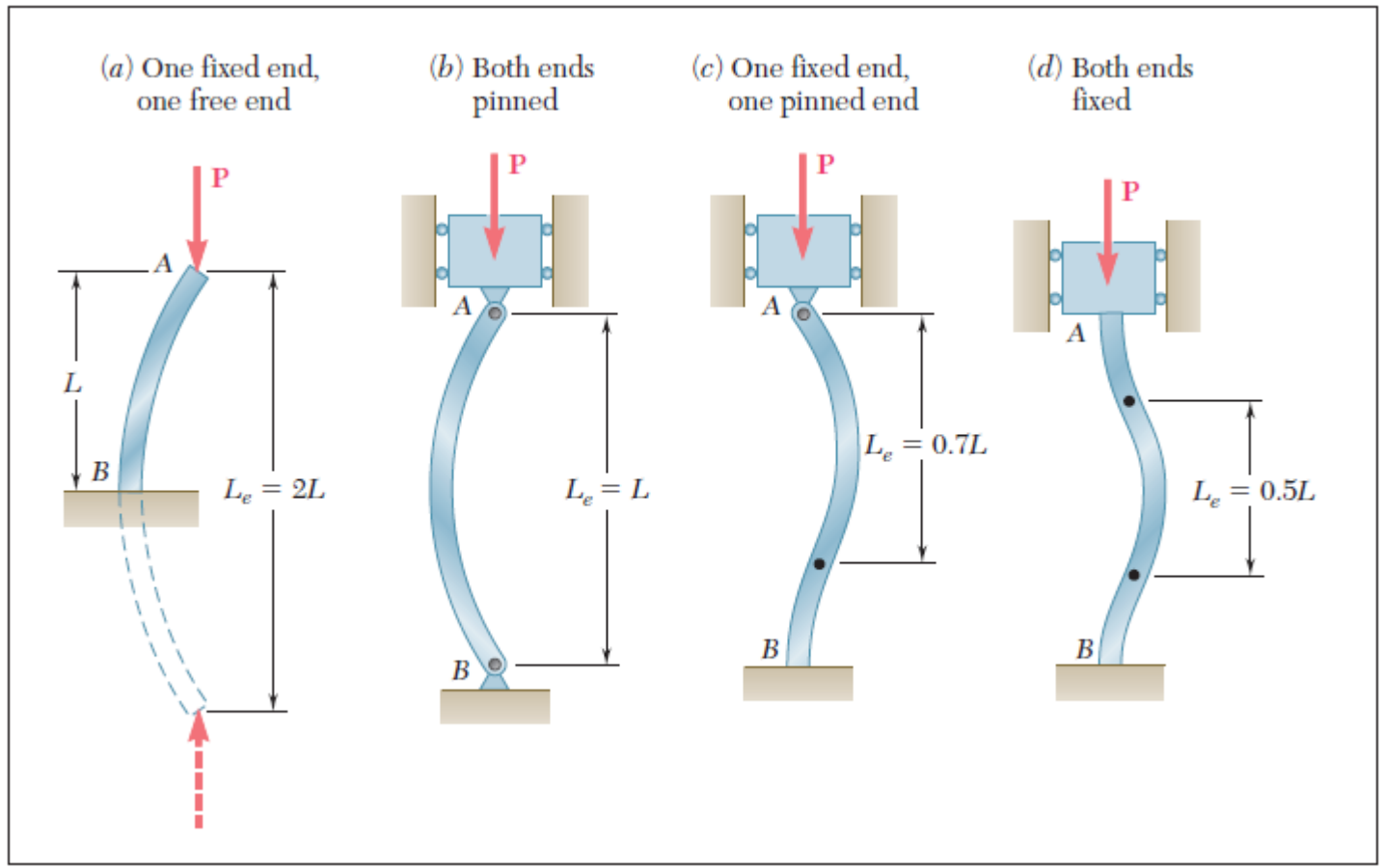
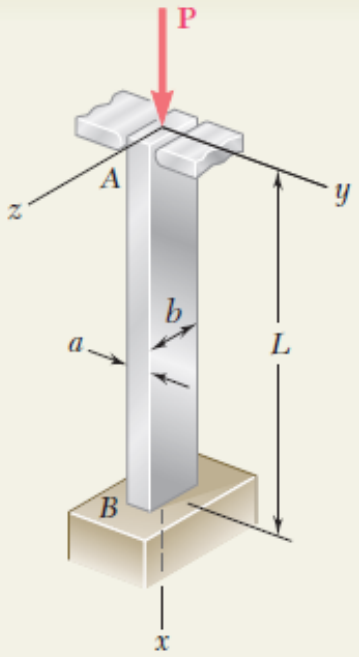


Fig. 10.13


(a) One fixed end, one free end
(b) Both ends pinned
(c) One fixed end, one pinned end
(d) Both ends fixed
P
A
B
L
$L_e = 2L$
$L_e = L$
$L_e = 0.7L$
$L_e = 0.5L$



SAMPLE PROBLEM 10.1

An aluminum column of length L and rectangular cross section has a fixed end B and supports a centric load at A. Two smooth and rounded fixed plates restrain end A from moving in one of the vertical planes of symmetry of the column, but allow it to move in the other plane. (a) Determine the ratio a/b of the two sides of the cross section corresponding to the most efficient design against buckling. (b) Design the most efficient cross section for the column, knowing that $L = 20$ in., $E = 10.1 \times 10^6$ psi, $P = 5$ kips, and that a factor of safety of 2.5 is required.

SOLUTION

Buckling in *xy* Plane. Referring to Fig. 10.17, we note that the effective length of the column with respect to buckling in this plane is $L_e = 0.7L$. The radius of gyration r_z of the cross section is obtained by writing

$$I_x = \tfrac{1}{12}ba^3 \qquad A = ab$$

and, since $I_z = Ar_z^2$,

$$r_z^2 = \frac{I_z}{A} = \frac{\frac{1}{12}ba^3}{ab} = \frac{a^2}{12} \qquad r_z = a/\sqrt{12}$$

The effective slenderness ratio of the column with respect to buckling in the xy plane is

$$\frac{L_e}{r_z} = \frac{0.7L}{a/\sqrt{12}} \tag{1}$$

Buckling in *xz* Plane. The effective length of the column with respect to buckling in this plane is $L_e = 2L$, and the corresponding radius of gyration is $r_y = b/\sqrt{12}$. Thus,

$$\frac{L_e}{r_y} = \frac{2L}{b/\sqrt{12}} \tag{2}$$

a. Most Efficient Design. The most efficient design is that for which the critical stresses corresponding to the two possible modes of buckling are equal. Referring to Eq. (10.13′), we note that this will be the case if the two values obtained above for the effective slenderness ratio are equal. We write

$$\frac{0.7L}{a/\sqrt{12}} = \frac{2L}{b/\sqrt{12}}$$

and, solving for the ratio a/b,

$$\frac{a}{b} = \frac{0.7}{2} \qquad \frac{a}{b} = 0.35 \blacktriangleleft$$

b. Design for Given Data. Since $F.S. = 2.5$ is required,

$$P_{cr} = (F.S.)P = (2.5)(5 \text{ kips}) = 12.5 \text{ kips}$$

Using $a = 0.35b$, we have $A = ab = 0.35b^2$ and

$$\sigma_{cr} = \frac{P_{cr}}{A} = \frac{12{,}500 \text{ lb}}{0.35b^2}$$

Making $L = 20$ in. in Eq. (2), we have $L_e/r_y = 138.6/b$. Substituting for E, L_e/r, and σ_{cr} into Eq. (10.13′), we write

$$\sigma_{cr} = \frac{\pi^2 E}{(L_e/r)^2} \qquad \frac{12{,}500 \text{ lb}}{0.35b^2} = \frac{\pi^2(10.1 \times 10^6 \text{ psi})}{(138.6/b)^2}$$

$$b = 1.620 \text{ in.} \qquad a = 0.35b = 0.567 \text{ in.} \blacktriangleleft$$