Math-201
Straight Lines

1. Shifting origin (α, β)

$$x = \alpha + x'$$
$$y = \beta + y'$$

2. Rotating axis (θ)

$$x = x'\cos\theta - y'\sin\theta$$
$$y = y'\cos\theta + x'\sin\theta$$

inverse :

$$x' = x\cos\theta + y\sin\theta$$
$$y' = y\cos\theta - x\sin\theta$$

3. Origin shift and axis rotation

$$x = \alpha + x'\cos\theta - y'\sin\theta$$
$$y = \beta + y'\cos\theta + x'\sin\theta$$

4. Second degree homogenous equation represents a pair of straight line

$$ax^2 + 2hxy + by^2 = 0 \quad \text{——— (1)}$$

→ diride the equation by $\boxed{bx^2}$

→ The roots of the equation are m_1 and m_2 ; $m_1+m_2 = -b/a$, $m_1m_2 = c/a$

→ replace the roots and obtain

$y = m_1x$ and $y = m_2x$

5. Pair of straight lines perpendicular to the lines represented by equation $ax^2+2hxy+by^2=0$

→ equation of straight lines are

$y = m_1x$ and $y = m_2x$

→ lines perpendicular to the lines are

$$y = -\frac{1}{m_1}x \text{ and } y = -\frac{1}{m_2}x$$

→ $(x+m_1y)(x+m_2y) = 0$

→ replace the values of m_1+m_2 and m_1m_2

6. Remove first degree terms from an equation

→ Convert the equation into summation of perfect squares. $(x+\alpha)^2+(y+\beta)^2=0$

→ Assume $(x+\alpha) = x'$ and $(y+\beta) = y'$

→ Shift the origin to $(\alpha, -\beta)$ to remove first degree terms.

7. The equation of bisector of angles between the lines represented by

$$ax^2+2hxy+by^2=0$$

→ Straight lines are $y-m_1x=0$ and $y-m_2x=0$

→ equation of bisector of angle between two lines $a_1x+b_1y+c_1=0$ and $a_2x+b_2y+c_2=0$ is

$$\frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}} = \pm \frac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b_2^2}}$$

→ That is $$\frac{y-m_1x}{\sqrt{1+m_1^2}} = \pm \frac{y-m_2x}{\sqrt{1+m_2^2}}$$

→ lastly, $$\boxed{\frac{x^2-y^2}{a-b} = \frac{xy}{h}}$$

8. By transforming the origin to (h,k) the equation can be reduced to one only containing the terms of 2nd degree.

$$12x^2 - 10xy + 2y^2 + 11x - 5y + 2 = 0$$

→ Since origin is shifting,

$$x = x' + h \; ; \; y = y' + k$$

replace the values of x and y in the equation.

→ Expanding the terms, the coefficients of x' and y' and the constant term is considered 0.

→ Solving the equations obtained from considering the coefficients of x' and y' equals to zero, we get the value of h and k and hence the transformed origin.

9. Find the condition that the general equation of second degree may represent a straight line.

2nd degree general equation –

$$ax^2+2hxy+by^2+2gx+2fy+c=0$$

→ Shift the origin to (α,β).

→ replace the values $x=x'+\alpha$ and $y=y'+\beta$ in equation and expand.

→ The equation must be homogenous to x' and y'. Therefore the coefficients of x', y' and the constant term must be zero.

→ We get 2 equations from coefficients of x' and y' and the third equation will be obtained using the constant term by simplifying it

→ When we get three equations, the determinant of the coefficients must be equal to zero to find the condition.

$$\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

10. Show that the equation $ax^2+2hxy+by^2+2gx+2fy+c=0$ represents two parallel lines if $a:h=h:b=g:f$. Also show that the perpendicular distance between two lines is $2\sqrt{\frac{g^2-ac}{a(a+b)}}$

→ Start with the condition of parallel lines $\boxed{ab=h^2}$

→ $$\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

The ratio will be solved through this

→ assume the parallel lines are

$$lx+my+n=0 \text{ and } lx+my+n_1=0$$

$$\therefore ax^2+2hxy+by^2+2gx+2fy+c = (lx+my+n)(lx+my+n_1)$$

expand and equate.

→ Use the formula to find the perpendicular distance between two parallel lines

$$\boxed{\text{distance} = \frac{n_1-n}{\sqrt{l^2+m^2}}}$$

$$\boxed{n_1-n=\sqrt{(n_1+n)^2-4nn_1}}$$

→ replace the values of terms containing l, m, n and n_1 into a, b, c, h, g, f and c.

11. If the pair of straight line $x^2 - 2axy - y^2 = 0$ and $x^2 - 2bxy - y^2 = 0$ be such that each pair bisects the angle between the other pair, prove that $ab = -1$

→ take equation(i)

→ take equation of bisectors of angles

$$\frac{x^2 - y^2}{1 - (-1)} = \frac{xy}{-a}$$

or, $$\frac{x^2 - y^2}{xy} = -\frac{2}{a} \quad \text{——— (iii)}$$

→ rearrange equation (ii) in a form

$$\frac{x^2 - y^2}{xy} = 2b \quad \text{——— (iv)}$$

→ comparing (iii) and (iv), $\boxed{ab = -1}$

12. Prove that the straight lines represented by $ax^2+2hxy+by^2+2gx+2fy+c=0$ will be equidistant from origin if $f^4-g^4 = c(bf^2-ag^2)$

→ The lines are $lx+my+n=0$ and

$$l_1x+m_1y+n_1=0$$

→ distance of the lines from origin

$$\frac{n}{\sqrt{l^2+m^2}} = \frac{n_1}{\sqrt{l_1^2+m_1^2}}$$

expand upto

$$(n_1l+nl_1)\sqrt{(n_1l+nl_1)^2-4nn_1ll_1} = (m_1n+mn_1)\sqrt{(m_1n+mn_1)^2-4mm_1nn_1} \quad \text{——(1)}$$

→ $ax^2+2hxy+by^2+2gx+2fy+c = (lx+my+n)(l_1x+m_1y+n_1)$

expand and equate the coefficients of x^2, y^2, xy, x, y and constant.

→ put it in the equation(1)

Math-201
Conics

1. second degree general equation

$$ax^2+2hxy+by^2+2gx+2fy+c=0$$

In case of conics,

$ab-h^2=0 \longrightarrow$ parabola
$ab-h^2>0 \longrightarrow$ ellipse
$ab-h^2<0 \longrightarrow$ hyperbola

2. Removing xy term

when there is a xy term in the equation of a conic, the axis of the conic is rotated at θ angle with respect to the coordinate system.

$$\theta = \frac{1}{2}\tan^{-1}\frac{2h}{a-b}$$

$\rightarrow$
$$x = x'\cos\theta - y'\sin\theta$$
$$y = y'\cos\theta + x'\sin\theta$$

replacing and simplifying, the equation gets free from xy term.

3. Find the equation of conic referred to center as origin

$$F = ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \quad \text{---(I)}$$

$$\frac{\partial F}{\partial x} = 2ax + 2hy + 2g = 0$$

i.e. $ax + hy + g = 0$ ---(II)

$$\frac{\partial F}{\partial y} = 2hx + 2by + 2f = 0$$

i.e. $hx + by + f = 0$ ---(III)

→ Solving the two equations we get

$$\boxed{(x_1, y_1)}$$

→ The constant term when the center of the conic is at origin is

$$\boxed{c_1 = gx_1 + fy_1 + c}$$

→ The required equation is

$$\boxed{ax^2 + 2hxy + by^2 + c_1 = 0}$$

Math-201
Circle

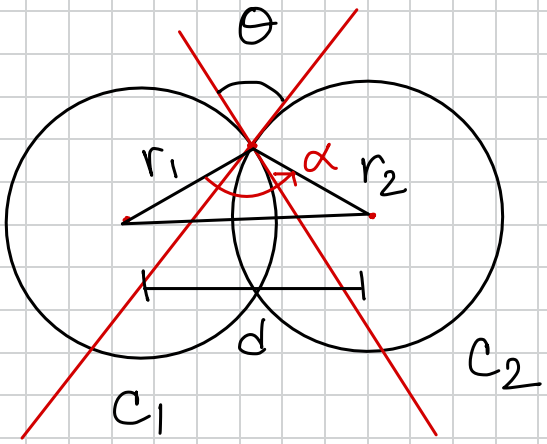
1. Common chord

$$S_1 = x^2+y^2+2g_1x+2f_1y+c_1 = 0$$
$$S_2 = x^2+y^2+2g_2x+2f_2y+c_2 = 0$$

equation of common chord is

$$\boxed{S_1 - S_2 = 0}$$

2. Angle between two circles



$$\boxed{\cos\theta = \frac{r_1^2+r_2^2-d^2}{2r_1r_2}}$$

3. Orthogonal Circle

– when angle between two circles is 90°.

$$\text{i.e. } \cos 90^\circ = \frac{r_1^2 + r_2^2 - d^2}{2r_1 r_2}$$

$$\text{or, } \boxed{r_1^2 + r_2^2 = d^2} \quad \text{——(1)}$$

For circles,

$$S_1 = x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$$

and $S_2 = x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$

$$r_1 = \sqrt{g_1^2 + f_1^2 - c_1} \; ; \; r_2 = \sqrt{g_2^2 + f_2^2 - c_2}$$

and $$d = \sqrt{(-g_1 + g_2)^2 + (-f_1 + f_2)^2}$$

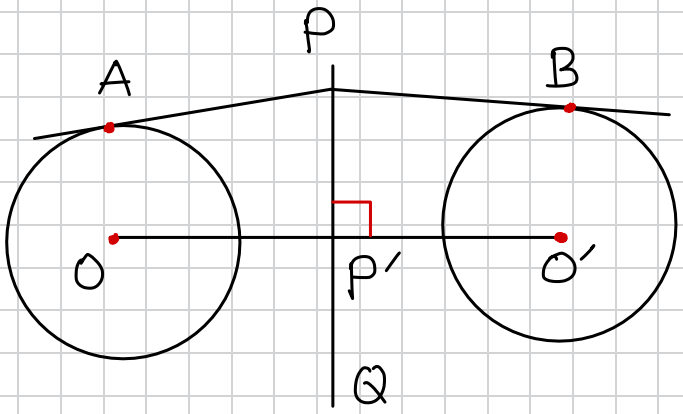
putting the value in equation (1) and further simplifying, we get

$$\boxed{2g_1g_2 + 2f_1f_2 = c_1 + c_2}$$

which is the condition for orthogonality of two circles.

4. Radical Axis

— locus of a point which moves so that the tangent drawn from that point to two given circles are equal and it is a straight line perpendicular to the line which joins the centers of the two circles.



$PA = PB$ and $PP' \perp OO'$

$\therefore$ PQ is the radical axis for the two circles.

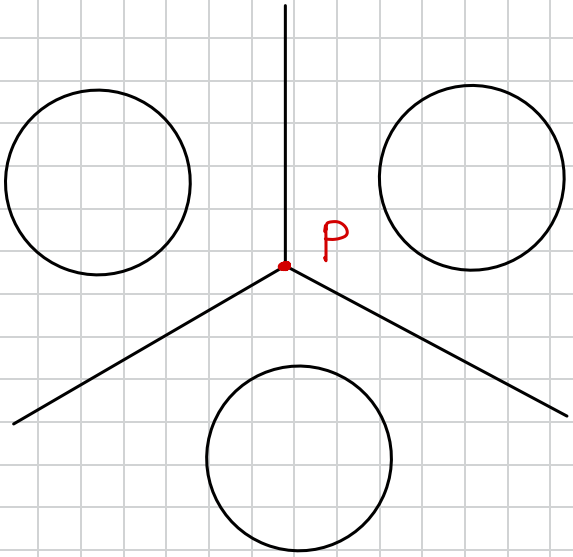
For two circles,

$$S_1 = x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$$
$$S_2 = x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$$

Radical axis is $\boxed{S_1 - S_2 = 0}$

5. Radical Center

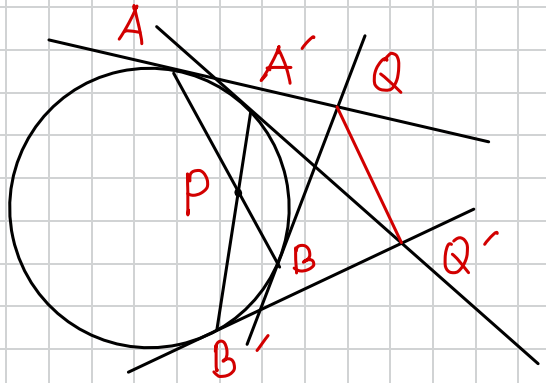
The point of concurrence of the radical axes of each pair of circles whose centers are not co-linear is called radical center



6. Find equation of circle whose center lies on the radical axis of circles S_1 and S_2.

$$S_1 + \lambda(S_1 - S_2) = 0$$

7. Pole and Polar



Here, P is the pole and QQ' is the polar

→ equation of polar of the circle $x^2+y^2=r^2$ having pole $P(x_1,y_1)$ is

$$\boxed{xx_1+yy_1=r^2}$$

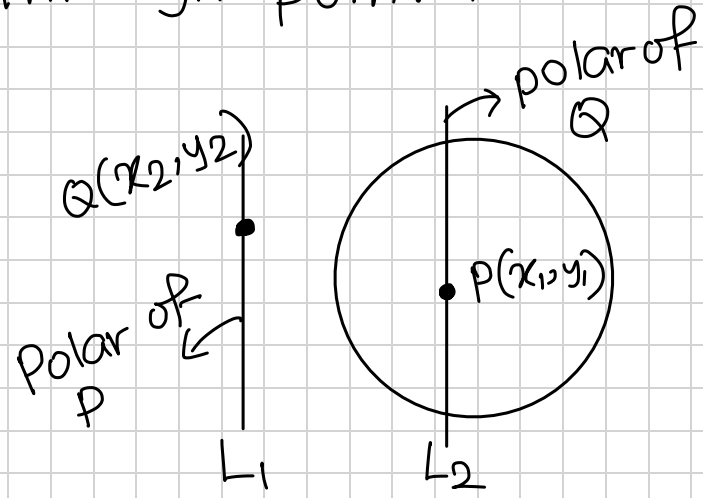
→ equation of polar of the circle $x^2+y^2+2gx+2fy+c=0$ having pole $P(x_1,y_1)$ is

$$xx_1+yy_1+2g\left(\frac{x+x_1}{2}\right)+2f\left(\frac{y+y_1}{2}\right)+c=0$$

Properties :

- If polar of a point P with respect to the circle passes through point Q, the polar of point Q will pass through point P.

- P and Q are called conjugate points and L_1 and L_2 are called conjugate lines.



- Line joining the pole and center of the circle will be perpendicular to the polar.

8. Coaxial circles

A system of circles where every pair of circle has the same radical axis is called the coaxial system.

9. Limiting points

Limiting points are the center of the point circle belonging to the family.

Finding Limiting points :

→ At first find the equation of radical axis. $[S_1 - S_2 = 0]$

→ $S_1 + \lambda(S_1 - S_2) = 0$, find out the coefficients of x and y and the constant term.

→ find center of the circle and radius, r.

→ $r = 0$, find value of λ from it's quadratic equation. $\lambda = a, b$

→ put the value of λ in the center coordinates to find the two limiting points of the coaxial system.

10. Finding pole from equation of circle and polar

→ From equation of circle, find equation of polar. But the equation of polar is given.

→ Compare between the coefficients of x, y and constant to find out the co-ordinates of the pole $P(x_1, y_1)$

Math-201
Solid Geometry

1. Direction Cosine

If a line $\overrightarrow{OP}$ makes angle α, β and γ with the positive direction of X, Y and Z axes respectively, then $\cos\alpha$, $\cos\beta$ and $\cos\gamma$ are the direction cosines of the line $\overrightarrow{OP}$ which are generally denoted by the letters l, m and n respectively.

$$\boxed{l^2+m^2+n^2 = 1}$$

2. Direction Ratios

Any three numbers which are proportional to the direction cosines of a line are called direction ratios of the line.

$$a \propto l \qquad b \propto m \qquad c \propto n$$

$$a = kl \qquad b = km \qquad c = kn$$

$$\text{i.e. } \frac{a}{l} = \frac{b}{m} = \frac{c}{n} = \frac{\sqrt{a^2+b^2+c^2}}{\sqrt{l^2+m^2+n^2}}$$

$$\therefore\ l = \frac{a}{\sqrt{a^2+b^2+c^2}}\ ;\ m = \frac{b}{\sqrt{a^2+b^2+c^2}}$$

$$n = \frac{c}{\sqrt{a^2+b^2+c^2}}$$

- Direction ratios are component of vectors.

- Parallel lines have same set of direction cosines.

3. Angle between two lines

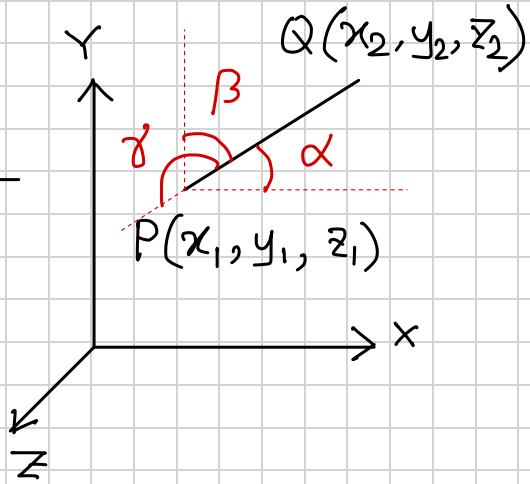
$$\vec{V_1} = a_1\hat{i} + b_1\hat{j} + c_1\hat{k}$$

$$\vec{V_2} = a_2\hat{i} + b_2\hat{j} + c_2\hat{k}$$

$$\cos\theta = \frac{\vec{V_1}\cdot\vec{V_2}}{|\vec{V_1}|\cdot|\vec{V_2}|} = \frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}$$

4. Relation between Projection and Direction cosine

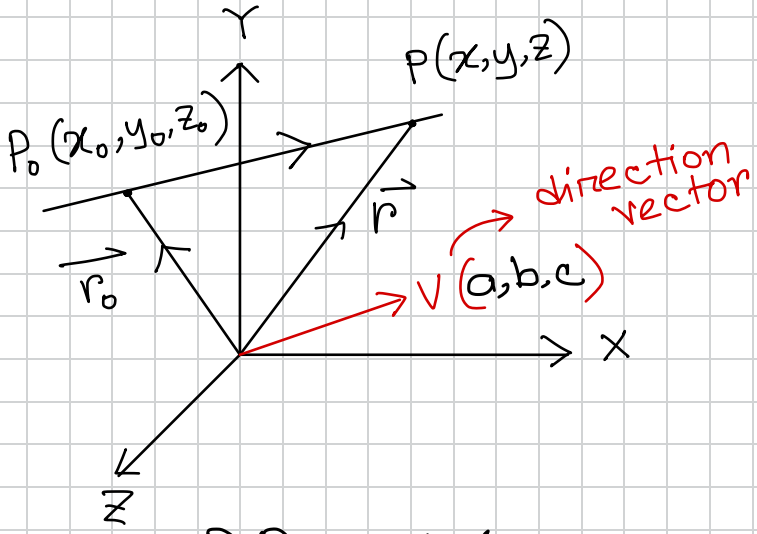
$$PQ = \frac{x_2-x_1}{l} = \frac{y_2-y_1}{m} = \frac{z_2-z_1}{n}$$



5. Equation of Lines in 3D

Standard equation of a vector line is

$$\vec{r} = \vec{r_0} + \vec{v}t$$



$$x = x_0 + at$$
$$y = y_0 + bt$$
$$z = z_0 + ct$$

$$P_0P = \langle x-x_0, y-y_0, z-z_0 \rangle$$

$$\therefore (x,y,z) = (x_0+at,\ y_0+bt,\ z_0+ct)$$

$$\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c} = t$$

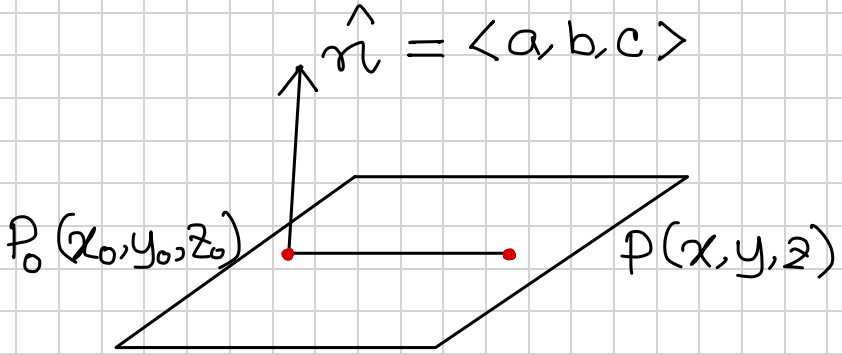
The symmetric equation of a vector is

$$\frac{x-x_0}{l} = \frac{y-y_0}{m} = \frac{z-z_0}{n}$$

where l, m and n are the direction cosines of the the vector.

6. The Plane

A plane is a surface such that if any two points are taken on it, the straight line joining them lies wholly on the surface i.e every point on the line joining the two points will be on the plane.



$$P_0P = \langle x - x_0, y - y_0, z - z_0 \rangle$$

$$P_0P \cdot \hat{n} = 0$$

or, $ax + by + cz - ax_0 - by_0 - cz_0 = 0$

d

i.e.

$$\boxed{ax + by + cz + d = 0}$$

which is the equation of a plane.

- parallel planes have same normal vector. i.e. the coefficients of x, y and z are identical.

7. Equation of plane when two points on the plane and and any plane perpendicular to it is given.

Let the points are $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$

$\therefore \overrightarrow{PQ} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$

and the equation of plane perpendicular to the required plane is

$$ax + by + cz + d = 0$$

$\therefore$ normal vector, $n_1 = \langle a, b, c \rangle$

Since the planes are perpendicular, the normal vector n_1 is coplanar with required plane.

$\therefore$ normal vector to the required plane

$$n = \begin{vmatrix} i & j & k \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a & b & c \end{vmatrix}$$

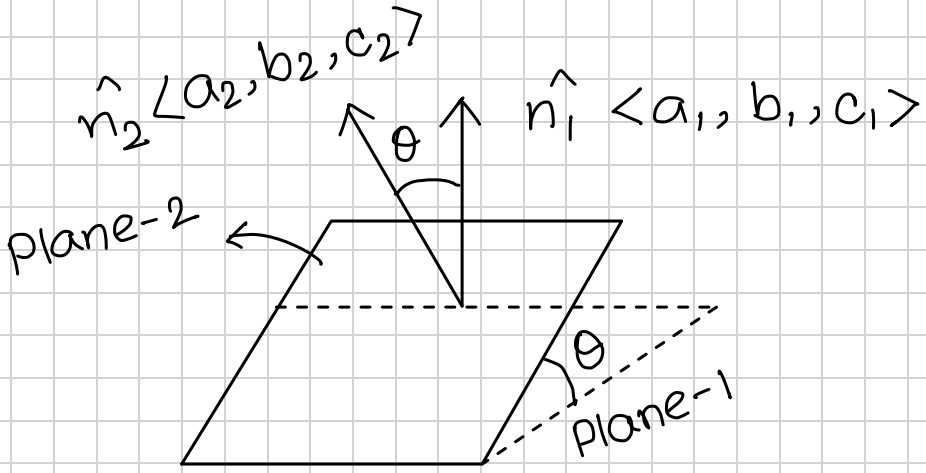
$$= \langle a', b', c' \rangle$$

∴ The equation of the plane having normal vector $\langle a', b', c' \rangle$ and passing through the point $P(x_1, y_1, z_1)$ is

$$\boxed{a'(x-x_1)+b'(y-y_1)+c'(z-z_1)=0}$$

8. Angle between two planes

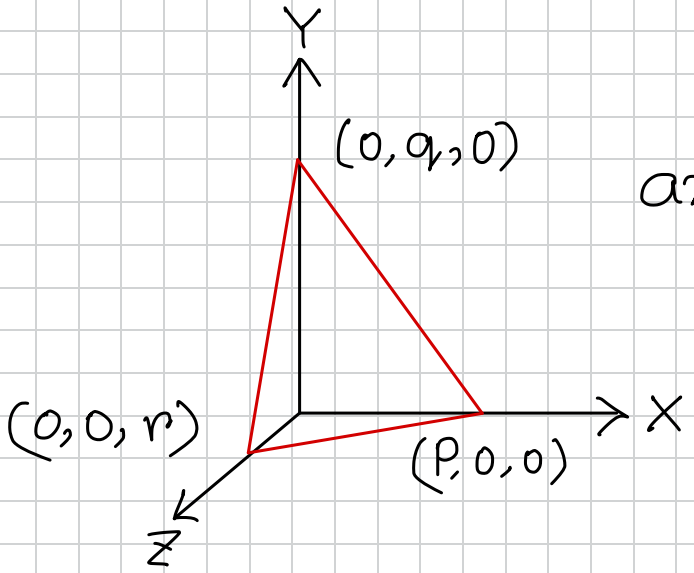
$$a_1x + b_1y + c_1z + d_1 = 0$$
$$a_2x + b_2y + c_2z + d_2 = 0$$



$$\cos\theta = \frac{\hat{n}_1 \cdot \hat{n}_2}{|\hat{n}_1||\hat{n}_2|}$$

$$\text{i.e. } \theta = \cos^{-1} \frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{(a_1^2+b_1^2+c_1^2)(a_2^2+b_2^2+c_2^2)}}$$

9. Plane in intercept form



$$ax+by+cz+d=0 \quad \text{——(1)}$$

plotting points p, q and r in equation (1),

$ap=-d$ $\quad$ $bq=-d$ $\quad$ $cr=-d$

or, $a=-d/p$ $\quad$ or, $b=-d/q$ $\quad$ or, $c=-d/r$

which implies,

$$-\frac{d}{p}x-\frac{d}{q}y-\frac{d}{r}z+d=0$$

i.e. $$\boxed{\frac{x}{p}+\frac{y}{q}+\frac{z}{r}=0}$$

10. Perpendicular distance of a point from a plane

point, $P(x_1, y_1, z_1)$

plane $\rightarrow$ $ax+by+cz+d=0$

$$\boxed{\text{distance} = \frac{|ax_1+by_1+cz_1+d|}{\sqrt{a^2+b^2+c^2}}}$$

11. Coplanar line

If two lines passes through points (x_1, y_1, z_1) and (x_2, y_2, z_2) respectively and their direction cosines are (l_1, m_1, n_1) and (l_2, m_2, n_2) respectively, then the lines are coplanar if

$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

12. Distance from a point (x_1, y_1, z_1) to a point where a line $\frac{x-p}{l} = \frac{y-q}{m} = \frac{z-r}{n}$ meets the plane

$$ax+by+cz+d = 0$$

$$\frac{x-p}{l} = \frac{y-q}{m} = \frac{z-r}{n} = k$$

or, $(x, y, z) = (kl+p, km+q, kn+r)$

Since intersects the plane,

$$a(kl+p) + b(km+q) + c(kn+r) + d = 0$$

Simplifying, $k =$ _____

∴ point of intersection found and distance measured.

13. Shortest Distance

Straight line which is the perpendicular to two non-intersecting lines is called the line of shortest distance.

Shortest distance between lines $\frac{x-x_1}{l_1} = \frac{y-y_1}{m_1} = \frac{z-z_1}{n_1}$ and $\frac{x-x_2}{l_2} = \frac{y-y_2}{m_2} = \frac{z-z_2}{n_2}$:

Let, $\frac{x-x_1}{l_1} = \frac{y-y_1}{m_1} = \frac{z-z_1}{n_1} = r_1$

or, $(x, y, z) = (l_1 r_1 + x_1, m_1 r_1 + y_1, n_1 r_1 + z_1)$ ——— (I)

again, $\frac{x-x_2}{l_2} = \frac{y-y_2}{m_2} = \frac{z-z_2}{n_2} = r_2$

or, $(x, y, z) = (l_2 r_2 + x_2, m_2 r_2 + y_2, n_2 r_2 + z_2)$ ——— (II)

Let the direction cosine of the line of shortest distance is (l, m, n)

$\therefore (l, m, n) = (l_2 r_2 - l_1 r_1 + x_2 - x_1, m_2 r_2 - m_1 r_1 + y_2 - y_1, n_2 r_2 - n_1 r_1 + z_2 - z_1)$

→ line of shortest distance is at right angle with both the lines.

$\therefore l_1 l + m_1 m + n_1 n = 0 \rightarrow r_1 = ____$

$l_2 l + m_2 m + n_2 n = 0 \rightarrow r_2 = ____$

∴ The points on the two lines are determined. Now use the formula of distance to find the shortest distance.

$$\text{distance} = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$$

14. Equation of line of shortest distance

Find the direction cosine of the line of shortest distance by putting the values of r_1 and r_2

$$(l,m,n) = (l_2r_2-l_1r_1+x_2-x_1,\ m_2r_2-m_1r_1+y_2-y_1,\ n_2r_2-n_1r_1+z_2-z_1)$$

The line passes through one of the two points obtained. (x_1, y_1, z_1)

∴ Equation of the line of shortest distance is

$$\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$$