Chapter 11 Energy Methods

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11.1 INTRODUCTION

In the previous chapter we were concerned with the relations existing between forces and deformations under various loading conditions. Our analysis was based on two fundamental concepts, the concept of stress (Chap. 1) and the concept of strain (Chap. 2). A third important concept, the concept of *strain energy*, will now be introduced.

In Sec. 11.2, the *strain energy* of a member will be defined as the increase in energy associated with the deformation of the member. You will see that the strain energy is equal to the work done by a slowly increasing load applied to the member. The *strain-energy density* of a material will be defined as the strain energy per unit volume; it will be seen that it is equal to the area under the stress-strain diagram of the material (Sec. 11.3). From the stress-strain diagram of a material two additional properties will be defined, namely, the *modulus of toughness* and the *modulus of resilience* of the material.

In Sec. 11.4 the elastic strain energy associated with *normal stresses* will be discussed, first in members under axial loading and then in members in bending. Later you will consider the elastic strain energy associated with shearing stresses such as occur in torsional loadings of shafts and in transverse loadings of beams (Sec. 11.5). Strain energy for a *general state of stress* will be considered in Sec. 11.6, where the *maximum-distortion-energy criterion* for yielding will be derived.

The effect of *impact loading* on members will be considered in Sec. 11.7. You will learn to calculate both the *maximum stress* and the *maximum deflection* caused by a moving mass impacting on a member. Properties that increase the ability of a structure to withstand impact loads effectively will be discussed in Sec. 11.8.

In Sec. 11.9 the elastic strain of a member subjected to a *single concentrated load* will be calculated, and in Sec. 11.10 the deflection at the point of application of a single load will be determined.

The last portion of the chapter will be devoted to the determination of the strain energy of structures subjected to *several loads* (Sec. 11.11). *Castigliano's theorem* will be derived in Sec. 11.12 and used in Sec. 11.13 to determine the deflection at a given point of a structure subjected to several loads. In the last section Castigliano's theorem will be applied to the analysis of indeterminate structures (Sec. 11.14).

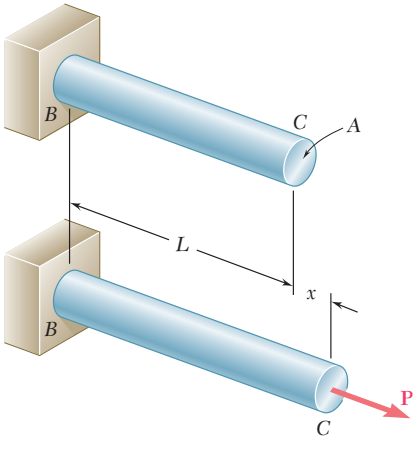


Fig. 11.1 Axially loaded rod.

11.2 STRAIN ENERGY

Consider a rod BC of length L and uniform cross-sectional area A, which is attached at B to a fixed support, and subjected at C to a *slowly increasing* axial load **P** (Fig. 11.1). As we noted in Sec. 2.2, by plotting the magnitude P of the load against the deformation x of the rod, we obtain a certain load-deformation diagram (Fig. 11.2) that is characteristic of the rod BC.

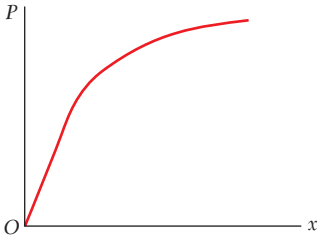


Fig. 11.2 Load-deformation diagram.

Let us now consider the work dU done by the load **P** as the rod elongates by a small amount dx. This *elementary work* is equal to the product of the magnitude P of the load and of the small elongation dx. We write

$$dU = P\,dx \tag{11.1}$$

and note that the expression obtained is equal to the element of area of width dx located under the load-deformation diagram (Fig. 11.3). The *total work* U done by the load as the rod undergoes a deformation x_1 is thus

$$U = \int_0^{x_1} P\,dx$$

and is equal to the area under the load-deformation diagram between $x = 0$ and $x = x_1$.

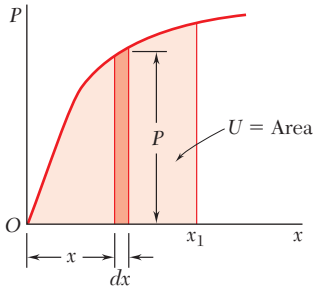


Fig. 11.3 Work due to load **P.**

The work done by the load **P** as it is slowly applied to the rod must result in the increase of some energy associated with the deformation of the rod. This energy is referred to as the *strain energy* of the rod. We have, by definition,

$$\text{Strain energy} = U = \int_0^{x_1} P\,dx \tag{11.2}$$

We recall that work and energy should be expressed in units obtained by multiplying units of length by units of force. Thus, if SI metric units are used, work and energy are expressed in N · m; this unit is called a *joule* (J). If U.S. customary units are used, work and energy are expressed in ft · lb or in in · lb.

In the case of a linear and elastic deformation, the portion of the load-deformation diagram involved can be represented by a straight line of equation $P = kx$ (Fig. 11.4). Substituting for P in Eq. (11.2), we have

$$U = \int_0^{x_1} kx\,dx = \tfrac{1}{2}kx_1^2$$

or

$$U = \tfrac{1}{2}P_1x_1 \tag{11.3}$$

where P_1 is the value of the load corresponding to the deformation x_1.

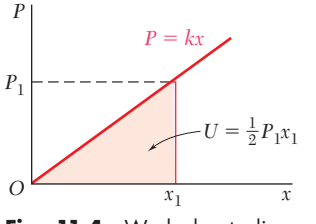


Fig. 11.4 Work due to linear, elastic deformation.

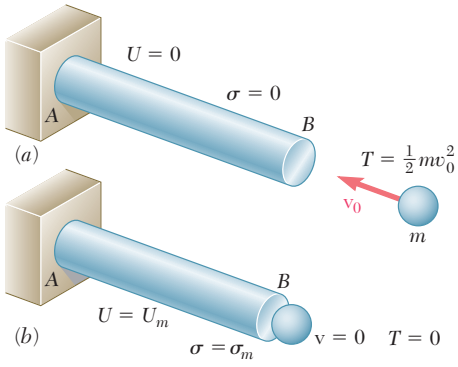


Fig. 11.5 Rod subject to impact loading.

The concept of strain energy is particularly useful in the determination of the effects of impact loadings on structures or machine components. Consider, for example, a body of mass m moving with a velocity $\mathbf{v}_0$ which strikes the end B of a rod AB (Fig. 11.5*a*). Neglecting the inertia of the elements of the rod, and assuming no dissipation of energy during the impact, we find that the maximum strain energy U_m acquired by the rod (Fig. 11.5*b*) is equal to the original kinetic energy $T = \frac{1}{2}mv_0^2$ of the moving body. We then determine the value P_m of the static load which would have produced the same strain energy in the rod, and obtain the value σ_m of the largest stress occurring in the rod by dividing P_m by the cross-sectional area of the rod.

11.3 STRAIN-ENERGY DENSITY

As we noted in Sec. 2.2, the load-deformation diagram for a rod BC depends upon the length L and the cross-sectional area A of the rod. The strain energy U defined by Eq. (11.2), therefore, will also depend upon the dimensions of the rod. In order to eliminate the effect of size from our discussion and direct our attention to the properties of the material, the strain energy per unit volume will be considered. Dividing the strain energy U by the volume $V = AL$ of the rod (Fig. 11.1), and using Eq. (11.2), we have

$$\frac{U}{V} = \int_0^{x_1} \frac{P}{A}\frac{dx}{L}$$

Recalling that P/A represents the normal stress σ_x in the rod, and x/L the normal strain ϵ_x, we write

$$\frac{U}{V} = \int_0^{\epsilon_1} \sigma_x \, d\epsilon_x$$

where ϵ_1 denotes the value of the strain corresponding to the elongation x_1. The strain energy per unit volume, U/V, is referred to as the *strain-energy density* and will be denoted by the letter u. We have, therefore,

$$\text{Strain-energy density} = u = \int_0^{\epsilon_1} \sigma_x \, d\epsilon_x \tag{11.4}$$

The strain-energy density u is expressed in units obtained by dividing units of energy by units of volume. Thus, if SI metric units are used, the strain-energy density is expressed in J/m^3 or its multiples kJ/m^3 and MJ/m^3; if U.S. customary units are used, it is expressed in in · lb/in^3.†

†We note that 1 J/m^3 and 1 Pa are both equal to 1 N/m^2, while 1 in · lb/in^3 and 1 psi are both equal to 1 lb/in^2. Thus, strain-energy density and stress are dimensionally equal and could be expressed in the same units.

Referring to Fig. 11.6, we note that the strain-energy density u is equal to the area under the stress-strain curve, measured from $\epsilon_x = 0$ to $\epsilon_x = \epsilon_1$. If the material is unloaded, the stress returns to zero, but there is a permanent deformation represented by the strain ϵ_p, and only the portion of the strain energy per unit volume corresponding to the triangular area is recovered. The remainder of the energy spent in deforming the material is dissipated in the form of heat.

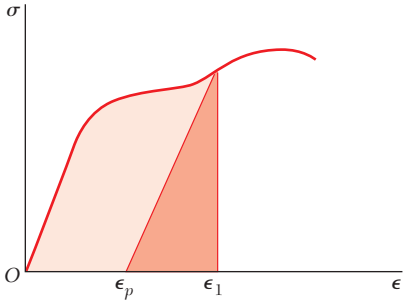


Fig. 11.6 Strain energy.

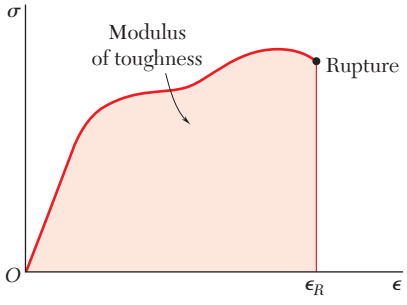


Fig. 11.7 Modulus of toughness.

The value of the strain-energy density obtained by setting $\epsilon_1 = \epsilon_R$ in Eq. (11.4), where ϵ_R is the strain at rupture, is known as the *modulus of toughness* of the material. It is equal to the area under the entire stress-strain diagram (Fig. 11.7) and represents the energy per unit volume required to cause the material to rupture. It is clear that the toughness of a material is related to its ductility as well as to its ultimate strength (Sec. 2.3), and that the capacity of a structure to withstand an impact load depends upon the toughness of the material used (Photo 11.1).

If the stress σ_x remains within the proportional limit of the material, Hooke's law applies and we write

$$\sigma_x = E\epsilon_x \tag{11.5}$$

Substituting for σ_x from (11.5) into (11.4), we have

$$u = \int_0^{\epsilon_1} E\epsilon_x \, d\epsilon_x = \frac{E\epsilon_1^2}{2} \tag{11.6}$$

or, using Eq. (11.5) to express ϵ_1 in terms of the corresponding stress σ_1,

$$u = \frac{\sigma_1^2}{2E} \tag{11.7}$$

The value u_Y of the strain-energy density obtained by setting $\sigma_1 = \sigma_Y$ in Eq. (11.7), where σ_Y is the yield strength, is called the *modulus of resilience* of the material. We have

$$u_Y = \frac{\sigma_Y^2}{2E} \tag{11.8}$$

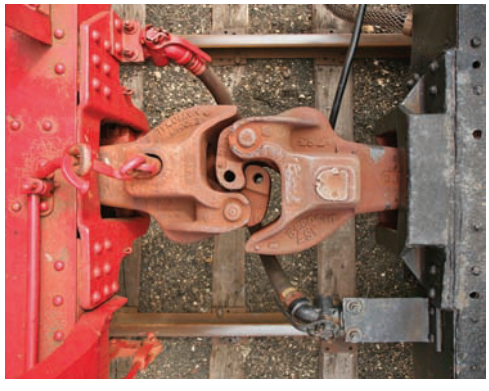


Photo 11.1 The railroad coupler is made of a ductile steel that has a large modulus of toughness.

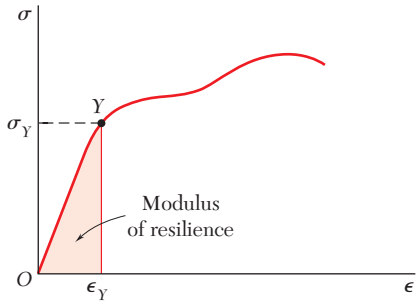


Fig. 11.8 Modulus of resilience.

The modulus of resilience is equal to the area under the straight-line portion OY of the stress-strain diagram (Fig. 11.8) and represents the energy per unit volume that the material can absorb without yielding. The capacity of a structure to withstand an impact load without being permanently deformed clearly depends upon the resilience of the material used.

Since the modulus of toughness and the modulus of resilience represent characteristic values of the strain-energy density of the material considered, they are both expressed in J/m^3 or its multiples if SI units are used, and in in · lb/in^3 if U.S. customary units are used.†

11.4 ELASTIC STRAIN ENERGY FOR NORMAL STRESSES

Since the rod considered in the preceding section was subjected to uniformly distributed stresses σ_x, the strain-energy density was constant throughout the rod and could be defined as the ratio U/V of the strain energy U and the volume V of the rod. In a structural element or machine part with a nonuniform stress distribution, the strain-energy density u can be defined by considering the strain energy of a small element of material of volume ΔV and writing

$$u = \lim_{\Delta V \to 0} \frac{\Delta U}{\Delta V}$$

or

$$u = \frac{dU}{dV} \tag{11.9}$$

The expression obtained for u in Sec. 11.3 in terms of σ_x and ϵ_x remains valid, i.e., we still have

$$u = \int_0^{\epsilon_x} \sigma_x \, d\epsilon_x \tag{11.10}$$

but the stress σ_x, the strain ϵ_x, and the strain-energy density u will generally vary from point to point.

For values of σ_x within the proportional limit, we may set $\sigma_x = E\epsilon_x$ in Eq. (11.10) and write

$$u = \frac{1}{2} E\epsilon_x^2 = \frac{1}{2}\sigma_x \epsilon_x = \frac{1}{2}\frac{\sigma_x^2}{E} \tag{11.11}$$

The value of the strain energy U of a body subjected to uniaxial normal stresses can be obtained by substituting for u from Eq. (11.11) into Eq. (11.9) and integrating both members. We have

$$U = \int \frac{\sigma_x^2}{2E} \, dV \tag{11.12}$$

The expression obtained is valid only for elastic deformations and is referred to as the *elastic strain energy* of the body.

†However, referring to the footnote on page 696, we note that the modulus of toughness and the modulus of resilience could be expressed in the same units as stress.

Strain Energy under Axial Loading. We recall from Sec. 2.17 that, when a rod is subjected to a centric axial loading, the normal stresses σ_x can be assumed uniformly distributed in any given transverse section. Denoting by A the area of the section located at a distance x from the end B of the rod (Fig. 11.9), and by P the internal force in that section, we write $\sigma_x = P/A$. Substituting for σ_x into Eq. (11.12), we have

$$U = \int \frac{P^2}{2EA^2}\,dV$$

or, setting $dV = A\,dx$,

$$U = \int_0^L \frac{P^2}{2AE}\,dx \tag{11.13}$$

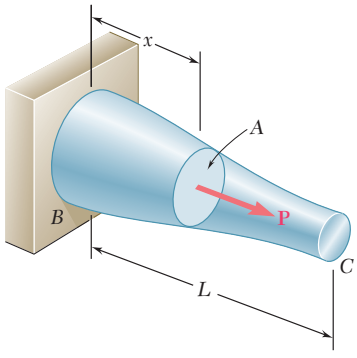


Fig. 11.9 Rod with centric axial load.

In the case of a rod of uniform cross section subjected at its ends to equal and opposite forces of magnitude P (Fig. 11.10), Eq. (11.13) yields

$$U = \frac{P^2L}{2AE} \tag{11.14}$$

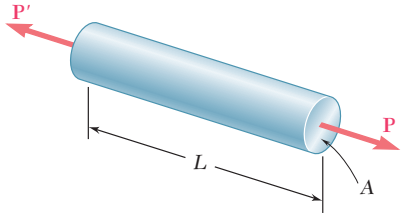


Fig. 11.10

EXAMPLE 11.01

A rod consists of two portions BC and CD of the same material and same length, but of different cross sections (Fig. 11.11). Determine the strain energy of the rod when it is subjected to a centric axial load **P,** expressing the result in terms of P, L, E, the cross-sectional area A of portion CD, and the ratio n of the two diameters.

We use Eq. (11.14) to compute the strain energy of each of the two portions, and add the expressions obtained:

$$U_n = \frac{P^2(\frac{1}{2}L)}{2AE} + \frac{P^2(\frac{1}{2}L)}{2(n^2A)E} = \frac{P^2L}{4AE}\left(1 + \frac{1}{n^2}\right)$$

or

$$U_n = \frac{1 + n^2}{2n^2}\frac{P^2L}{2AE} \tag{11.15}$$

We check that, for $n = 1$, we have

$$U_1 = \frac{P^2L}{2AE}$$

which is the expression given in Eq. (11.14) for a rod of length L and uniform cross section of area A. We also note that, for $n > 1$, we have $U_n < U_1$; for example, when $n = 2$, we have $U_2 = (\frac{5}{8})U_1$. Since the maximum stress occurs in portion CD of the rod and is equal to $\sigma_{max} = P/A$, it follows that, for a given allowable stress, increasing the diameter of portion BC of the rod results in a *decrease* of the overall energy-absorbing capacity of the rod. Unnecessary changes in cross-sectional area should therefore be avoided in the design of members that may be subjected to loadings, such as impact loadings, where the energy-absorbing capacity of the member is critical.

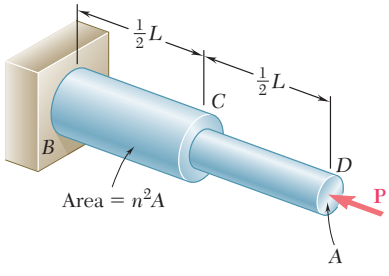


Fig. 11.11

EXAMPLE 11.02

A load **P** is supported at B by two rods of the same material and of the same uniform cross section of area A (Fig. 11.12). Determine the strain energy of the system.

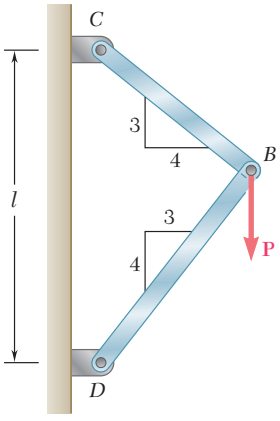


Fig. 11.12

Denoting by F_{BC} and F_{BD}, respectively, the forces in members BC and BD, and recalling Eq. (11.14), we express the strain energy of the system as

$$U = \frac{F_{BC}^2(BC)}{2AE} + \frac{F_{BD}^2(BD)}{2AE} \tag{11.16}$$

But we note from Fig. 11.12 that

$$BC = 0.6l \qquad BD = 0.8l$$

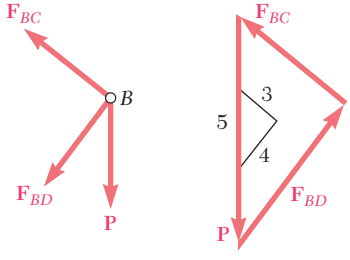


Fig. 11.13

and from the free-body diagram of pin B and the corresponding force triangle (Fig. 11.13) that

$$F_{BC} = +0.6P \qquad F_{BD} = -0.8P$$

Substituting into Eq. (11.16), we have

$$U = \frac{P^2l[(0.6)^3 + (0.8)^3]}{2AE} = 0.364\frac{P^2l}{AE}$$

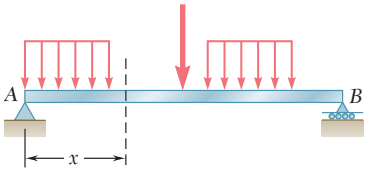


Fig. 11.14 Beam subject to transverse loads.

Strain Energy in Bending. Consider a beam AB subjected to a given loading (Fig. 11.14), and let M be the bending moment at a distance x from end A. Neglecting for the time being the effect of shear, and taking into account only the normal stresses $\sigma_x = My/I$, we substitute this expression into Eq. (11.12) and write

$$U = \int \frac{\sigma_x^2}{2E}\,dV = \int \frac{M^2y^2}{2EI^2}\,dV$$

Setting $dV = dA\,dx$, where dA represents an element of the cross-sectional area, and recalling that $M^2/2EI^2$ is a function of x alone, we have

$$U = \int_0^L \frac{M^2}{2EI^2}\left(\int y^2\,dA\right)dx$$

Recalling that the integral within the parentheses represents the moment of inertia I of the cross section about its neutral axis, we write

$$U = \int_0^L \frac{M^2}{2EI}\,dx \tag{11.17}$$

EXAMPLE 11.03

Determine the strain energy of the prismatic cantilever beam AB (Fig. 11.15), taking into account only the effect of the normal stresses.

The bending moment at a distance x from end A is $M = -Px$. Substituting this expression into Eq. (11.17), we write

$$U = \int_0^L \frac{P^2x^2}{2EI}\,dx = \frac{P^2L^3}{6EI}$$

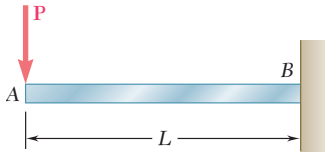


Fig. 11.15

11.5 ELASTIC STRAIN ENERGY FOR SHEARING STRESSES

When a material is subjected to plane shearing stresses τ_{xy}, the strain-energy density at a given point can be expressed as

$$u = \int_0^{\gamma_{xy}} \tau_{xy}\,d\gamma_{xy} \tag{11.18}$$

where γ_{xy} is the shearing strain corresponding to τ_{xy} (Fig. 11.16*a*). We note that the strain-energy density u is equal to the area under the shearing-stress-strain diagram (Fig. 11.16*b*).

For values of τ_{xy} within the proportional limit, we have $\tau_{xy} = G\gamma_{xy}$, where G is the modulus of rigidity of the material. Substituting for τ_{xy} into Eq. (11.18) and performing the integration, we write

$$u = \frac{1}{2}G\gamma_{xy}^2 = \frac{1}{2}\tau_{xy}\gamma_{xy} = \frac{\tau_{xy}^2}{2G} \tag{11.19}$$

The value of the strain energy U of a body subjected to plane shearing stresses can be obtained by recalling from Sec. 11.4 that

$$u = \frac{dU}{dV} \tag{11.9}$$

Substituting for u from Eq. (11.19) into Eq. (11.9) and integrating both members, we have

$$U = \int \frac{\tau_{xy}^2}{2G}\,dV \tag{11.20}$$

This expression defines the elastic strain associated with the shear deformations of the body. Like the similar expression obtained in Sec. 11.4 for uniaxial normal stresses, it is valid only for elastic deformations.

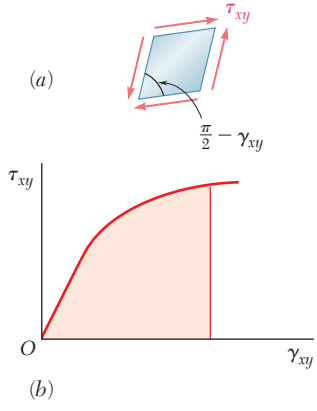


Fig. 11.16 Strain energy due to shear.

Strain Energy in Torsion. Consider a shaft BC of length L subjected to one or several twisting couples. Denoting by J the polar moment of inertia of the cross section located at a distance x from B (Fig. 11.17), and by T the internal torque in that section, we recall

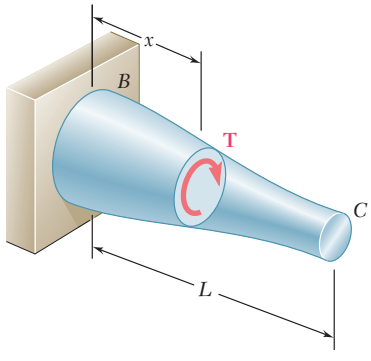


Fig. 11.17 Shaft subject to torque.

that the shearing stresses in the section are $\tau_{xy} = T\rho/J$. Substituting for τ_{xy} into Eq. (11.20), we have

$$U = \int \frac{\tau_{xy}^2}{2G}\,dV = \int \frac{T^2\rho^2}{2GJ^2}\,dV$$

Setting $dV = dA\,dx$, where dA represents an element of the cross-sectional area, and observing that $T^2/2GJ^2$ is a function of x alone, we write

$$U = \int_0^L \frac{T^2}{2GJ^2}\left(\int \rho^2\,dA\right)dx$$

Recalling that the integral within the parentheses represents the polar moment of inertia J of the cross section, we have

$$U = \int_0^L \frac{T^2}{2GJ}\,dx \tag{11.21}$$

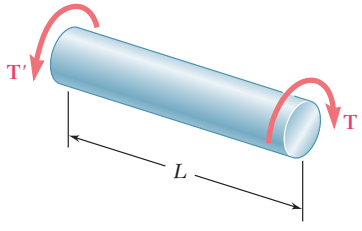


Fig. 11.18

In the case of a shaft of uniform cross section subjected at its ends to equal and opposite couples of magnitude T (Fig. 11.18), Eq. (11.21) yields

$$U = \frac{T^2L}{2GJ} \tag{11.22}$$

EXAMPLE 11.04

A circular shaft consists of two portions BC and CD of the same material and same length, but of different cross sections (Fig. 11.19). Determine the strain energy of the shaft when it is subjected to a twisting couple **T** at end D, expressing the result in terms of T, L, G, the polar moment of inertia J of the smaller cross section, and the ratio n of the two diameters.

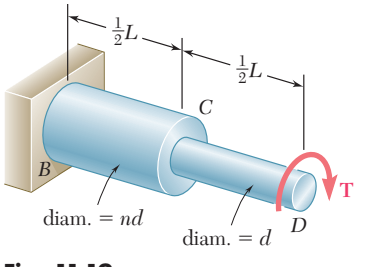


Fig. 11.19

We use Eq. (11.22) to compute the strain energy of each of the two portions of shaft, and add the expressions obtained. Noting that the polar moment of inertia of portion BC is equal to n^4J, we write

$$U_n = \frac{T^2(\frac{1}{2}L)}{2GJ} + \frac{T^2(\frac{1}{2}L)}{2G(n^4J)} = \frac{T^2L}{4GJ}\left(1 + \frac{1}{n^4}\right)$$

or

$$U_n = \frac{1 + n^4}{2n^4}\frac{T^2L}{2GJ} \tag{11.23}$$

We check that, for $n = 1$, we have

$$U_1 = \frac{T^2L}{2GJ}$$

which is the expression given in Eq. (11.22) for a shaft of length L and uniform cross section. We also note that, for $n > 1$, we have $U_n < U_1$; for example, when $n = 2$, we have $U_2 = (\frac{17}{32})U_1$. Since the maximum shearing stress occurs in the portion CD of the shaft and is proportional to the torque T, we note as we did earlier in the case of the axial loading of a rod that, for a given allowable stress, increasing the diameter of portion BC of the shaft results in a *decrease* of the overall energy-absorbing capacity of the shaft.

Strain Energy under Transverse Loading. In Sec. 11.4 we obtained an expression for the strain energy of a beam subjected to a transverse loading. However, in deriving that expression we took into account only the effect of the normal stresses due to bending and neglected the effect of the shearing stresses. In Example 11.05 both types of stresses will be taken into account.

EXAMPLE 11.05

Determine the strain energy of the rectangular cantilever beam AB (Fig. 11.20), taking into account the effect of both normal and shearing stresses.

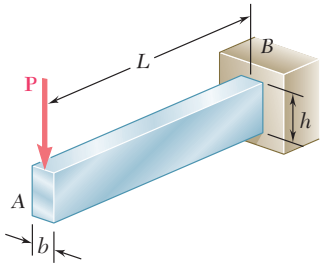


Fig. 11.20

We first recall from Example 11.03 that the strain energy due to the normal stresses σ_x is

$$U_\sigma = \frac{P^2L^3}{6EI}$$

To determine the strain energy U_τ due to the shearing stresses τ_{xy}, we recall Eq. (6.9) of Sec. 6.4 and find that, for a beam with a rectangular cross section of width b and depth h,

$$\tau_{xy} = \frac{3}{2}\frac{V}{A}\left(1 - \frac{y^2}{c^2}\right) = \frac{3}{2}\frac{P}{bh}\left(1 - \frac{y^2}{c^2}\right)$$

Substituting for τ_{xy} into Eq. (11.20), we write

$$U_\tau = \frac{1}{2G}\left(\frac{3}{2}\frac{P}{bh}\right)^2 \int\left(1 - \frac{y^2}{c^2}\right)^2 dV$$

or, setting $dV = b\,dy\,dx$, and after reductions,

$$U_\tau = \frac{9P^2}{8Gbh^2}\int_{-c}^{c}\left(1 - 2\frac{y^2}{c^2} + \frac{y^4}{c^4}\right)dy\int_0^L dx$$

Performing the integrations, and recalling that $c = h/2$, we have

$$U_\tau = \frac{9P^2L}{8Gbh^2}\left[y - \frac{2}{3}\frac{y^3}{c^2} + \frac{1}{5}\frac{y^5}{c^4}\right]_{-c}^{+c} = \frac{3P^2L}{5Gbh} = \frac{3P^2L}{5GA}$$

The total strain energy of the beam is thus

$$U = U_\sigma + U_\tau = \frac{P^2L^3}{6EI} + \frac{3P^2L}{5GA}$$

or, noting that $I/A = h^2/12$ and factoring the expression for U_σ,

$$U = \frac{P^2L^3}{6EI}\left(1 + \frac{3Eh^2}{10GL^2}\right) = U_\sigma\left(1 + \frac{3Eh^2}{10GL^2}\right) \qquad (11.24)$$

Recalling from Sec. 2.14 that $G \geq E/3$, we conclude that the parenthesis in the expression obtained is less than $1 + 0.9(h/L)^2$ and, thus, that the relative error is less than $0.9(h/L)^2$ when the effect of shear is neglected. For a beam with a ratio h/L less than $\frac{1}{10}$, the percentage error is less than 0.9%. It is therefore customary in engineering practice to neglect the effect of shear in computing the strain energy of slender beams.

11.6 STRAIN ENERGY FOR A GENERAL STATE OF STRESS

In the preceding sections, we determined the strain energy of a body in a state of uniaxial stress (Sec. 11.4) and in a state of plane shearing stress (Sec. 11.5). In the case of a body in a general state of stress characterized by the six stress components σ_x, σ_y, σ_z, τ_{xy}, τ_{yz}, and τ_{zx}, the strain-energy density can be obtained by adding the expressions given in Eqs. (11.10) and (11.18), as well as the four other expressions obtained through a permutation of the subscripts.

In the case of the elastic deformation of an isotropic body, each of the six stress-strain relations involved is linear, and the strain-energy density can be expressed as

$$u = \tfrac{1}{2}(\sigma_x\epsilon_x + \sigma_y\epsilon_y + \sigma_z\epsilon_z + \tau_{xy}\gamma_{xy} + \tau_{yz}\gamma_{yz} + \tau_{zx}\gamma_{zx}) \tag{11.25}$$

Recalling the relations (2.38) obtained in Sec. 2.14, and substituting for the strain components into (11.25), we have, for the most general state of stress at a given point of an elastic isotropic body,

$$u = \frac{1}{2E}[\sigma_x^2 + \sigma_y^2 + \sigma_z^2 - 2\nu(\sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x)] + \frac{1}{2G}(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2) \tag{11.26}$$

If the principal axes at the given point are used as coordinate axes, the shearing stresses become zero and Eq. (11.26) reduces to

$$u = \frac{1}{2E}[\sigma_a^2 + \sigma_b^2 + \sigma_c^2 - 2\nu(\sigma_a\sigma_b + \sigma_b\sigma_c + \sigma_c\sigma_a)] \tag{11.27}$$

where σ_a, σ_b, and σ_c are the principal stresses at the given point.

We now recall from Sec. 7.7 that one of the criteria used to predict whether a given state of stress will cause a ductile material to yield, namely, the maximum-distortion-energy criterion, is based on the determination of the energy per unit volume associated with the distortion, or change in shape, of that material. Let us, therefore, attempt to separate the strain-energy density u at a given point into two parts, a part u_v associated with a change in volume of the material at that point, and a part u_d associated with a distortion, or change in shape, of the material at the same point. We write

$$u = u_v + u_d \tag{11.28}$$

In order to determine u_v and u_d, we introduce the *average value* $\overline{\sigma}$ of the principal stresses at the point considered,

$$\overline{\sigma} = \frac{\sigma_a + \sigma_b + \sigma_c}{3} \tag{11.29}$$

and set

$$\sigma_a = \overline{\sigma} + \sigma_a' \qquad \sigma_b = \overline{\sigma} + \sigma_b' \qquad \sigma_c = \overline{\sigma} + \sigma_c' \tag{11.30}$$

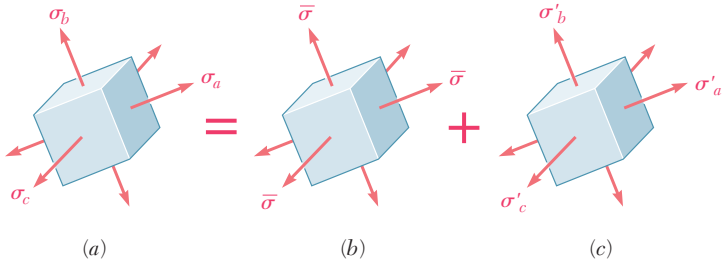


Fig. 11.21 Element subject to multiaxial stress.

Thus, the given state of stress (Fig. 11.21*a*) can be obtained by superposing the states of stress shown in Fig. 11.21*b* and *c*. We note that the state of stress described in Fig. 11.21*b* tends to change the volume of the element of material, but not its shape, since all the faces of the element are subjected to the same stress $\overline{\sigma}$. On the other hand, it follows from Eqs. (11.29) and (11.30) that

$$\sigma'_a + \sigma'_b + \sigma'_c = 0 \tag{11.31}$$

which indicates that some of the stresses shown in Fig. 11.21*c* are tensile and others compressive. Thus, this state of stress tends to change the shape of the element. However, it does not tend to change its volume. Indeed, recalling Eq. (2.31) of Sec. 2.13, we note that the dilatation e (i.e., the change in volume per unit volume) caused by this state of stress is

$$e = \frac{1 - 2\nu}{E}(\sigma'_a + \sigma'_b + \sigma'_c)$$

or $e = 0$, in view of Eq. (11.31). We conclude from these observations that the portion u_v of the strain-energy density must be associated with the state of stress shown in Fig. 11.21*b*, while the portion u_d must be associated with the state of stress shown in Fig. 11.21*c*.

It follows that the portion u_v of the strain-energy density corresponding to a change in volume of the element can be obtained by substituting $\overline{\sigma}$ for each of the principal stresses in Eq. (11.27). We have

$$u_v = \frac{1}{2E}[3\overline{\sigma}^2 - 2\nu(3\overline{\sigma}^2)] = \frac{3(1 - 2\nu)}{2E}\overline{\sigma}^2$$

or, recalling Eq. (11.29),

$$u_v = \frac{1 - 2\nu}{6E}(\sigma_a + \sigma_b + \sigma_c)^2 \tag{11.32}$$

The portion of the strain-energy density corresponding to the distortion of the element is obtained by solving Eq. (11.28) for u_d

and substituting for u and u_v from Eqs. (11.27) and (11.32), respectively. We write

$$u_d = u - u_v = \frac{1}{6E}\left[3(\sigma_a^2 + \sigma_b^2 + \sigma_c^2) - 6\nu(\sigma_a\sigma_b + \sigma_b\sigma_c + \sigma_c\sigma_a) - (1 - 2\nu)(\sigma_a + \sigma_b + \sigma_c)^2\right]$$

Expanding the square and rearranging terms, we have

$$u_d = \frac{1 + \nu}{6E}\left[(\sigma_a^2 - 2\sigma_a\sigma_b + \sigma_b^2) + (\sigma_b^2 - 2\sigma_b\sigma_c + \sigma_c^2) + (\sigma_c^2 - 2\sigma_c\sigma_a + \sigma_a^2)\right]$$

Noting that each of the parentheses inside the bracket is a perfect square, and recalling from Eq. (2.43) of Sec. 2.15 that the coefficient in front of the bracket is equal to $1/12G$, we obtain the following expression for the portion u_d of the strain-energy density, i.e., for the distortion energy per unit volume,

$$u_d = \frac{1}{12G}\left[(\sigma_a - \sigma_b)^2 + (\sigma_b - \sigma_c)^2 + (\sigma_c - \sigma_a)^2\right] \qquad (11.33)$$

In the case of *plane stress*, and assuming that the c axis is perpendicular to the plane of stress, we have $\sigma_c = 0$ and Eq. (11.33) reduces to

$$u_d = \frac{1}{6G}(\sigma_a^2 - \sigma_a\sigma_b + \sigma_b^2) \qquad (11.34)$$

Considering the particular case of a tensile-test specimen, we note that, at yield, we have $\sigma_a = \sigma_Y$, $\sigma_b = 0$, and thus $(u_d)_Y = \sigma_Y^2/6G$. The maximum-distortion-energy criterion for plane stress indicates that a given state of stress is safe as long as $u_d < (u_d)_Y$ or, substituting for u_d from Eq. (11.34), as long as

$$\sigma_a^2 - \sigma_a\sigma_b + \sigma_b^2 < \sigma_Y^2 \qquad (7.26)$$

which is the condition stated in Sec. 7.7 and represented graphically by the ellipse of Fig. 7.39. In the case of a general state of stress, the expression (11.33) obtained for u_d should be used. The maximum-distortion-energy criterion is then expressed by the condition.

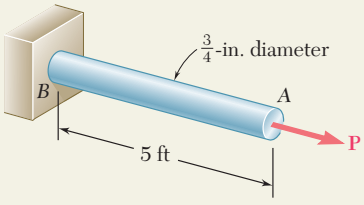
$$(\sigma_a - \sigma_b)^2 + (\sigma_b - \sigma_c)^2 + (\sigma_c - \sigma_a)^2 < 2\sigma_Y^2 \qquad (11.35)$$

which indicates that a given state of stress is safe if the point of coordinates σ_a, σ_b, σ_c is located within the surface defined by the equation

$$(\sigma_a - \sigma_b)^2 + (\sigma_b - \sigma_c)^2 + (\sigma_c - \sigma_a)^2 = 2\sigma_Y^2 \qquad (11.36)$$

This surface is a circular cylinder of radius $\sqrt{2/3}\,\sigma_Y$ with an axis of symmetry forming equal angles with the three principal axes of stress.

SAMPLE PROBLEM 11.1



During a routine manufacturing operation, rod *AB* must acquire an elastic strain energy of 120 in · lb. Using $E = 29 \times 10^6$ psi, determine the required yield strength of the steel if the factor of safety with respect to permanent deformation is to be five.

SOLUTION

Factor of Safety. Since a factor of safety of five is required, the rod should be designed for a strain energy of

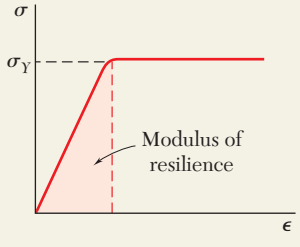
$$U = 5(120 \text{ in} \cdot \text{lb}) = 600 \text{ in} \cdot \text{lb}$$

Strain-Energy Density. The volume of the rod is

$$V = AL = \frac{\pi}{4}(0.75 \text{ in.})^2(60 \text{ in.}) = 26.5 \text{ in}^3$$

Since the rod is of uniform cross section, the required strain-energy density is

$$u = \frac{U}{V} = \frac{600 \text{ in} \cdot \text{lb}}{26.5 \text{ in}^3} = 22.6 \text{ in} \cdot \text{lb/in}^3$$

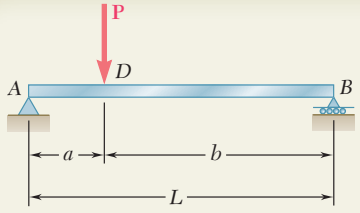


Yield Strength. We recall that the modulus of resilience is equal to the strain-energy density when the maximum stress is equal to σ_Y. Using Eq. (11.8), we write

$$u = \frac{\sigma_Y^2}{2E}$$

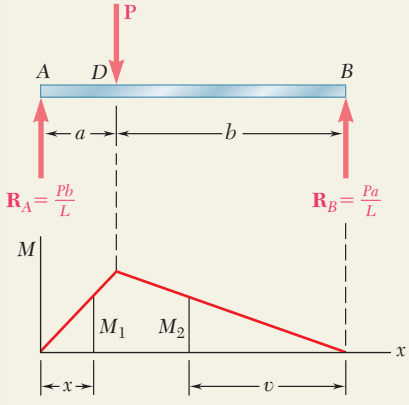
$$22.6 \text{ in} \cdot \text{lb/in}^3 = \frac{\sigma_Y^2}{2(29 \times 10^6 \text{ psi})} \qquad \sigma_Y = 36.2 \text{ ksi}$$

Comment. It is important to note that, since energy loads are not linearly related to the stresses they produce, factors of safety associated with energy loads should be applied to the energy loads and not to the stresses.



SAMPLE PROBLEM 11.2

(*a*) Taking into account only the effect of normal stresses due to bending, determine the strain energy of the prismatic beam *AB* for the loading shown. (*b*) Evaluate the strain energy, knowing that the beam is a W10 × 45, $P = 40$ kips, $L = 12$ ft, $a = 3$ ft, $b = 9$ ft, and $E = 29 \times 10^6$ psi.



SOLUTION

Bending Moment. Using the free-body diagram of the entire beam, we determine the reactions

$$\mathbf{R}_A = \frac{Pb}{L}\uparrow \qquad \mathbf{R}_B = \frac{Pa}{L}\uparrow$$

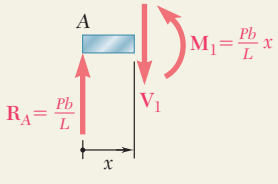
For portion *AD* of the beam, the bending moment is

$$M_1 = \frac{Pb}{L}x$$

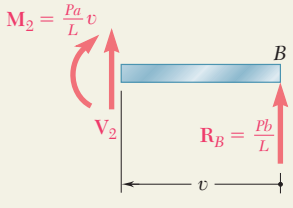
For portion *DB*, the bending moment at a distance v from end *B* is

$$M_2 = \frac{Pa}{L}v$$

From *A* to *D*:



From *B* to *D*:



a. Strain Energy. Since strain energy is a scalar quantity, we add the strain energy of portion *AD* to that of portion *DB* to obtain the total strain energy of the beam. Using Eq. (11.17), we write

$$\begin{aligned} U &= U_{AD} + U_{DB} \\ &= \int_0^a \frac{M_1^2}{2EI}\,dx + \int_0^b \frac{M_2^2}{2EI}\,dv \\ &= \frac{1}{2EI}\int_0^a \left(\frac{Pb}{L}x\right)^2 dx + \frac{1}{2EI}\int_0^b \left(\frac{Pa}{L}v\right)^2 dv \\ &= \frac{1}{2EI}\frac{P^2}{L^2}\left(\frac{b^2a^3}{3} + \frac{a^2b^3}{3}\right) = \frac{P^2a^2b^2}{6EIL^2}(a+b) \end{aligned}$$

or, since $(a + b) = L$,

$$U = \frac{P^2a^2b^2}{6EIL}$$

b. Evaluation of the Strain Energy. The moment of inertia of a W10 × 45 rolled-steel shape is obtained from Appendix C and the given data is restated using units of kips and inches.

$P = 40$ kips $\qquad L = 12$ ft $= 144$ in.
$a = 3$ ft $= 36$ in. $\qquad b = 9$ ft $= 108$ in.
$E = 29 \times 10^6$ psi $= 29 \times 10^3$ ksi $\qquad I = 248$ in^4

Substituting into the expression for U, we have

$$U = \frac{(40\text{ kips})^2(36\text{ in.})^2(108\text{ in.})^2}{6(29 \times 10^3\text{ ksi})(248\text{ in}^4)(144\text{ in.})} \qquad U = 3.89\text{ in} \cdot \text{kips}$$