[4] Find an equation for the bisectors of the angles between the lines represented by $ax^2+2hxy+by^2=0$.

Soln: The given eqn is,

$$ax^2+2hxy+by^2=0 \quad \text{—①}$$

We know the lines represented by ① are,

$$y-m_1x=0 \text{ and } y-m_2x=0$$

then; $m_1+m_2=-\frac{2h}{b}$; $m_1m_2=\frac{a}{b}$

Thus, the equation of the required bisectors are, $\frac{y-m_1x}{\sqrt{1^2+(-m_1)^2}} = \pm \frac{y-m_2x}{\sqrt{1^2+(-m_2)^2}}$

$$\Rightarrow \frac{(y-m_1x)^2}{1+m_1^2} = \frac{(y-m_2x)^2}{1+m_2^2}$$

$$\Rightarrow (1+m_2^2)(y-m_1x)^2 = (1+m_1^2)(y-m_2x)^2$$

$$\Rightarrow (1+m_2^2)(y^2-2xym_1+m_1^2x^2) = (1+m_1^2)(y^2-2xym_2+m_2^2x^2)$$

$$\Rightarrow \{(1+m_2^2)-(1+m_1^2)\}y^2-2\{m_1(1+m_2^2)-m_2(1+m_1^2)\}xy + \{m_1^2(1+m_2^2)-m_2^2(1+m_1^2)\}x^2=0$$

$$\Rightarrow (m_2^2-m_1^2)y^2-2\{(m_1-m_2)+m_1m_2(m_2-m_1)\}xy + (m_1^2-m_2^2)x^2=0$$

$\Rightarrow (x^2-y^2)(m_1^2-m_2^2) - 2\{(m_1-m_2) - m_1m_2(m_1-m_2)\}xy = 0$

$\Rightarrow (x^2-y^2)(m_1^2-m_2^2) - 2(m_1-m_2)(1-m_1m_2)xy = 0$

$\Rightarrow (x^2-y^2)(m_1+m_2) - 2(1-m_1m_2)xy = 0$

$\Rightarrow (x^2-y^2)\left(-\frac{2h}{b}\right) - 2\left(1-\frac{a}{b}\right)xy = 0$

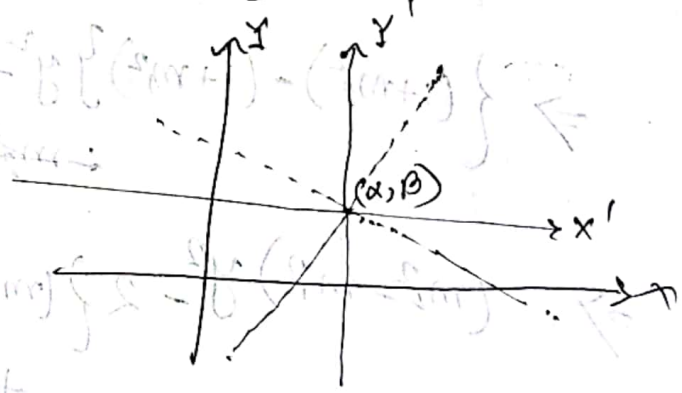
$\Rightarrow \frac{x^2-y^2}{a-b} = \frac{xy}{h}$

Which is the required equation of bisectors.

* Find the condition that the general eqn of second degree may represent a pair of straight line.

Soln: Let us consider the general eqn of second degree, $ax^2+2hxy+by^2+2gx+2fy+c=0$ —— (1) represents a pair of straight lines. Let, their point of intersection be (α,β).

By transforming the Origin to (α,β) keeping the direction of the axes unaltered.



Let (x,y) and (x',y') reffered to the same point with original coordinate and transformed coordinate system. then,

$$x = x' + \alpha$$
$$y = y' + \beta$$

then from (1) we get,

$$a(x'+\alpha)^2 + 2h(x'+\alpha)(y'+\beta) + b(y'+\beta)^2 + 2g(x'+\alpha) + 2f(y'+\beta) + c = 0$$

$$\Rightarrow a(x'^2 + 2x'\alpha + \alpha^2) + 2h(x'y' + y'\alpha + x'\beta + \alpha\beta) + b(y'^2 + 2\beta y' + \beta^2) + 2gx' + 2g\alpha + 2fy' + 2f\beta + c = 0$$

$$\Rightarrow ax'^2 + 2hx'y' + by'^2 + 2(a\alpha + h\beta + g)x' + 2(h\alpha + b\beta + f)y' + a\alpha^2 + 2h\alpha\beta + b\beta^2 + 2g\alpha + 2f\beta + c = 0 \quad \text{---(3)}$$

Since eqn (3) represents a pair of straight lines through the new origin, it must be homogeneous in x' and y'.

Therefore the first order in x', y' and the constant term must be disappeared and leave only,

$$ax'^2 + 2hx'y' + by'^2 = 0 \quad \text{---(4)}$$

Then from eqn ③ we get,

$$a\alpha + h\beta + g = 0 \quad \text{—⑤}$$
$$h\alpha + b\beta + f = 0 \quad \text{—⑥}$$

and $a\alpha^2 + 2h\alpha\beta + b\beta^2 + 2g\alpha + 2f\beta + c = 0$

$\Rightarrow \alpha(a\alpha + h\beta + g) + \beta(h\alpha + b\beta + f) + g\alpha + f\beta + c = 0$

$\Rightarrow \alpha \cdot 0 + \beta \cdot 0 + g\alpha + f\beta + c = 0$ [using ⑤, ⑥]

$\Rightarrow g\alpha + f\beta + c = 0 \quad \text{—⑦}$

Now eliminating α, β from eqn ⑤, ⑥, ⑦ the required condition is,

$$\Delta \equiv \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

Or, $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$

Which is the required condition.

Note: $ax + by = 0 \quad \text{—①}$
$cx + dy = 0 \quad \text{—②}$

Now eliminating x, y from ① and ②

① imply $x = -\frac{b}{a}y$

② imply, $c\left(-\frac{b}{a}\right)y + dy = 0$

$\Rightarrow (-bc + ad)y = 0$

since, $y \neq 0$

$\therefore ad - bc = 0$

$\Rightarrow \begin{vmatrix} a & b \\ c & d \end{vmatrix} = 0$

Problem 01. Show that, $3x^2 - 14xy - 5y^2 - 54x - 2y + 51 = 0$ represents a pair of straight lines.

Soln: Given,

$$3x^2 - 14xy - 5y^2 - 54x - 2y + 51 = 0 \quad \text{—(1)}$$

Comparing (1) with general eqn of second degree we can write,

$$a = 3,\ h = -7,\ b = -5,\ g = -27,\ f = -1,$$
$$c = 51$$

Now, (1) will represent a pair of straight lines if

$$\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

Now,

$$\Delta = \begin{vmatrix} 3 & -7 & -27 \\ -7 & -5 & -1 \\ -27 & -1 & 51 \end{vmatrix}$$

$$= 3(-255-1) - (-7)(-357-27) + (-27)(7-135)$$

$$= -768 - 2688 + 3456$$

$$= 0$$

Since $\Delta = 0$

$\therefore$ eqn (1) represents a pair of straight line.

Problem 02: Find the value of λ so that the equation $\lambda x^2+4xy+y^2-4x-2y-3=0$ may represent a pair of straight line.

Soln: Given,

the eqn is,

$$\lambda x^2+4xy+y^2-4x-2y-3=0 \quad \text{— (1)}$$

Comparing (1) with general eqn of second degree we can write,

$$a=\lambda,\ h=2,\ b=1,\ g=-2,\ f=-1,\ c=-3$$

Since, (1) represents a pair of straight lines,

$$\therefore \Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} \lambda & 2 & -2 \\ 2 & 1 & -1 \\ -2 & -1 & -3 \end{vmatrix} = 0$$

$$\Rightarrow \lambda(-3-1)-2(-6-2)-2(-2+2)=0$$

$$\Rightarrow 4\lambda = 16$$

$$\Rightarrow \lambda = 4 \quad \text{(Ans)}$$

Problem-03: Show that the equation $ax^2+2hxy+by^2+2gx+2fy+c=0$ represents two parallel lines if $a:h = h:b = g:f$. Also show that the distance between them is $2\sqrt{\frac{g^2-ac}{a(a+b)}}$

Soln: We know ~~that~~ the condition that the general equation represents two straight lines if,

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

$\Rightarrow abc + 2fgh - af^2 - bg^2 - ch^2 = 0$ ——— (1)

since the lines are parallel, so, $ab = h^2$

from (1) we can write,

$$h^2c + 2fg\sqrt{ab} - af^2 - bg^2 - ch^2 = 0$$

$\Rightarrow af^2 - 2fg\sqrt{ab} + bg^2 = 0$

$\Rightarrow (\sqrt{a}f)^2 - 2\sqrt{a}f\sqrt{b}g + (\sqrt{b}g)^2 = 0$

$\Rightarrow (\sqrt{a}f - \sqrt{b}g)^2 = 0$

$\Rightarrow \sqrt{a}f = \sqrt{b}g$

$\Rightarrow \frac{g}{f} = \frac{\sqrt{a}}{\sqrt{b}} = \frac{\sqrt{ab}}{b}$

$\Rightarrow \frac{g}{f} = \frac{h}{b} \Rightarrow g:f = h:b$ ——— (2)

Again,

$$ab = h^2$$

$$\Rightarrow \frac{a}{h} = \frac{h}{b}$$

$$\Rightarrow a : h = h : b \quad \text{— (3)}$$

Combining (2) and (3) we can write,

$$a : h = h : b = g : f$$

2nd part: Let the two ~~lines~~ parallel lines be,

$$lx + my + n = 0 \text{ and } lx + my + n_1 = 0$$

$$\therefore ax^2 + 2hxy + by^2 + 2gx + 2fy + c = (lx + my + n)(lx + my + n_1)$$

$$= l^2x^2 + m^2y^2 + (lm + ml)xy + (ln + ln_1)x + (mn + mn_1)y + nn_1 = 0$$

Equating like terms we can write,

$$l^2 = a,\ m^2 = b,\ nn_1 = c,\ l(n_1 + n) = 2g$$

Now, the distance between two lines ~~are~~ is:

$$\frac{n_1 - n}{\sqrt{l^2 + m^2}} = \frac{\sqrt{(n_1 + n)^2 - 4nn_1}}{\sqrt{l^2 + m^2}} = \frac{\sqrt{\left(\frac{2g}{l}\right)^2 - 4c}}{\sqrt{a + b}}$$

$$= \sqrt{\frac{4g^2 - 4cl^2}{l^2(a + b)}} \quad \cancel{= \frac{2\sqrt{g^2 - ac}}{}}$$

$$= 2\sqrt{\frac{g^2 - ac}{a(a + b)}} \quad \text{[showed]}$$