where the constants are placed in front of the second integral, which is equal to zero because, by definition, a product of inertia for a principal axis vanishes.

By virtue of the foregoing, the restriction placed on the elastic flexure formula at the beginning of the chapter limiting it to applications for symmetric cross sections can be removed. However, in the application of Eq. 9-1, *the principal axes for a cross section must be used.* A procedure for bypassing this requirement is given in Section 9-8.

Example 9-1

The 100-by-150-mm wooden beam shown in Fig. 9-4(a) is used to support a uniformly distributed load of 4 kN (total) on a simple span of 3 m. The applied load acts in a plane making an angle of 30° with the vertical, as shown in Fig. 9-4(b) and again in Fig. 9-4(c). Calculate the maximum bending stress at midspan, and, for the same section, locate the neutral axis. Neglect the weight of the beam.

SOLUTION

The maximum bending *in the plane of the applied load* occurs at midspan, and, according to Example 7-8, it is equal to $w_o L^2/8$ or $WL/8$, where W is the total load on span L. Hence,

$$M = \frac{WL}{8} = \frac{4 \times 3}{8} = 1.5 \text{ kN} \cdot \text{m}$$

Next, this moment is resolved into components acting around the respective axes, and I_z and I_y are calculated:

$$M_z = M \cos \alpha = 1.5 \times \sqrt{3}/2 = 1.3 \text{ kN} \cdot \text{m}$$

$$M_y = M \sin \alpha = 1.5 \times 0.5 = 0.75 \text{ kN} \cdot \text{m}$$

$$I_z = 100 \times 150^3/12 = 28.1 \times 10^6 \text{ mm}^4$$

$$I_y = 150 \times 100^3/12 = 12.5 \times 10^6 \text{ mm}^4$$

By considering the sense of the moment components, it can be concluded that the maximum tensile stress occurs at A. Similar reasoning applies when considering the other corner points. Alternatively, the values for the coordinate points can be substituted directly into Eq. 9-1. On either basis,

$$\sigma_A = -\frac{M_z(-c_1)}{I_z} + \frac{M_y c_2}{I_y} = \frac{1.3 \times 10^6 \times 75}{28.1 \times 10^6} + \frac{0.75 \times 10^6 \times 50}{12.5 \times 10^6}$$

$$= +3.47 + 3.00 = +6.47 \text{ MPa}$$

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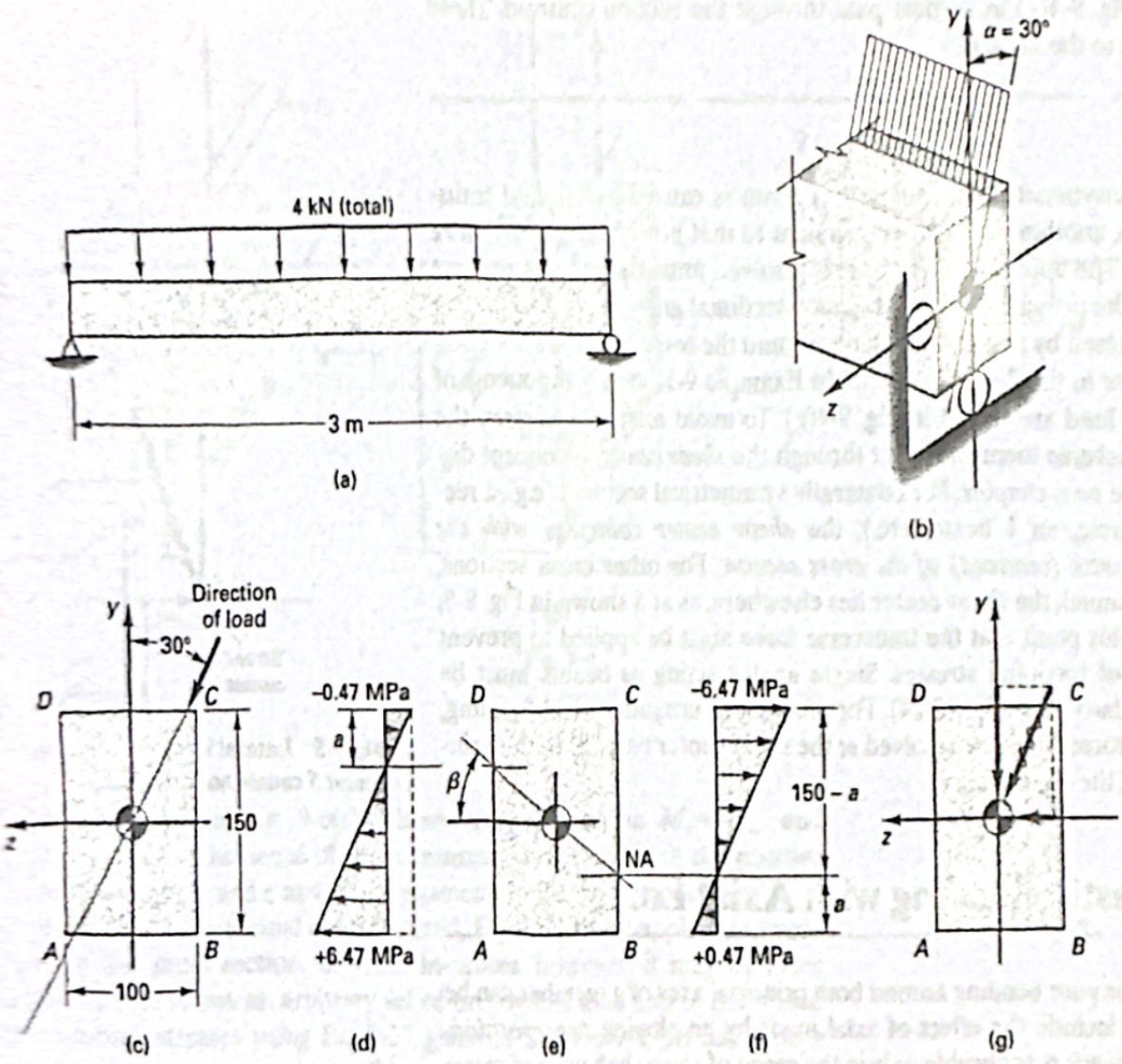


Fig. 9-4

$$\sigma_B = +3.47 - 3.00 = +0.47 \text{ MPa}$$

$$\sigma_C = -3.47 - 3.00 = -6.47 \text{ MPa}$$

$$\sigma_D = -3.47 + 3.00 = -0.47 \text{ MPa}$$

Note that the stress magnitudes on diametrically opposite corners are numerically equal.

The neutral axis is located by the angle β, using Eq. 9-3:

$$\tan \beta = \frac{28.1 \times 10^6}{12.5 \times 10^6} \tan 30° = 1.30 \quad \text{or} \quad \beta = 52.4°$$

Alternatively, it can be found from the stress distribution, which varies linearly between any two points. For example, from similar triangles, $a/(150 - a) = 0.47/6.47$, giving $a = 10.2$ mm. This locates the neutral axis

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plane. The same is true for *elastic* stresses. These inclined planes intersect the reference plane in a line. This line of *zero stress* or *strain* is analogous to the neutral axis occurring in pure bending. Unlike the former case, however, when $P \neq 0$, this line does not pass through the centroid of a section. For large axial forces and small bending moments, the line of zero stress or strain may lie outside a cross section. The significance of this line is that the normal stresses or strains vary from it linearly.

It should be noted that in many instances, the bending moment in a member is caused by transverse forces rather than by an eccentrically applied axial force, such as illustrated in Fig. 9-6. In such cases, Eq. 9-5 remains applicable.

Several illustrative examples follow, beginning with situations where bending takes place only around one of the principal axes.

Example 9-2

A 50-by-75-mm, 1.5-m-long elastic bar of negligible weight is loaded as shown in mm in Fig. 9-7(a). Determine the maximum tensile and compressive stresses acting normal to the section through the beam.

SOLUTION

To emphasize the method of superposition, this problem is solved by dividing it into two parts. In Fig. 9-7(b), the bar is shown subjected only to the axial force, and in Fig. 9-7(c) the same bar is shown subjected only to the transverse force. For the axial force, the normal stress throughout the length of the bar is

$$\sigma = \frac{P}{A} = \frac{25 \times 10^3}{50 \times 75} = 6.67 \text{ MPa} \quad \text{(tension)}$$

This result is indicated in Fig. 9-7(d). The normal stresses due to the transverse force depend on the magnitude of the bending moment, and the maximum bending moment occurs at the applied force. As the left reaction is 2.7 kN, $M_{max} = 2.7 \times 10^3 \times 375 = 1.013 \times 10^6$ N · mm. From the flexure formula, the maximum stresses at the extreme fibers caused by this moment are

$$\sigma = \frac{Mc}{I} = \frac{6M}{bh^2} = \frac{61 \times 1.013 \times 10^6}{50 \times 75^2} = \pm 21.6 \text{ MPa}$$

These stresses act normal to the section of the beam and decrease linearly toward the neutral axis, as in Fig. 9-7(e). Then, to obtain the compound stress for any particular element, bending stresses must be added algebraically to the direct tensile stress. Thus, as may be seen from Fig. 9-7(f), at point A, the resultant normal stress is 14.9 MPa compression, and at B, it is

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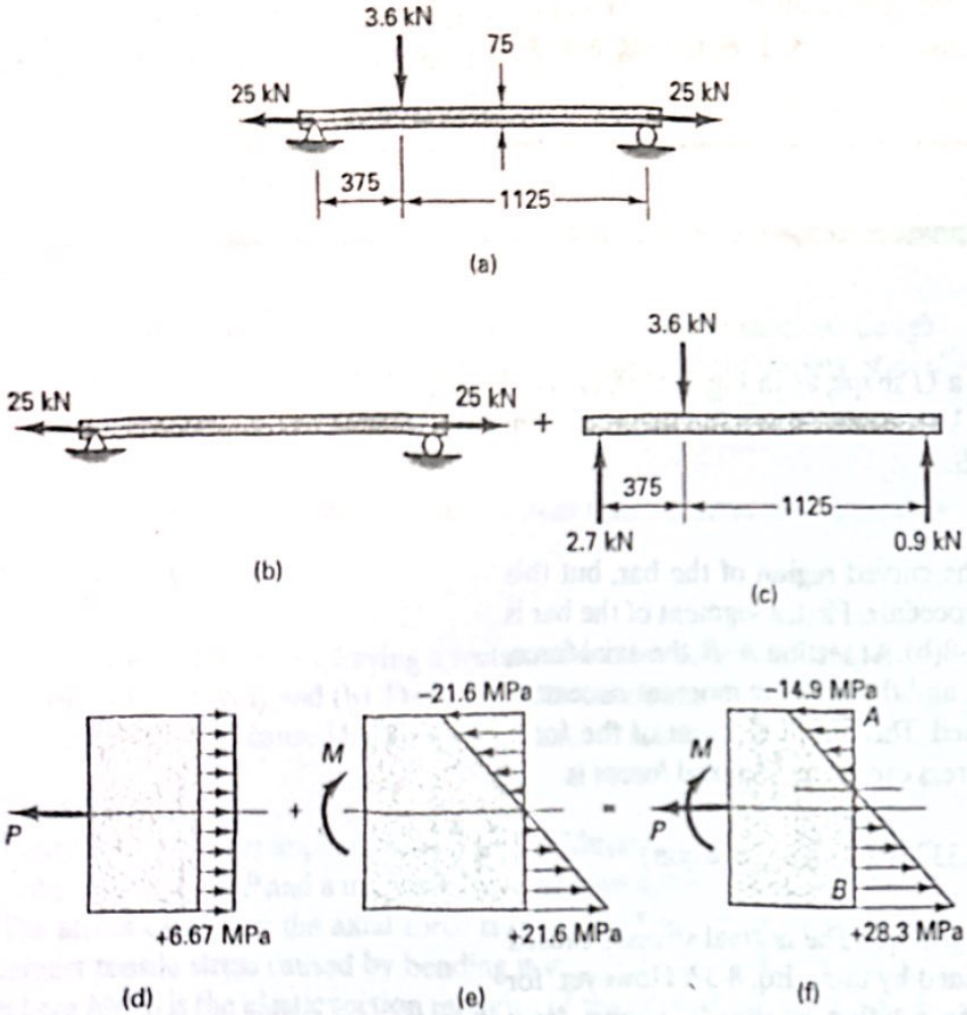


Fig. 9-7

28.3 MPa tension. Side views of the stress vectors as commonly drawn are shown in the figure.

Although in this problem the given axial force is larger than the transverse force, bending causes higher stresses. However, the reader is cautioned not to regard slender compression members in the same light.

Note that in the final result, the line of zero stress, which is located at the centroid of the section for flexure, moves upward. Also note that the local stresses, caused by the concentrated force, which act normal to the top surface of the beam, were not considered. Generally, these stresses are treated independently as local bearing stresses.

The stress distribution shown in Fig. 9-7(f) would change if instead of the axial tensile forces applied at the ends, compressive forces of the same magnitude were acting on the member. The maximum tensile stress would be reduced to 14.9 MPa from 28.3 MPa, which would be desirable in a beam made of a material weak in tension and carrying a transverse load. This idea is utilized in prestressed construction. Tendons made of high-strength steel rods or cable passing through a beam with anchorages at the

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ends are used to precompress concrete beams. Such artificially applied forces inhibit the development of tensile stresses. Prestressing also has been used in racing-car frames.

Example 9-3

A 50-by-50-mm elastic bar bent into a U shape, as in Fig. 9-8(a), is acted upon by two opposing forces P of 8.33 kN each. Determine the maximum normal stress occurring at section A–B.

SOLUTION

The section to be investigated is in the curved region of the bar, but this makes no essential difference in the procedure. First, a segment of the bar is taken as a free body, as shown in Fig. 9-8(b). At section A–B, the axial force, applied at the centroid of the section, and the bending moment necessary to maintain equilibrium are determined. Then, each element of the force system is considered separately. The stress caused by the axial forces is

$$\sigma = \frac{P}{A} = \frac{8.33 \times 10^3}{50 \times 50} = 3.33 \text{ MPa} \quad \text{(compression)}$$

and is shown in the first diagram of Fig. 9-8(c). The normal stresses caused by the bending moment may be obtained by using Eq. 8-37. However, for this bar, bent to a 75-mm radius, the solution is already known from Example 8-14. The stress distribution corresponding to this case is shown in the second diagram of Fig. 9-8(c). By superposing the results of these two solutions, the compound stress distribution is obtained. This is shown in the

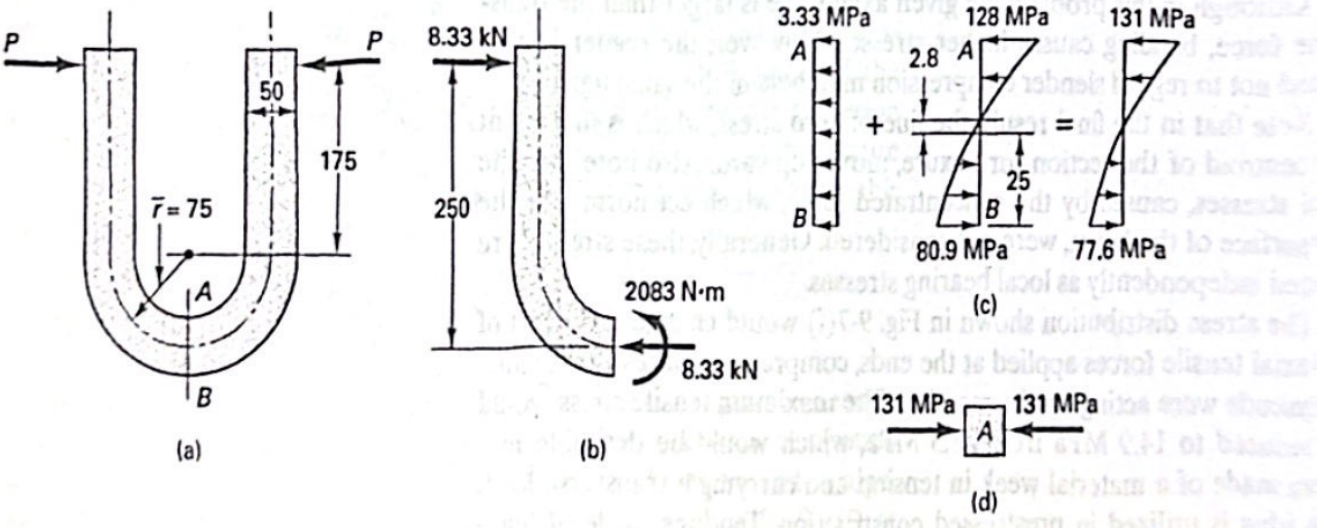


Fig. 9-8

third diagram of Fig. 9-8(c). The maximum compressive stress occurs at A and is 131 MPa. An isolated element for point A is shown in Fig. 9-8(d). Shear stresses are absent at section A–B as no shear force is necessary to maintain equilibrium of the segment shown in Fig. 9-8(b). The relative insignificance of the stress caused by the axial force is striking.

Problems similar to the preceding commonly occur in machine design. Hooks, C clamps, frames of punch presses, etc. illustrate the variety of situations to which the foregoing methods of analysis must be applied.

Example 9-4

Consider a tapered block having a rectangular cross section at the base, as shown in Figs. 9-9(a) and (b). Determine the maximum eccentricity e such that the stress at B caused by the applied force P is zero.

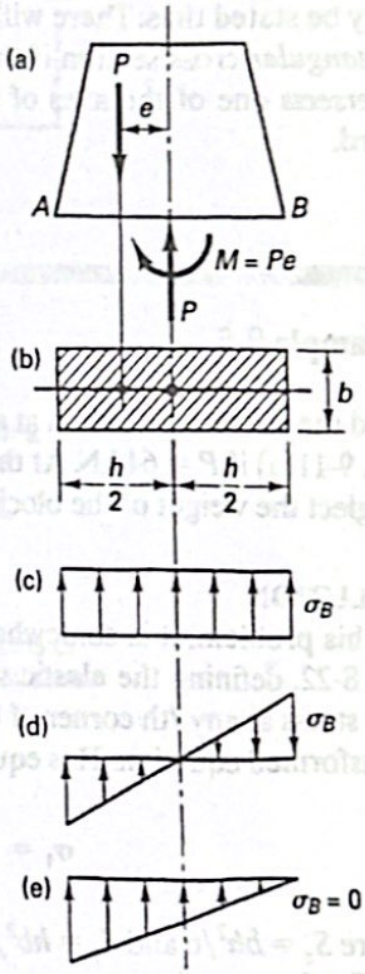


Fig. 9-9 Location of force P causing zero stress at B.

SOLUTION

In order to maintain applied force P in equilibrium, there must be an axial compressive force P and a moment Pe at the base having the senses shown. The stress caused by the axial force is $\sigma = -P/A = -P/bh$, whereas the largest tensile stress caused by bending is $\sigma_{max} = Mc/I = M/S = 6Pe/bh^2$, where $bh^2/6$ is the elastic section modulus of the rectangular cross section. To satisfy the condition for having stress at B equal to zero, it follows that

$$\sigma_B = -\frac{P}{bh} + \frac{6Pe}{bh^2} = 0 \quad \text{or} \quad e = \frac{h}{6}$$

which means that if force P is applied at a distance of $h/6$ from the centroidal axis of the cross section, the stress at B is just zero. Stress distributions across the base corresponding, respectively, to the axial force and bending moment are shown in Figs. 9-9(c) and (d), and their algebraic sum in Fig. 9-9(e).

In the preceding problem, if force P were applied closer to the centroid of the section, a smaller bending moment would be developed at a section A–B, and there would be some compression stress at B. The same argument may be repeated for the force acting to the right of the centroidal axis. Hence, a practical rule, much used by the early designers of masonry structures, may be formulated thus: *If the resultant of all vertical forces acts within the middle third of the rectangular cross section, there is no tension in the material at that section.* It is understood that the resultant acts in a vertical plane containing one of the axes of symmetry of the rectangular cross-sectional area.

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The foregoing discussion may be generalized in order to apply to any planar system of forces acting on a member. The resultant of these forces may be made to intersect the plane of the cross section, as is shown in Fig. 9-10. At the point of intersection of this resultant with the section, it may be resolved into horizontal and vertical components. If the vertical component of the resultant fulfills the conditions of the former problem, no tension will be developed at point B, as the horizontal component causes only shear stresses. Hence, a more general "middle-third" rule may be stated thus: There will be no tension at a section of a member of a *rectangular* cross section if the resultant of the forces above this section *intersects* one of the axes of symmetry of the section within the middle third.

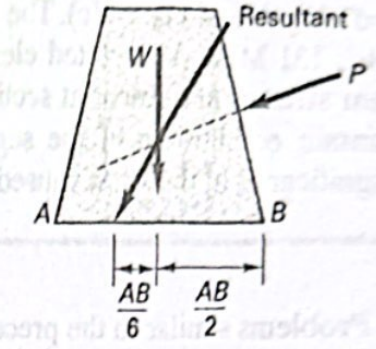


Fig. 9-10 Resultant causing no tension at B.

Example 9-5

Find the stress distribution at section $ABCD$ for the block shown in mm in Fig. 9-11(a) if $P = 64$ kN. At the same section, locate the line of zero stress. Neglect the weight of the block.

SOLUTION

In this problem, it is somewhat simpler to recast Eq. 9-5 with the aid of Eq. 8-22, defining the elastic section modulus $S = I/c$ as $bh^2/6$. The normal stress at any Ith corner of the block can be found directly from such a transformed equation. This equation reads

$$\sigma_I = \frac{P}{A} - \frac{M_z}{S_z} + \frac{M_y}{S_y} \qquad (9\text{-}8)$$

where $S_z = bh^2/6$ and $S_y = hb^2/6$.

The forces acting on section $ABCD$, Fig. 9-11(c), are $P = -64 \times 10^3$ N, $M_y = -64 \times 10^3 \times 150 = -9.6 \times 10^6$ N · mm, and $M_z = -64 \times 10^3 \times (75 + 75) = -9.6 \times 10^6$ N · mm. The cross-sectional area has the following properties: $A = 150 \times 300 = 45 \times 10^3$ mm^2, $S_z = 300 \times 150^2/6 = 1.125 \times 10^6$ mm^3, and $S_y = 150 \times 300^2/6 = 2.25 \times 10^6$ mm^3.

The normal stresses at the corners are found using Eq. 9-8, assigning signs for the stresses caused by moments by inspection. For example, from Fig. 9-11(c), it can be seen that due to M_y, the stresses at corners A and D are compressive. Other cases are treated similarly. Using this approach,

$$\sigma_A = -\frac{64 \times 10^3}{45 \times 10^3} - \frac{9.6 \times 10^6}{1.125 \times 10^6} - \frac{9.6 \times 10^6}{2.25 \times 10^6}$$

$$= -1.42 - 8.53 - 4.27 = -14.2 \text{ MPa}$$

$$\sigma_B = -1.42 - 8.53 + 4.27 = -5.7 \text{ MPa}$$

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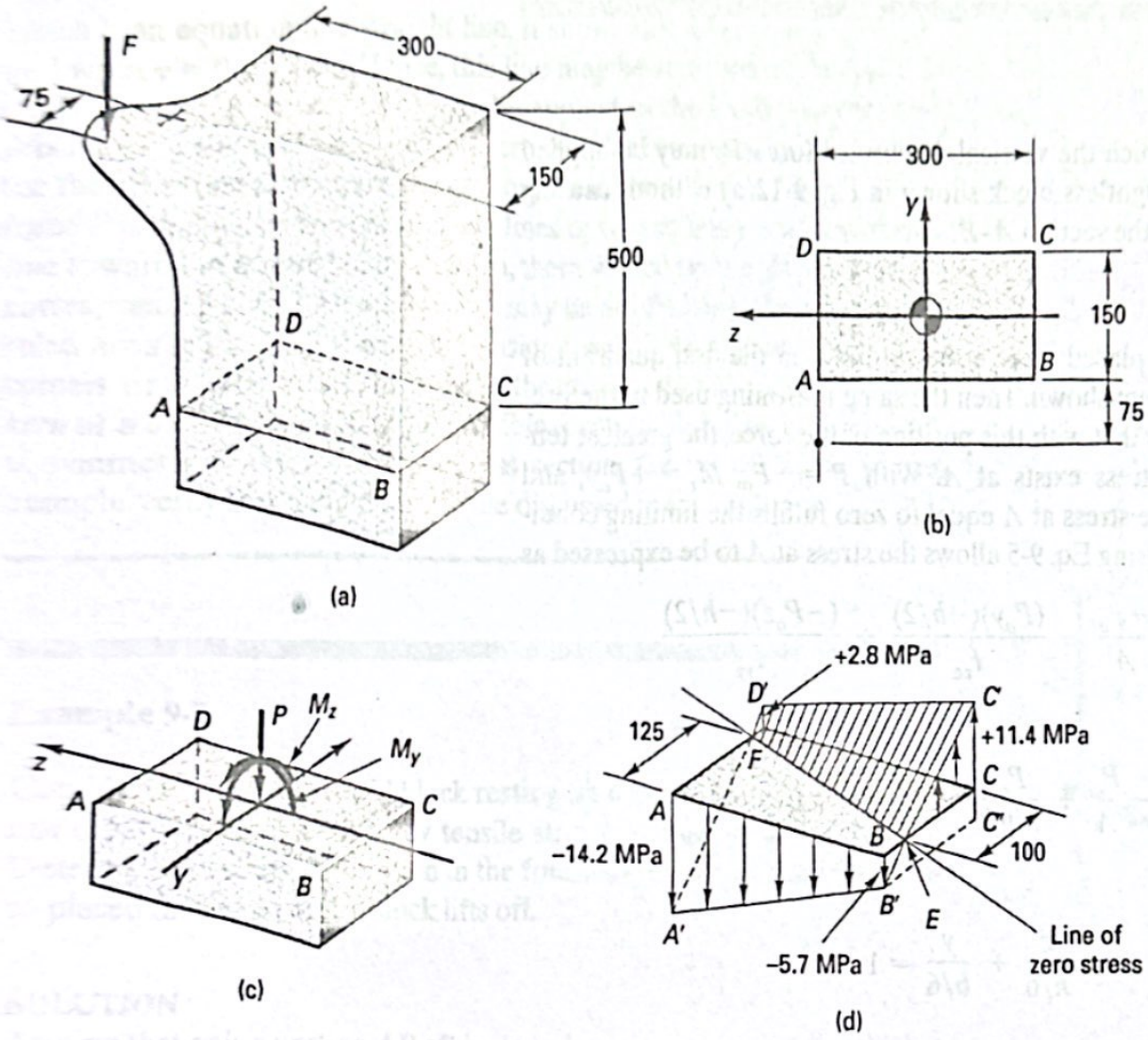


Fig. 9-11

$$\sigma_C = -1.42 + 8.53 + 4.27 = +11.4 \text{ MPa}$$

$$\sigma_D = -1.42 + 8.53 - 4.27 = +2.8 \text{ MPa}$$

These stresses are shown in Fig. 9-11(d). The ends of these four stress vectors at A', B', C', and D' lie in the plane $A'B'C'D'$. The vertical distance between planes $ABCD$ and $A'B'C'D'$ defines the total stress on any point on the cross section. The intersection of plane $A'B'C'D'$ with plane $ABCD$ locates the line of zero stress FE.

By drawing a line $B'C''$ parallel to BC, similar triangles $C'B'C''$ and $C'EC$ are obtained; thus, the distance $CE = [11.4/(11.4 + 5.7)]150 = 100$ mm. Similarly, distance AF is found to be 125 mm. Points E and F locate the line of zero stress. If the weight of the block is neglected, the stress distribution on any other section parallel to $ABCD$ is the same.

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Example 9-6

Find the zone over which the vertical downward force P_o may be applied to the rectangular weightless block shown in Fig. 9-12(a) without causing any tensile stresses at the section A–B.

SOLUTION

The force $P = -P_o$ is placed at an arbitrary point in the first quadrant of the yz coordinate system shown. Then the same reasoning used in the preceding example shows that with this position of the force, the greatest tendency for a tensile stress exists at A. With $P = -P_o$, $M_z = +P_o y$, and $M_y = -P_o z$, setting the stress at A equal to zero fulfills the limiting condition of the problem. Using Eq. 9-5 allows the stress at A to be expressed as

$$\sigma_A = 0 = \frac{-P_o}{A} - \frac{(P_o y)(-b/2)}{I_{zz}} + \frac{(-P_o z)(-h/2)}{I_{yy}}$$

or

$$-\frac{P_o}{A} + \frac{P_o y}{b^2 h/6} + \frac{P_o z}{bh^2/6} = 0$$

Simplifying,

$$\frac{z}{h/6} + \frac{y}{b/6} = 1$$

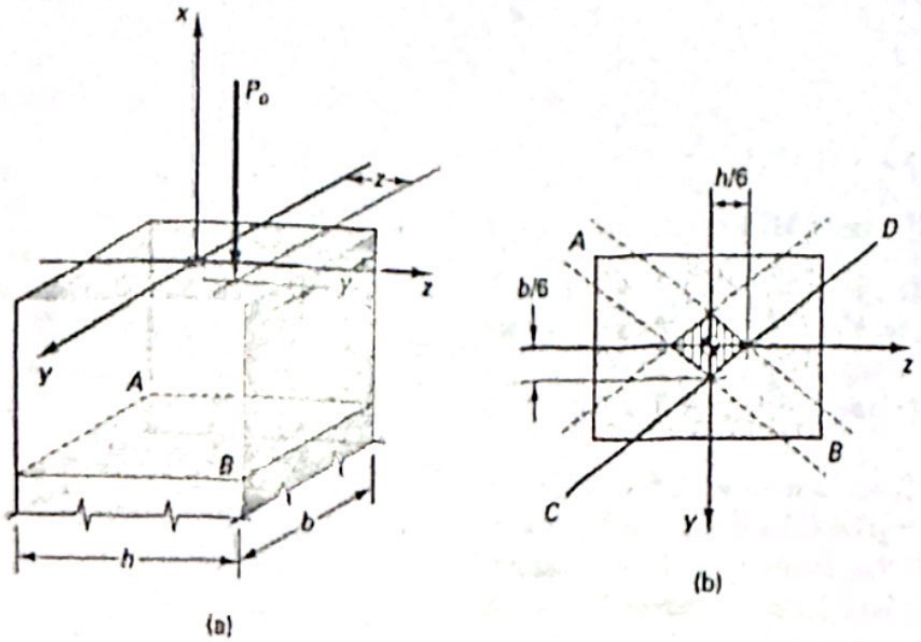


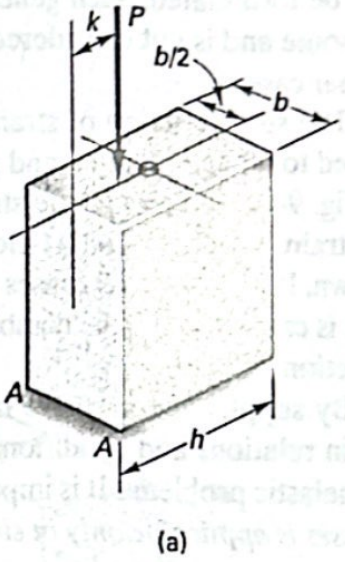
Fig. 9-12

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which is an equation of a straight line. It shows that when $z = 0, y = b/6$; and when $y = 0, z = h/6$. Hence, this line may be represented by line CD in Fig. 9-12(b). A vertical force may be applied to the block anywhere on this line and the stress at A will be zero. Similar lines may be established for the other three corners of the section; these are shown in Fig. 9-12(b). If force P is applied on any one of these lines or on any line parallel to such a line toward the centroid of the section, there will be no tensile stress at the corresponding corner. Hence, force P may be applied anywhere within the ruled area in Fig. 9-12(b) without causing tensile stress at any of the four corners or anywhere else. This zone of the cross-sectional area is called the *kern* of a section. By limiting the possible location of the force to the lines of symmetry of the rectangular cross section, the results found in this example verify the "middle-third" rule discussed in Example 9-4.

Example 9-7

Consider a "weightless" rigid block resting on a linearly elastic foundation not capable of transmitting any tensile stresses, as shown in Fig. 9-13(a). Determine the stress distribution in the foundation when applied force P is so placed that a part of the block lifts off.



(a)

SOLUTION

Assume that only a portion AB of the foundation of length x and width b is effective in resisting applied force P. This corresponds to the shaded area in Fig. 9-13(c). The stress along line B–B is zero by definition. Hence, the following equation for the stress at B may be written:

$$\sigma_B = -\frac{P}{xb} + P\left(\frac{x}{2} - k\right)\frac{6}{bx^2} = 0$$

where $x/2 - k$ is the eccentricity of the applied force with respect to the centroidal axis of the shaded contact area, and $bx^2/6$ is its section modulus. By solving for x, it is found that $x = 3k$ and the pressure distribution will be "triangular," as shown in Fig. 9-13(b) (why?). As k decreases, the intensity of pressure on line A–A increases; when k is zero, the block becomes unstable.

Problems such as this arise, for example, in the design of foundations for chimneys, as no tensile stresses can develop at the contact surface of a concrete pad with soil. Similar problems arise in foundations for heavy machinery. Similar reasoning can be applied where a number of forces are acting on a member and the contact area is of any shape.

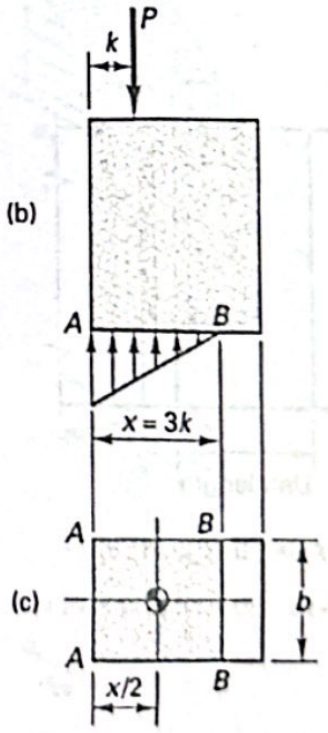


Fig. 9-13 Stresses between two contacting surfaces that do not transmit tensile forces.

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PROBLEMS

Section 9-2

9-1. Rework Example 9-1 by assuming that the span is 6000 mm, the beam is 150×200 mm, and α is 20°.

9-2. A 150×200 mm beam spanning 6000 mm is loaded in the middle of the span with an inclined force of 5 kN along the diagonal of the cross section, as shown in the figure. Determine the largest bending stresses and locate the neutral axis.

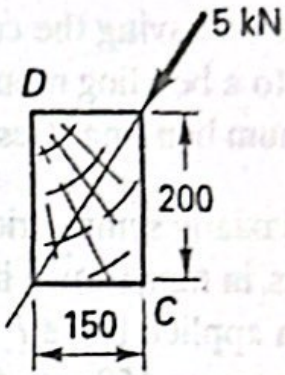


Fig. P9-2

9-3. A box beam made from an extruded aluminum alloy having 5 414mm wall thickness is loaded by an inclined force $P = 100$ N, as shown in the figure. Determine the maximum stress on section *a–b*. For the box beam, use centerline dimensions for finding *I*. (Note that the stress is far away from the support, where large local stress concentrations are unlikely.)

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