Pole and Polar of a circle

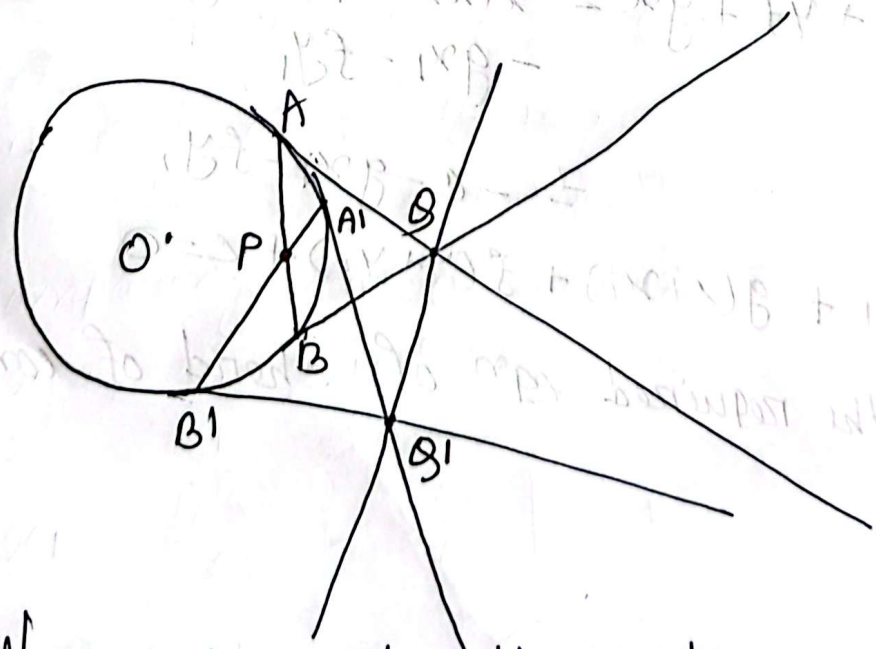
Pole: Let P be any point inside, outside or on the circle (except the center). If we draw any chord AB and A'B' passing through P. If tangents to the circle at A and B meet at Q and tangents to the circle at A' and B' meet at Q', then the straight line QQ' is called the polar with P as its pole.

The eqn of polar w.r.to circle $x^2+y^2=a^2$ is

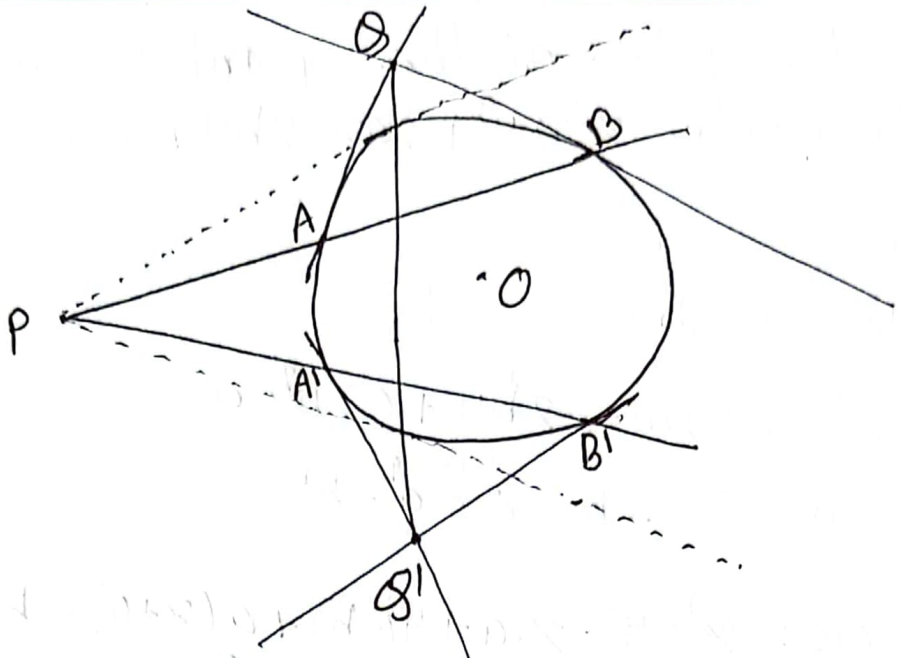
$$xx_1+yy_1=a^2$$

The eqn of polar w.r.to the circle $x^2+y^2+2gx+2fy+c=0$ is $xx_1+yy_1+g(x+x_1)+f(y+y_1)+c=0$.

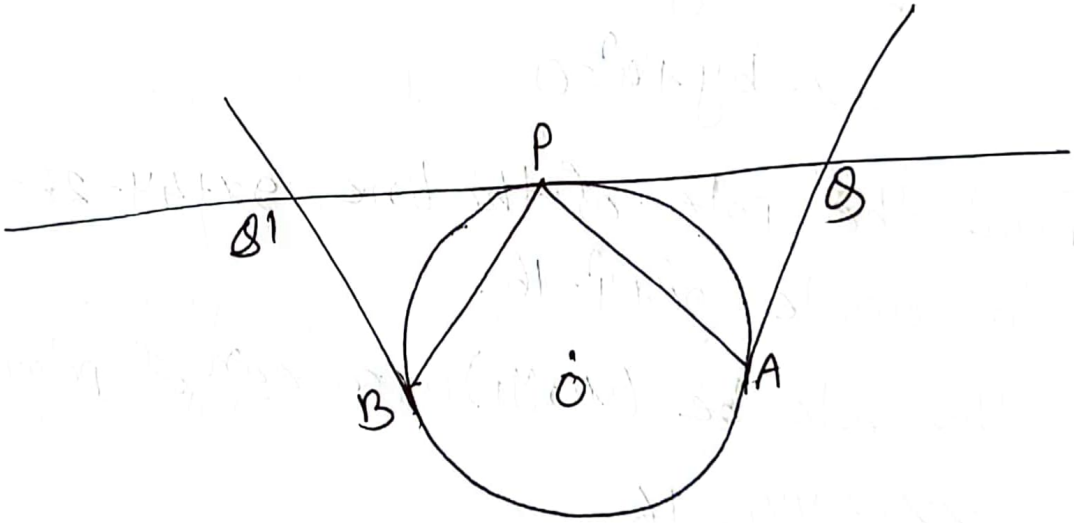
when (x_1,y_1) is the pole.



When P is inside the circle



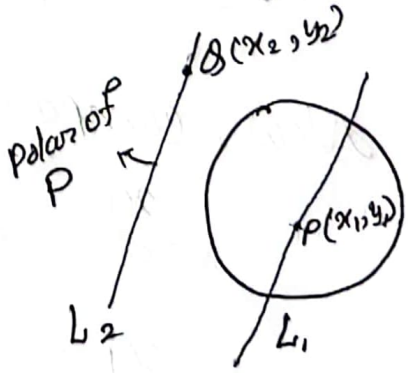
When P is outside the circle



When P is on the circle

Properties:

* If polar of a point 'P' wrt the circle passes through the point 'Q' then polar of point 'Q' will pass through the point P.
* Then P and Q are called conjugate points, the two lines (L_1, L_2) are called conjugate lines.
* Line joining the pole and the center of the circle will be perpendicular to the polar.



Ex-01: Find the polar of the point $(a,-b)$ with respect to the circle $x^2+y^2+2ax-2by+a^2-b^2=0$.

Soln: Given,

pole $(a,-b)$

eqn of circle: $x^2+y^2+2ax-2by+a^2-b^2=0$

$g=a,\ f=-b,\ c=a^2-b^2$

$\therefore$ eqn of polar is: ~~x^2+y^2~~ $\cdot x.a+y(-b)+a(x+a)-b(y-b)+a^2-b^2=0$

$\Rightarrow 2ax-2by+2a^2=0$

$\Rightarrow ax-by+a^2=0$

Ex-02: Find the pole of the line $9x+4y-28=0$ w.r.t. to the circle $x^2+y^2=16$.

Soln: Let, the pole be (x_1,y_1), so eqn of polar is:

$xx_1+yy_1=16$

$\Rightarrow xx_1+yy_1-16=0$ ——— (1)

Again eqn of polar is,

$9x+4y-28=0$ ——— (2)

Now, comparing eqn (1) and (2) we can write,

$$\frac{x_1}{9}=\frac{y_1}{4}=\frac{16}{28}$$

$$\Rightarrow x_1=\frac{16\times 9}{28}=\frac{36}{7} \quad | \quad y_1=\frac{16\times 4}{28}=\frac{16}{7}$$

$\therefore$ the pole is $\left(\frac{36}{7},\frac{16}{7}\right)$.

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Ex-03: Find the pole of the straight line $2x+y+12=0$ w.r.to the circle $x^2+y^2-4x+3y-1=0$.

Soln: Here, $g=-2$, $f=\frac{3}{2}$, $c=-1$

Let, the pole be (x_1, y_1)

∴ the eqn of polar w.r.to the given circle is,

$$xx_1+yy_1-2(x+x_1)+\frac{3}{2}(y+y_1)-1=0$$

$$\Rightarrow (x_1-2)x+\left(y_1+\frac{3}{2}\right)y-2x_1+\frac{3}{2}y_1-1=0 \quad \text{—(1)}$$

Again the eqn of polar is $2x+y+12=0$ —(2)

Now, comparing eqn (1) and (2) we can write,

$$\frac{x_1-2}{2}=\frac{y_1+\frac{3}{2}}{1}=\frac{-2x_1+\frac{3}{2}y_1-1}{12}$$

$$\therefore \frac{x_1-2}{2}=\frac{-2x_1+\frac{3}{2}y_1-1}{12}$$

$$\Rightarrow 16x_1-3y_1=22 \quad \text{—(3)}$$

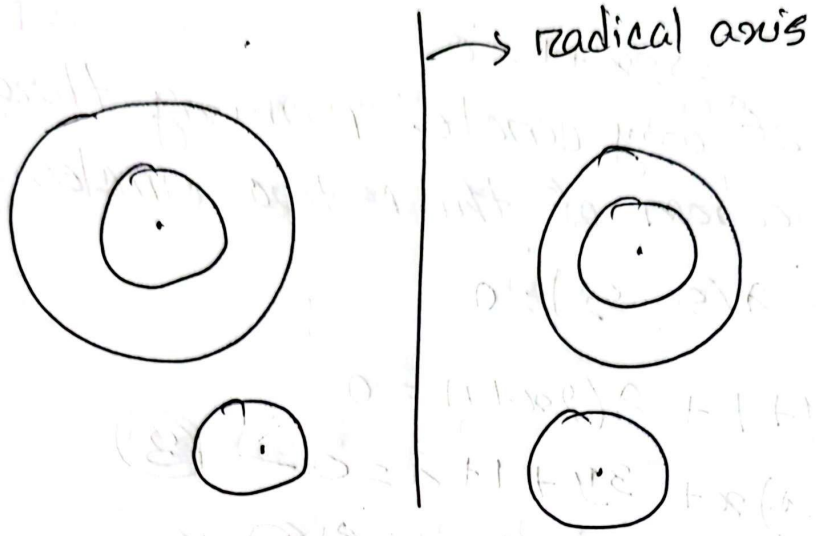
and $$\frac{y_1+\frac{3}{2}}{1}=\frac{-2x_1+\frac{3}{2}y_1-1}{12}$$

$$\Rightarrow 21y_1+4x_1=-38 \quad \text{—(4)}$$

Solving (3) and (4) we can write,

$$(x_1, y_1)=(1,-2).$$

Coaxial circles: A system of circles in which every pair of circles has the same radical axis is called the coaxial system.



Limiting points: Limiting points of co-axial circle are the center of the point circle belonging to the family.

Ex-01: Find the coordinates of the limiting points of the coaxial system determined by the circles $x^2+y^2+4x+2y+5=0$ and $x^2+y^2+2x+4y+7=0$.

Soln: Let, $S_1 = x^2+y^2+4x+2y+5=0$ —— (1)

$S_2 = x^2+y^2+2x+4y+7=0$ —— (2)

the radical axis is:

$$S_1 - S_2 = 2x-2y-2=0$$

$$\Rightarrow x-y-1=0$$

Now the eqn of any circle coaxial with given circle is:

$$x^2+y^2+4x+2y+5+\lambda(x-y-1)=0$$

$$\Rightarrow x^2+y^2+(4+\lambda)x+(2-\lambda)y+5-\lambda=0 \quad \text{——} (3)$$

the center of (3) is $\left(-\frac{4+\lambda}{2}, -\frac{2-\lambda}{2}\right)$

radius is: $\sqrt{\left(\frac{4+\lambda}{2}\right)^2+\left(\frac{2-\lambda}{2}\right)^2-5+\lambda}$

$$= \sqrt{\frac{\lambda^2+4\lambda}{2}}$$

Now the circle be a point circle if the radius is zero,

$$\therefore \sqrt{\frac{\lambda^2+4\lambda}{2}} = 0$$

$$\Rightarrow \lambda(\lambda+4)=0$$

$$\Rightarrow \lambda = 0, -4$$

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∴ the coordinates of the center of eqn (3) are:

$$\left(-\frac{4+0}{2}, -\frac{2-0}{2}\right) = (-2,-1) \text{ and}$$

$$\left(-\frac{4-4}{2}, -\frac{2+4}{2}\right) = (0,-3)$$

Which are the required limiting points.

Ex-02: Find the radical axis of co-axial system of circles whose limiting points are (-1,2) and (2,3).

Soln: Since (-1,+2) and (2,3) are limiting points.

$$\therefore (x+1)^2 + (y-2)^2 = 0$$

$$\Rightarrow x^2+y^2+2x-4y+5=0 \longleftarrow (1)$$

and $(x-2)^2+(y-3)^2=0$

$$\Rightarrow x^2+y^2-4x-6y+13=0 \longleftarrow (2)$$

∴ (1) and (2) belongs to the coaxial system

∴ the radical axis is: $S_1 - S_2 = 0$

$$\Rightarrow 6x+2y-8=0$$

$$\Rightarrow 3x+y-4=0$$

Ex-03: If origin be one limiting points of which $x^2+y^2+3x+4y+25=0$ is a member, Find the other limiting point.

Soln: Given,

$x^2+y^2+3x+4y+25=0$ —— (1) is a member of the coaxial system and origin is a limiting points

So $(x-0)^2+(y-0)^2=0$

$\Rightarrow x^2+y^2=0$ —— (2)

(1) − (2) implies,

$3x+4y+25=0$ —— (3)

∴ The family of circle is,

$x^2+y^2+3x+4y+25+\lambda(3x+4y+25)=0$

$\Rightarrow x^2+y^2+3(1+\lambda)x+4(1+\lambda)y+25(1+\lambda)=0$ —— (4)

center is $\left(-\frac{3}{2}(1+\lambda), -2(1+\lambda)\right)$

radius is : $\sqrt{\left\{\frac{3}{2}(1+\lambda)\right\}^2+4(1+\lambda)^2-25(1+\lambda)}$

$=\sqrt{25(1+\lambda)^2-100(1+\lambda)}$

for limiting points, radius $=0$

$\therefore 25(1+\lambda)^2-100(1+\lambda)=0$

$\Rightarrow \lambda=3$

∴ the other limiting point is: $\left(-\frac{3}{2}(1+3), -2(1+3)\right)$

$=(-6,-8)$

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Ex-04: Find the eqn of the system of circles coaxial with $x^2+y^2+4x+2y+1=0$ and $x^2+y^2-2x+6y-6=0$. Also find the particular circle whose center lies on the radical axis.

soln: the radical axis is:

$$6x-4y+7=0$$

$\therefore$ The system of circle is:

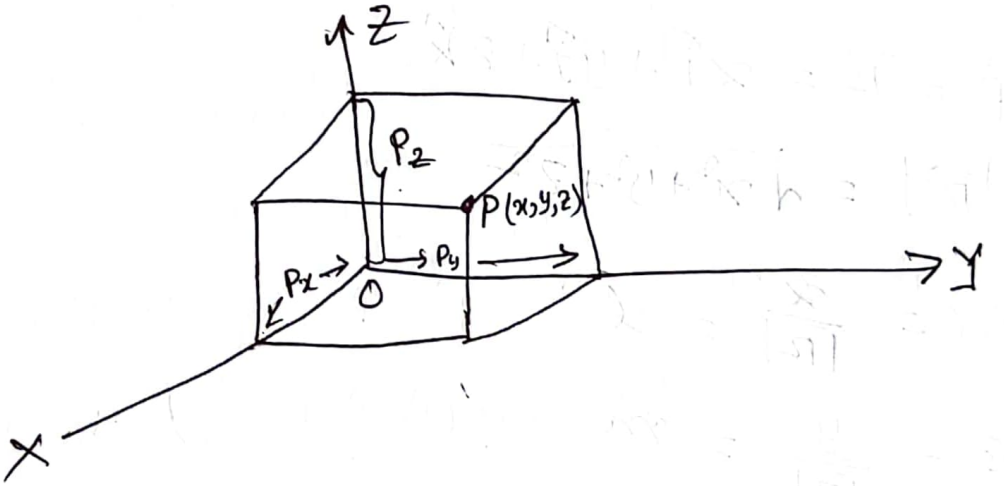
$$S_1+(S_1-S_2)\lambda=0$$

$$\Rightarrow x^2+y^2+4x+2y+1+\lambda(6x-4y+7)=0$$

2nd part: (same as before).

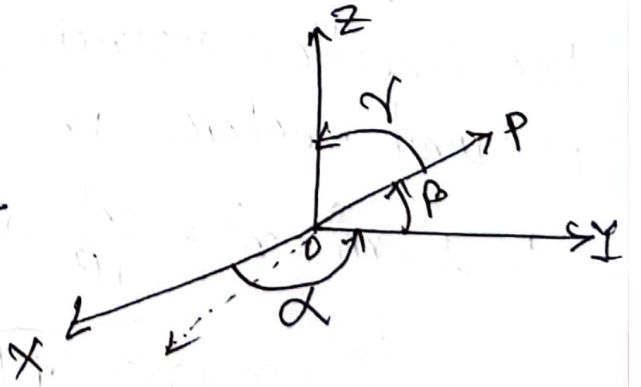
Solid Geometry (3D coordinate system)

The position of a point in space may be defined by refering it to three fixed planes. Those planes have a common point, known as origin and intersect in pair in three straight lines, known as coordinate axes. The planes themselves are called the coordinate planes. When the three co-ordinate planes and consequently the three co-ordinate axes are mutually at right angles, the axes are said to be Rectangular. The three planes divides the space into eight part known as the octants.



* Direction cosine of a line:

If a given line OP makes angles α, β, γ with the positive direction of axes of X, Y, Z respectively then $\cos\alpha, \cos\beta, \cos\gamma$ are the direction cosines of the line OP and generally denoted by the letters l, m, n respectively.

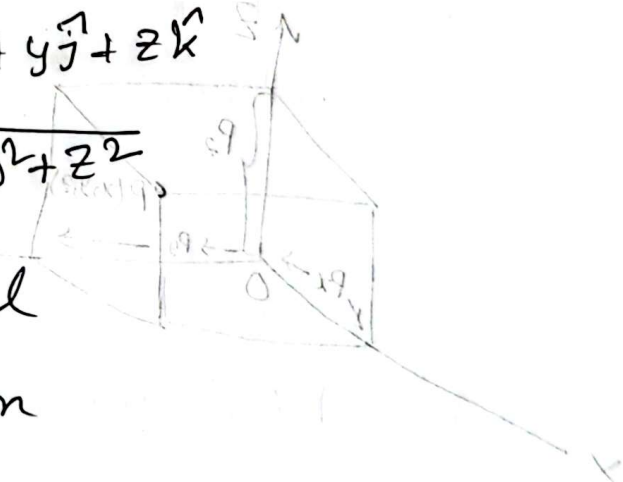


Again the direction cosines of the line PO will be $\cos(180°+\alpha)$, $\cos(180°+\beta)$ and $\cos(180°+\gamma)$ that is, $(-l, -m, -n)$

The line X axis makes angle 0°, 90° and 90° with X, Y and Z axis respectively.

The direction cosines of x axis be $(\cos 0°, \cos 90°, \cos 90°)$ i.e $(1, 0, 0)$

The angles α, β, γ of the line OP are called the direction angle of OP.



Let, $\overrightarrow{OP} = \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$$\Rightarrow |\vec{r}| = \sqrt{x^2+y^2+z^2}$$

$$\therefore \cos\alpha = \frac{x}{|\vec{r}|} = l$$

$$\cos\beta = \frac{y}{|\vec{r}|} = m$$

$$\cos\gamma = \frac{z}{|\vec{r}|} = n$$

* Here, $l\hat{i} + m\hat{j} + n\hat{k}$ is the unit vector of r the direction cosines of a liner are the component of unit vector in the direction of the line.

$$\therefore l^2+m^2+n^2 = \frac{x^2}{r^2} + \frac{y^2}{r^2} + \frac{z^2}{r^2} = \frac{x^2+y^2+z^2}{r^2} = \frac{r^2}{r^2} = 1$$

$$\therefore \boxed{l^2+m^2+n^2 = 1}$$

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Direction Ratio's: Any three numbers which are proportional to the direction cosines of the line is called the direction ratios of the line.

Let, the numbers are : a, b, c

$\therefore a \propto l,\ b \propto m$ and $c \propto n$

$\therefore a = kl;\ b = km$ and $c = kn$

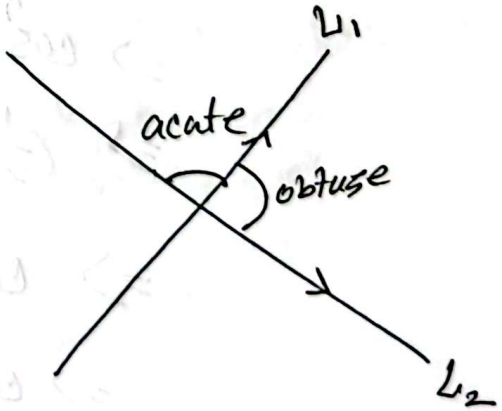
$$\therefore \frac{l}{a} = \frac{m}{b} = \frac{n}{c} = \frac{\sqrt{l^2+m^2+n^2}}{\sqrt{a^2+b^2+c^2}} = \frac{1}{\sqrt{a^2+b^2+c^2}}$$

$$\therefore l = \frac{a}{\sqrt{a^2+b^2+c^2}},\ m = \frac{b}{\sqrt{a^2+b^2+c^2}},\ n = \frac{c}{\sqrt{a^2+b^2+c^2}}$$

* Direction ratios are the component of vectors.
* Two parallel line have the same set of direction cosines.
* For any line, if a, b, c are direction ratio, then ka, kb and kc are also direction ratio.

* **Angle between two lines:**

Angle between two lines are the angle between the direction of unit vector of the lines.



$$\vec{v}_1 = l_1\hat{i} + m_1\hat{j} + n_1\hat{k}$$

$$\vec{v}_2 = l_2\hat{i} + m_2\hat{j} + n_2\hat{k}$$

$$\therefore \cos\theta = \frac{\vec{v}_1 \cdot \vec{v}_2}{|\vec{v}_1| \cdot |\vec{v}_2|} = l_1l_2 + m_1m_2 + n_1n_2$$

since $|\vec{v}_1| = 1$ and $|\vec{v}_2| = 1$

Ex-01: If a line makes angles α, β, γ with the axis X, Y, Z respectively, then Prove that, $\sin^2\alpha + \sin^2\beta + \sin^2\gamma = 2$.

Soln: We know that

$$l^2 + m^2 + n^2 = 1$$

$$\Rightarrow \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

$$\Rightarrow 1 - \sin^2\alpha + 1 - \sin^2\beta + 1 - \sin^2\gamma = 1$$

$$\Rightarrow \sin^2\alpha + \sin^2\beta + \sin^2\gamma = 2.$$

[Proved]

Ex-02: A directed line makes angles 60°, 45° with axes of X and Y respectively, what angle does it makes with Z-axis?

Soln: We know,

$$l^2 + m^2 + n^2 = 1$$

$$\Rightarrow \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

$$\Rightarrow \cos^2 60^\circ + \cos^2 45^\circ + \cos^2\gamma = 1$$

$$\Rightarrow \left(\frac{1}{2}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 + \cos^2\gamma = 1$$

$$\Rightarrow \cos^2\gamma = \frac{1}{4}$$

$$\Rightarrow \cos\gamma = \pm\frac{1}{2}$$

$$\Rightarrow \gamma = 60^\circ \text{ or } 120^\circ.$$

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Example-03: Find the angle between the lines whose direction ratios are $(2,-1,3)$ and $(-1,3,4)$.

Soln: If the angle between the lines is θ and if the direction cosines of the lines are (l_1, m_1, n_1) and (l_2, m_2, n_2) then,

$$\cos\theta = l_1 l_2 + m_1 m_2 + n_1 n_2$$

Let $(a_1, b_1, c_1) = (2,-1,3)$ and $(a_2, b_2, c_2) = (-1,3,4)$ are the direction ratios related to (l_1, m_1, n_1) and (l_2, m_2, n_2).

$$\therefore \cos\theta = \frac{a_1 a_2 + b_1 b_2 + c_1 c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\ \sqrt{a_2^2 + b_2^2 + c_2^2}}$$

$$= \frac{-2 - 3 + 12}{\sqrt{14}\ \sqrt{26}}$$

$$\cos\theta = \sqrt{\frac{7}{52}}$$

$$\Rightarrow \theta = \cos^{-1}\left(\sqrt{\frac{7}{52}}\right)$$

Ex-04: Let α be the angle between the lines whose direction cosines satisfy the eqn $l+m-n=0$ and $l^2+m^2-n^2=0$. Then find the value of $\sin^4\alpha + \cos^4\alpha$.

Soln: Given,

$$l+m-n=0 \quad \text{——} (1)$$

$$l^2+m^2-n^2=0 \quad \text{——} (2)$$

We know that, $l^2+m^2+n^2=1$ —— (3)

from ①, $l+m=n$ — ④

$\therefore l^2+m^2-(l+m)^2=0$

$\Rightarrow ml=0$

$\Rightarrow m=0$ or $l=0$

for $m=0$ ④ implies $l=n$

①$\Rightarrow \therefore 2l^2=1 \Rightarrow l=\pm\frac{1}{\sqrt{2}}=n$

$\therefore (l,m,n)=\left(\pm\frac{1}{\sqrt{2}}, 0, \pm\frac{1}{\sqrt{2}}\right)$

Again, $l=0$, ④ implies $m=n$

$\therefore$ ① $\Rightarrow 2m^2=1 \Rightarrow m=\pm\frac{1}{\sqrt{2}}$

$\therefore (l,m,n)=\left(0, \pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}\right)$

$\therefore \cos\theta = l_1l_2+m_1m_2+n_1n_2$

$= 0+0+\frac{1}{2}$

$\Rightarrow \cos\alpha = \frac{1}{2}$

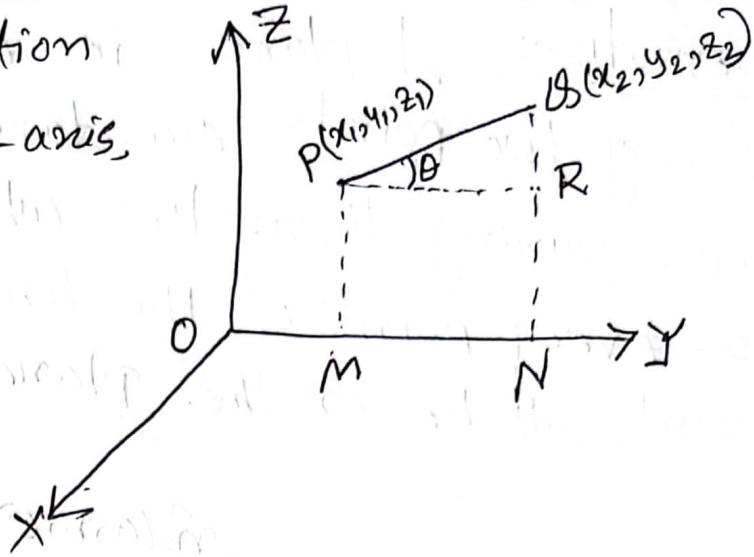
$\Rightarrow \alpha = 60^\circ$

$\therefore \sin^4\alpha+\cos^4\alpha = \sin^4 60^\circ+\cos^4 60^\circ$

$= \left(\frac{\sqrt{3}}{2}\right)^4+\left(\frac{1}{2}\right)^4$

$= \frac{5}{8}$ (Ans)

* Relation Between Projection and Direction Cosine *



Here, MN is the projection of the line PQ on Y-axis,

$\therefore MN = y_2 - y_1$

from figure;

$\cos\theta = \frac{PR}{PQ}$

$\Rightarrow \cos\theta = \frac{MN}{PQ}$

$\Rightarrow PQ = \frac{MN}{\cos\theta} = \frac{y_2 - y_1}{m}$

similarly, $PQ = \frac{x_2 - x_1}{l} = \frac{z_2 - z_1}{n}$

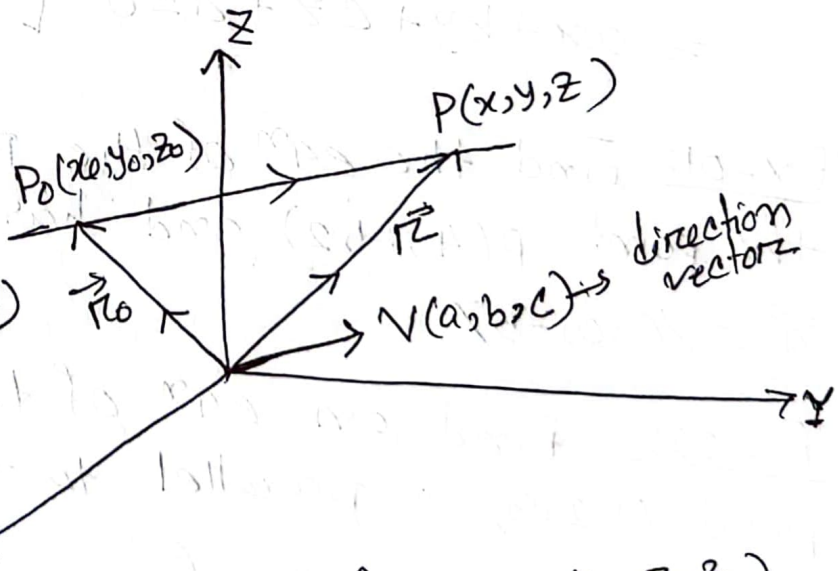
$\therefore$ We can write,

$$\boxed{\frac{x_2 - x_1}{l} = \frac{y_2 - y_1}{m} = \frac{z_2 - z_1}{n}}$$

Eqn of lines in 3D:

We can write:

$\vec{r} = \vec{r_0} + t\vec{v}$



$\Rightarrow (x, y, z) = (x_0 + at, y_0 + bt, z_0 + ct)$

$\Rightarrow \frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c} = t$

$\Rightarrow \boxed{\frac{x - x_0}{l} = \frac{y - y_0}{m} = \frac{z - z_0}{c}}$

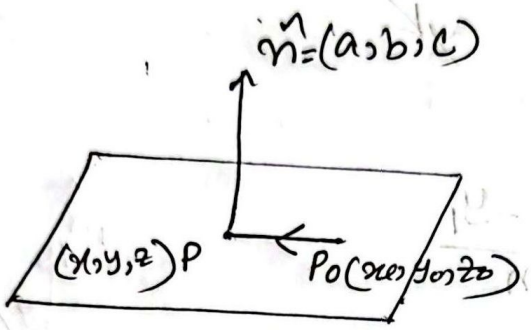
$P_0P = (x - x_0, y - y_0, z - z_0)$

which is the standard eqn of line.

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The Plane

Definition: A plane is a surface such that if any two points are taken on it, the straight line joining them lies wholly and the surface i.e, every point of the line joining the two point will be on the plane.



$$\hat{n} \cdot \overrightarrow{P_0P} = 0$$

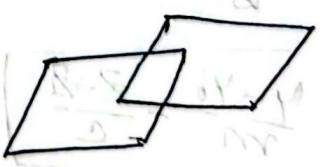
$$\Rightarrow (a,b,c) \cdot (x-x_0, y-y_0, z-z_0) = 0$$

$$\Rightarrow a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$$

$$\Rightarrow ax + by + cz + d = 0 \quad [\text{Here } d = -(ax_0 + by_0 + cz_0)]$$

Ex-01: Find the eqn of the plane that goes through the point $P(4,-1,2)$ and has a normal vector $\vec{n} = \langle 6,0,-5 \rangle$.

Ex-02: Find an eqn of the plane containing the point $(1,2,3)$ & parallel to the plane $3x-y+z=5$.



Soln: Let, the eqn of the plane is $ax+by+cz+d=0$ ——— (1)

since the two planes are parallel, so having same normal vector.

$\therefore (a,b,c) = (3,-1,1)$

$\therefore 3x - y + z + d = 0$ — (2)

(2) passes through $(1,2,3)$

$\therefore 3.1 - 2 + 3 + d = 0$

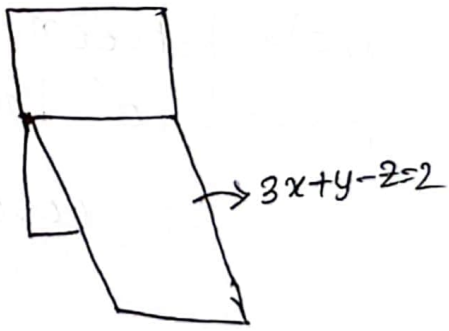
$\Rightarrow d = -4$

$\therefore$ the eqn of plane is: $3x - y + z - 4 = 0$.

Ex-03: Find the eqn of the plane that passes through the points $(4,-2,1)$ and $(1,1,0)$ and is perpendicular to the plane $3x + y - z = 2$

Soln: The passing points are:

$(4,-2,1)$ and $(1,1,0)$



Let, $\vec{V}_1 = \langle (4,-2,1) - (1,1,0) \rangle$

$= \langle 3,-3,1 \rangle$

$\vec{V}_2 = (3,1,-1)$

$\therefore$ since the plane is perpendicular to the plane $3x + y - z = 2$, so, $\vec{V}_2 = (3,1,-1)$ is lies on the plane.

$\therefore$ The normal vector of the plane is: $\vec{V}_1 \times \vec{V}_2 = \hat{n}$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -3 & 1 \\ 3 & 1 & -1 \end{vmatrix}$$

$\therefore \hat{n} = \langle 2,6,12 \rangle$

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The plane passes through $(1,1,0)$ and the normal vector is $\langle 2,6,12\rangle$.

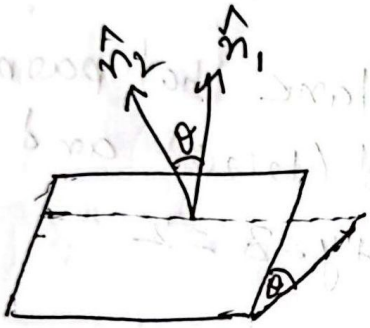
$\therefore$ The required eqn of plane is: $2(x-1)+6(y-1)+12(z-0)=0$

$$\Rightarrow x+3y+6z-4=0$$

* Angle between two planes:

Let the planes are: $a_1x+b_1y+c_1z+d_1=0$

$a_2x+b_2y+c_2z+d_2=0$



angle between two planes is the angle between two normals of the planes.

$$\therefore \cos\theta = \frac{\hat{n}_1 \cdot \hat{n}_2}{|\hat{n}_1||\hat{n}_2|}$$

Here, $\hat{n}_1 = (a_1, b_1, c_1)$

$\hat{n}_2 = (a_2, b_2, c_2)$

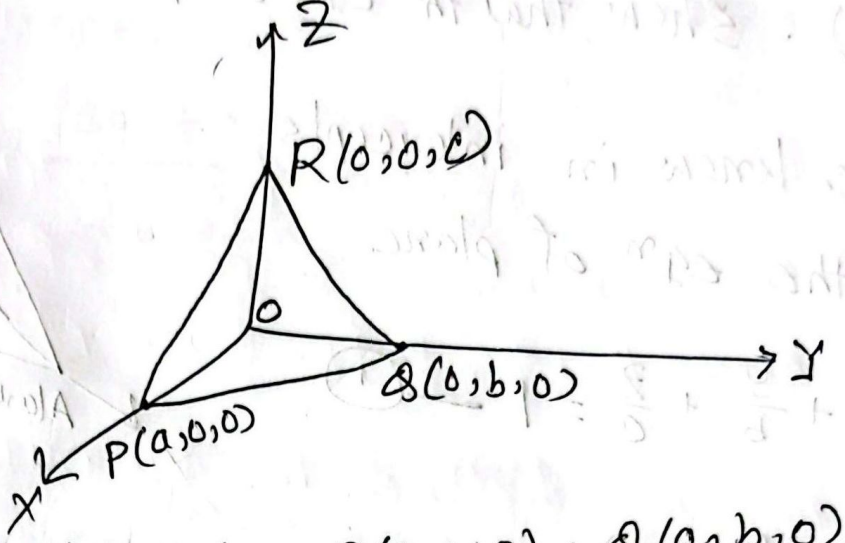
$$= \frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}$$

Ex-4: Find the angle between the planes $2x-3y+4z=5$ and $3x+5y-2z=7$.

Ans: 120.8°.

Plane is intercept form:

Let, the eqn of plane is: $Ax + By + Cz + D = 0$ — (1)



(1) passing through $P(a,0,0)$, $Q(0,b,0)$ and $R(0,0,c)$

$\Rightarrow Aa + D = 0 \Rightarrow A = -\frac{D}{a}$

Similary, $B = -\frac{D}{b}$, $C = -\frac{D}{c}$

$\therefore$ (1) implies

$$-\frac{Dx}{a} - \frac{Dy}{b} - \frac{Dz}{c} = -D$$

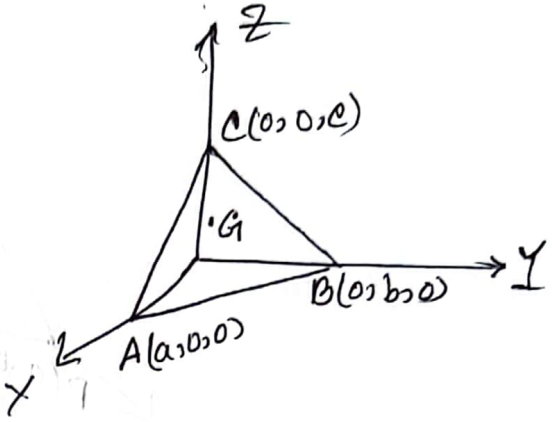
$$\Rightarrow \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

Here, a, b, c are length ~~of~~ of ~~x, z, z~~ x, y, z axes from the origin.

Ex-05: Plane meets coordinate axis in A, B, C such that centroid of the triangle ABC is (p, q, r). Show that the eqn of plane is $\frac{x}{p} + \frac{y}{q} + \frac{z}{r} = 3$.

Soln: We know in intercepts form the eqn of plane is:

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \quad \text{---} (1)$$



Here $A = (a, 0, 0)$

$B = (0, b, 0)$

$C = (0, 0, c)$

$\therefore$ the coordinate of centroid of the triangle ABC is

$$\left(\frac{a+0+0}{3}, \frac{0+b+0}{3}, \frac{0+0+c}{3}\right) = \left(\frac{a}{3}, \frac{b}{3}, \frac{c}{3}\right)$$

By question, $\left(\frac{a}{3}, \frac{b}{3}, \frac{c}{3}\right) = (p, q, r)$

$\therefore \frac{a}{3} = p \Rightarrow a = 3p$

$b = 3q$

$c = 3r$

then (1) implies,

$$\frac{x}{3p} + \frac{y}{3q} + \frac{z}{3r} = 1$$

$$\Rightarrow \frac{x}{p} + \frac{y}{q} + \frac{z}{r} = 3$$

[Proved]

* Perpendicular distance of a point from a plane:

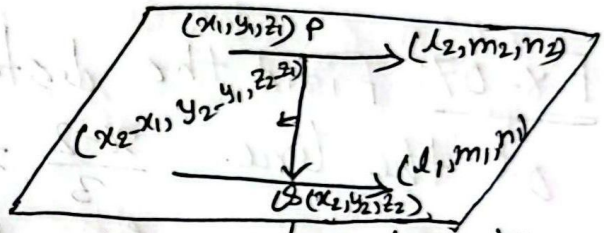
Point (x_1, y_1, z_1), Plane $ax+by+cz+d=0$

$\therefore$ distance $= \dfrac{|ax_1+by_1+cz_1+d|}{\sqrt{a^2+b^2+c^2}}$

* Coplannar line:

If two lines passes through points (x_1, y_1, z_1) and (x_2, y_2, z_2) and their direction cosine be (l_1, m_1, n_1) and (l_2, m_2, n_2). Then the two lines are coplannar if

$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$



three vectors lies in same plane, so the determinant of their coefficient is zero.

Ex-06: If the axes are rectangular, find the distance from the point $(3,4,5)$ to the point where the line $\frac{x-3}{1} = \frac{y-4}{2} = \frac{z-5}{2}$ meets the plane $x+y+z-2=0$.

Soln: The eqn of plane is: $x+y+z-2=0$

The given eqn of line is,

$$\frac{x-3}{1} = \frac{y-4}{2} = \frac{z-5}{2} = r \text{ (say)} \quad \text{—(1)}$$

Then any point on the line (1) is $(r+3, 2r+4, 2r+5)$ —(2)

If the point lies on the given plane, then we have,

$$r+3+2r+4+2r+5-2=0$$

$$\Rightarrow 5r+10=0$$

$$\Rightarrow r=-2$$

(2) implies, the point is $(-2+3, -4+4, -4+5)$ i.e $(1,0,1)$, which is the point of intersection of the given line and the given plane.

So, the required distance is: $\sqrt{(3-1)^2+(4-0)^2+(5-1)^2}$

$$=6.$$

Ex-07: Find the distance of the point of intersection of the line $\frac{x-2}{3}=\frac{y+1}{4}=\frac{z-2}{12}$ and the plane $x-y+z-5=0$ from the point $(-1,-5,-10)$. (Ans; 13)

Ex-08: Prove that, the lines $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}$ and $\frac{x-2}{3}=\frac{y-3}{4}=\frac{z-4}{5}$ are coplannar. Also find their point of intersection.

Soln: Here, $(x_1, y_1, z_1)=(1,2,3)$

$(x_2, y_2, z_2)=(2,3,4)$

$(l_1, m_1, n_1)=(2,3,4)$

$(l_2, m_2, n_2)=(3,4,5)$

Given,

$$\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4} \quad \text{---} (1)$$

and $$\frac{x-2}{3}=\frac{y-3}{4}=\frac{z-4}{5} \quad \text{---} (2)$$

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$$\therefore \begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}$$

$$= \begin{vmatrix} 2-1 & 3-2 & 4-3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix}$$

$$= -1+2-1$$

$$= 0$$

since, the above determinant is zero.

$\therefore$ the two lines are coplanar.

2nd part:

Any point

Let, $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4} = r$ and

$\frac{x-2}{3} = \frac{y-3}{4} = \frac{z-4}{5} = r_1$

On line ① is $(2r+1, 3r+2, 4r+3)$ and any point on line ② is $(3r_1+2, 4r_1+3, 5r_1+4)$

If the two ~~lines~~ lines meet in a point, then for some values of r and r_1, the point given by above is identical.

$\therefore$ $2r+1 = 3r_1+2 \Rightarrow 2r-3r_1 = 1$ — ③

$3r+2 = 4r_1+3 \Rightarrow 3r-4r_1 = 1$ — ④

$4r+3 = 5r_1+4 \Rightarrow 4r-5r_1 = 1$ — ⑤

Solving ③ and ④ we can write

$r = r_1 = -1$

putting the values of r and r_1 in the point we get, $(-2+1, -3+2, -4+3)$ and $(-3+2, -4+3, -5+4)$

$\Rightarrow (-1, -1, -1)$ and $(-1, -1, -1)$

∴ The required point of intersection is $(-1, -1, -1)$.

Shortest distance

Defn: The straight line which is perpendicular to each of the two skew lines (or non-intersecting lines) is called the line of shortest distance.

Ex-09: Find the shortest distance between two straight lines, $\frac{x-3}{3} = \frac{y-8}{-1} = \frac{z-3}{1}$ and $\frac{x+3}{-3} = \frac{y+7}{2} = \frac{z-6}{4}$.

Also find the eqn of the shortest distance.

Soln: The given two lines are:

$$\frac{x-3}{3} = \frac{y-8}{-1} = \frac{z-3}{1} \quad \text{—— (1)}$$

and $$\frac{x+3}{-3} = \frac{y+7}{2} = \frac{z-6}{4} \quad \text{—— (2)}$$

the direction cosines of line ① and ② are $(3,-1,1)$ and $(-3,2,4)$ respectively.

Let, $\frac{x-3}{3} = \frac{y-8}{-1} = \frac{z-3}{1} = r_1$

and $\frac{x+3}{-3} = \frac{y+7}{2} = \frac{z-6}{4} = r_2$

so, any point on line ① is $P(3r_1+3, -r_1+8, r_1+3)$ and line ② is $Q(-3r_2-3, 2r_2-7, 4r_2+6)$.

Now the direction cosines of the line PQ are

$(-3r_1-3r_2-6, r_1+2r_2-15, -r_1+4r_2+3)$

Let, the line PQ be the lines of shortest distance of the lines ① & ②.

so, PQ is perpendicular to the both given lines ① and ②

By condition of perpendicularity,

$$l_1l_2 + m_1m_2 + n_1n_2 = 0 \quad [\because \cos 90° = 0]$$

So by condition,

$3(-3r_1-3r_2-6) - 1(r_1+2r_2-15) + 1(-r_1+4r_2+3) = 0$

$\Rightarrow 11r_1 + 7r_2 = 0$ —— ③

and, $-3(-3r_1-3r_2-6) + 2(r_1+2r_2+15) + 4(-r_1+4r_2+3) = 0$

$\Rightarrow 7r_1 + 29r_2 = 0$ —— ④

Solving (3) and (4) we get, $r_1 = 0$ and $r_2 = 0$.

So the points P and Q are $(3, 8, 3)$ and $(-3, -7, 6)$ respectively.

$\therefore$ shortest distance $PQ = \sqrt{(-3-3)^2 + (-7-8)^2 + (6-3)^2}$

$= \sqrt{270}$

$= 3\sqrt{30}$

2nd part:

The direction cosines of shortest distance line PQ is $(-6, -15, 3)$, and it is passing through the point $P(3, 8, 3)$.

$\therefore$ The eqn of shortest distance line PQ is

$$\frac{x-3}{-6} = \frac{y-8}{-15} = \frac{z-3}{3}$$

$$\Rightarrow \frac{x-3}{-2} = \frac{y-8}{-05} = \frac{z-3}{1}$$

which is the required eqn.

Ex-10: Find the shortest distance between the lines $\frac{x-6}{1} = \frac{y-2}{-2} = \frac{z-2}{2}$ and $\frac{x+4}{3} = \frac{y}{-2} = \frac{z+1}{-2}$.