09.07.24

- Vector function $\vec{r} = f(t)$

  || point function $\vec{F} = \vec{F}(P) = \vec{F}(x, y, z)$

x, y, z are co-ordinates of P

- Scalar point function $\phi = \phi(P) = \phi(x, y, z)$

- Derivative of a vector function with respect to a scalar,

$$\frac{d\vec{r}}{dt} = \lim_{\delta t \to 0} \frac{\vec{f}(t+\delta t) - f(t)}{\delta t}$$

- Gradient, Divergence & Curl:

1. Gradient of a scalar fuel:

$$\nabla\phi = \text{grand}\,\phi = i\frac{\delta\phi}{\delta x} + j\frac{\delta\phi}{\delta y} + k\frac{\delta\phi}{\delta z}$$

$\nabla\phi$ → vector quantity.

2. Divergence of a vector:

$$\nabla \cdot \vec{V} = \text{div}\vec{V} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot \vec{V}$$

$$= \hat{i}\frac{\partial \vec{V}}{\partial x} + \hat{j}\frac{\partial \vec{V}}{\partial y} + \hat{k}\frac{\partial \vec{V}}{\partial z}$$

3. Curl of a vector:

$$\nabla \times \vec{V} = \text{Curl}\,\vec{V} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \times \vec{V}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix}$$

Math: If $\phi = x^3 + y^3 + z^3 - 3xyz$ find gradϕ at $(1, -1, 2)$

$$\text{grad}\,\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^3 + y^3 + z^3 - 3xyz)$$

$$= \hat{i}\frac{\partial}{\partial x}(x^3 + y^3 + z^3 - 3xyz) + \hat{j}\frac{\partial}{\partial y}(x^3 + y^3 + z^3 - 3xyz)$$
$$+ \hat{k}\frac{\partial}{\partial z}(x^3 + y^3 + z^3 - 3xyz)$$

$$= \hat{i}(3x^2 - 3yz) + \hat{j}(3y^2 - 3zx) + \hat{k}(3z^2 - 3xy)$$

$$\text{grad}\,\phi_{(1,-1,2)} = \hat{i}\{3 - 3(-1)2\} + \hat{j}(3 - 3\times 2) + \hat{k}(3\times 2^2 - 3(-1))$$

$$= 9\hat{i} - 3\hat{j} + 15\hat{k}$$

Math: Calculate the curl of $\vec{F} = xyz\hat{i} + 3x^2y\hat{j} + (xz^2 - y^2z)\hat{k}$ at $(1,-1,1)$

$$\text{Curl}\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xyz & 3x^2y & xz^2 - y^2z \end{vmatrix}$$

$$= \hat{i}\left\{\frac{\partial}{\partial y}(xz^2 - y^2z) - \frac{\partial}{\partial z}3x^2y\right\} - \hat{j}\left\{\frac{\partial}{\partial x}(xz^2 - y^2z) - \frac{\partial}{\partial z}xyz\right\} + \hat{k}\left\{\frac{\partial}{\partial x}3x^2y - \frac{\partial}{\partial y}xyz\right\}$$

$$= \hat{i}(-2yz) - \hat{j}(z^2 - xy) + \hat{k}(6xy - xz)$$

$$= \hat{i}(-2\times -1\times 1) - \hat{j}(1 - 1\times -1) + \hat{k}(6\times 1\times -1 - 1\times 1)$$

$$= 2\hat{i} - 2\hat{j} - 7\hat{k}$$

Math: If $\vec{F} = x^3\hat{i} + y^3\hat{j} + z^3\hat{k}$ find $\text{div}(\text{Curl}\vec{F})$

$$\text{Curl}\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^3 & y^3 & z^3 \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial}{\partial y}z^3 - \frac{\partial}{\partial z}y^3\right) - \hat{j}\left(\frac{\partial}{\partial x}z^3 - \frac{\partial}{\partial z}x^3\right) + \hat{k}\left(\frac{\partial}{\partial x}y^3 - \frac{\partial}{\partial y}x^3\right) = 0$$

$$\text{div}(\text{Curl}\vec{F}) = 0$$

Math: If $\vec{F} = x^2y\hat{i} + y^2z\hat{j} + z^2x\hat{k}$ Find Curl(CurlF)

$$\text{Curl}\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y & y^2z & z^2x \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial}{\partial y}z^2x - \frac{\partial}{\partial z}y^2z\right) - \hat{j}\left(\frac{\partial}{\partial x}z^2x - \frac{\partial}{\partial z}x^2y\right) + \hat{k}\left(\frac{\partial}{\partial x}y^2z - \frac{\partial}{\partial y}x^2y\right)$$

$$= -y^2\hat{i} - z^2\hat{j} - x^2\hat{k}$$

$$\text{Curl}(\text{Curl}\vec{F}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y^2 & -z^2 & -x^2 \end{vmatrix}$$

$$= \hat{i}\left\{\frac{\partial}{\partial y}(-x^2) - \frac{\partial}{\partial z}(-z^2)\right\} - \hat{j}\left\{\frac{\partial}{\partial x}(-x^2) - \frac{\partial}{\partial z}(-y^2)\right\} + \hat{k}\left\{\frac{\partial}{\partial x}(-z^2) - \frac{\partial}{\partial y}(-y^2)\right\}$$

$$= 2z\hat{i} + 2x\hat{j} + 2y\hat{k}$$

$$= 2z\hat{i} + 2x\hat{j} + 2y\hat{k}$$

Solenoidal and Irrotational Vector Fields:

1. **Solenoidal Vector Field:** If $\text{div}\vec{F} = 0$, then $\vec{F}$ is called a solenoidal vector.

2. **Irrotational Vector Field:** If $\text{curl}\vec{F} = 0$, then $\vec{F}$ is called irrotational vector.

Math: Prove that $\vec{F} = z\hat{i} + x\hat{j} + y\hat{k}$ is solenoidal.

$$\text{div}\vec{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).(z\hat{i} + x\hat{j} + y\hat{k})$$

$$= 0$$

$\therefore$ Solenoidal.

Math: If $\vec{F} = (ax+3y+4z)\hat{i} + (x-2y+3z)\hat{j} + (3x+2y-z)\hat{k}$ is solenoidal find the value of a.

$$\text{div}\,\vec{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot \{(ax+3y+4z)\hat{i} + (x-2y+3z)\hat{j} + (3x+2y-z)\hat{k}\}$$

$$= a - 2 - 1 \Rightarrow$$

$$\text{div}\,\vec{F} = 0$$

$$\Rightarrow a - 2 - 1 = 0 \Rightarrow a = 3$$

3.9.24

Math: Find the constant a, b, c such that

$\vec{F} = (x+2y+az)\hat{i} + (bz-3y-z)\hat{j} + (4x+cy+2z)\hat{k}$

is irrotational.

$\vec{F} = (x+2y+az)\hat{i} + (bx-3y-z)\hat{k} + (4x+cy+2z)\hat{k}$

If $\vec{F}$ is irrotational, then $\nabla \times \vec{F} = 0$

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+2y+az & bx-3y-z & 4x+cy+2z \end{vmatrix} = 0$$

$$\Rightarrow \hat{i}(c+1) - \hat{j}(4-a) + \hat{k}(b-2) = 0\hat{i} + 0\hat{j} + 0\hat{k}$$

$c+1=0 \Rightarrow c=-1$

$4-a=0 \Rightarrow a=4$

$b-2=0 \Rightarrow b=2$

4.2.3

Given grad ϕ to find the scalar potential ϕ:

Math: If grad $\phi = (y^2 - 2xyz^3)\hat{i} + (3 + 2xy - x^2z^3)\hat{j} + (6z^3 - 3x^2yz^2)\hat{k}$

Find its scalar potential ϕ, $\phi(1,1,-2) = 4$

grad $\phi = (y^2 - 2xyz^3)\hat{i} + (3 + 2xy - x^2z^3)\hat{j} + (6z^3 - 3x^2yz^2)\hat{k}$

$\Rightarrow \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z} =$

$(y^2 - 2xyz^3)\hat{i} + (3 + 2xy - x^2z^3)\hat{j} + (6z^3 - 3x^2yz^2)\hat{k}$

$\frac{\partial\phi}{\partial x} = y^2 - 2xyz^3$

$\frac{\partial\phi}{\partial x} = y^2 - 2xyz^3$ ——— (1)

$\frac{\partial\phi}{\partial y} = 3 + 2xy - x^2z^3$ ——— (2)

$\frac{\partial\phi}{\partial z} = 6z^3 - 3x^2yz^2$ ——— (3)

Integrating (1) w.r.t to x,

$\phi(x,y,z) = xy^2 - x^2yz^3 + f(y,z)$ ——— (4)

Integrating (2) w.r.t to y,

$\phi = 3y + xy^2 - x^2y^3 + f(z,x)$ ——— (5)

Integrating (3) w.r.t to z,

$\phi = \frac{6}{4}z^4 - \frac{3}{4}x^2yz^3 + f(x,y)$ ——— (6)

From (4), (5), (6),

$\phi(x,y,z) = xy^2 - x^2yz^3 + 3y + \frac{3}{2}z^4 + C$

$\phi(1,1,-2) = 4$

$\Rightarrow 1 + 8 + 3 + 24 + C = 4$

$\Rightarrow C = -32$

Math: Prove that $\vec{F} = \nabla\phi = (y^2\cos x + z^3)\hat{i} + (2y\sin x - 4)\hat{j} + 3xz^2\hat{k}$ is irrotational. Hence find its scalar potential.

$$\text{Curl } \vec{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \times (y^2\cos x + z^3)\hat{i} + (2y\sin x - 4)\hat{j} + 3xz^2\hat{k}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2\cos x + z^3 & 2y\sin x - 4 & 3xz^2 \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial}{\partial y}3xz^2 - \frac{\partial}{\partial z}(2y\sin x - 4)\right) - \hat{j}\left(\frac{\partial}{\partial x}3xz^2 - \frac{\partial}{\partial z}(y^2\cos x + z^3)\right) + \hat{k}\left(\frac{\partial}{\partial x}(2y\sin x - 4) - \frac{\partial}{\partial y}(y^2\cos x + z^3)\right)$$

$$= -\hat{j}(3z^2 - 3z^2) + \hat{k}(2y\cos x - 2y\cos x)$$

$$= 0$$

$\therefore$ Irrotational.

Given that,

$$i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z} = (y^2\cos x + z^3)\hat{i} + (2y\sin x - 4)\hat{j} + 3xz^2\hat{k}$$

$$\frac{\partial\phi}{\partial x} = y^2\cos x + z^3$$

Integrating w.r.to x,

$$\phi = y^2\sin x + xz^3 + f(y,z) \quad \text{——} (1)$$

$$\frac{\partial\phi}{\partial y} = 2y\sin x - 4$$

Integrating w.r.t. y,

$$\phi = y^2\sin x - 4y + f(z,x) \quad \text{——} (2)$$

$$\frac{\partial\phi}{\partial z} = 3xz^2$$

Integrating w.r.t z,

$$\phi = xz^3 + f(x,y) \quad \text{——} (3)$$

$\phi = y^2\sin x + xz^2 - 4y + C$

Math: If $\vec{F} = (y+\sin z)\hat{i} + xy\hat{j} + x\cos z\hat{k}$ where $\vec{F} = \nabla\phi$, find its scalar potential

Given that,

$\hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z} = (y+\sin z)\hat{i} + xy\hat{j} + x\cos z\hat{k}$

$\frac{\partial\phi}{\partial x} = y + \sin z$

Integrating wrt x,

$\phi = xy + x\sin z$ ~~+ f(x)~~ ——①$ + f(y,z)$ ——①

$\frac{\partial\phi}{\partial y} = xy$

Integrating wrt y,

$\phi = \frac{xy^2}{2} + f(z,x)$ ——②

$\frac{\partial\phi}{\partial z} = x\cos z$

Integrating wrt z,

$\phi = x\sin z + f(x,y)$ ——③

From ①, ②, ③,

$\phi = xy + x\sin z + \frac{xy^2}{2} + C$

Math: If $\vec{F} = (6xy+z^3)\hat{i} + (3x^2-z)\hat{j} + (3xz^2-y)\hat{k}$ where $\vec{F} = \nabla\phi$ find its scalar potential

$\hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z} = (6xy+z^3)\hat{i} + (3x^2-z)\hat{j} + (3xz^2-y)\hat{k}$

$\frac{\partial\phi}{\partial x} = 6xy + z^3$

Integrating wrt x,

$\phi = 3x^2y + z^3x + F(y,z)$ ——①

$\frac{\partial\phi}{\partial y} = 3x^2 - z$

Integrating wrt y,

$\phi = 3x^2y - yz + F(z,x)$ ——②

$\frac{\partial\phi}{\partial z} = 3xz^2 - y$

Integrating wrt z,

$\phi = xz^3 - xy + F(x,y)$ ——③

From ①, ②, ③,

$\phi = 3x^2y + z^3x - xz - xy + c$

Unit vector normal to the surface:

$\hat{n} = \frac{\text{grad}\phi}{|\text{grad}\phi|} = \frac{\nabla\phi}{|\nabla\phi|}$ is the unit vector normal to the surface $\phi(x,y,z) = c$ at any point (x,y,z).

Math: Find that unit vector normal to the surface given by $x^2y + 2zx - 4 = 0$ at $(2,-2,3)$

$\phi = x^2y + 2zx - 4$

$\text{Grad}\phi = \hat{i}(2x+2z) + \hat{j}x^2 + 2x\hat{k}$

$\nabla\phi(2,-2,3) = -2\hat{i} + 4\hat{j} + 4\hat{k}$

$|\nabla\phi| = 6$

$\hat{n} = \frac{-2i + 4j + 4k}{6} = -\frac{1}{3}\hat{i} + \frac{2}{3}\hat{j} + \frac{2}{3}\hat{k}$

Angle between two surface:

$\hat{n}_1 = \frac{\text{grad}\phi_1}{|\text{grad}\phi_1|}$ $\hat{n}_2 = \frac{\text{grad}\phi_2}{|\text{grad}\phi_2|}$

ϕ θ be the ~~ab~~ angle between them.

$\hat{n}_1 \cdot \hat{n}_2 = \cos\theta$

Math: Find the angle between the surfaces, $x^2+y^2+z^2 = 9$. $z = x^2+y^2-3$ at $(2,-1,2)$

$\phi_1 = x^2+y^2+z^2-9$

$\text{grad}\phi_1 = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$

$\nabla\phi_1(2,-1,2) = 4\hat{i} - 2\hat{j} + 4\hat{k}$

$|\nabla\phi_1| = 6$

$\hat{n}_1 = \frac{1}{6}(4\hat{i} - 2\hat{j} + 4\hat{k})$

$\phi_2 = x^2+y^2-z-3$

$\nabla\phi_2 = 2x\hat{i} + 2y\hat{j} - \hat{k}$

$\nabla\phi_2(2,-1,3) = 4\hat{i} - 2\hat{j} - \hat{k}$

$|\nabla\phi_2| = \sqrt{21}$

$\hat{n}_2 = \frac{1}{\sqrt{21}}(4\hat{i} - 2\hat{j} - \hat{k})$

$$\eta_1 \cdot \eta_2 = \frac{1}{6\sqrt{21}}(16+9-9)$$

$$= \frac{8}{3\sqrt{21}}$$

$$\cos\theta = \frac{8}{3\sqrt{21}}$$

$$\Rightarrow \theta = \cos^{-1}\frac{8}{3\sqrt{21}}$$

Math: Find the angle between the surfaces $x\log z = y^2 - 1$ and $x^2 y = 2 - z$ at the point $(1,1,1)$

$$x\log z - y^2 + 1 = \phi_1$$

$$\nabla\phi_1 = (\log z)\hat{i} + \hat{j}(-2y) + \hat{k}\frac{x}{z}$$

$$\nabla\phi_{1\,(1,1,1)} = -2\hat{j} + \hat{k}$$

$$|\nabla\phi| = \sqrt{5}$$

$$\eta_1 = \frac{1}{\sqrt{5}}(-2\hat{j} + \hat{k})$$

$$\phi_2 = x^2 y + z - 2$$

$$\nabla\phi_2 = 2xy\hat{i} + x^2\hat{j} + \hat{k}$$

$$\nabla\phi_{2(1,1,1)} = 2\hat{i} + \hat{j} + \hat{k}$$

$$|\nabla\phi_2| = \sqrt{6}$$

$$\eta_2 = \frac{1}{\sqrt{6}}(2\hat{i} + \hat{j} + \hat{k})$$

$$\eta_1 \cdot \eta_2 = \frac{1}{\sqrt{30}}(-2+1) = -\frac{1}{\sqrt{30}}$$

$$\Rightarrow \cos\theta = -\frac{1}{\sqrt{30}}$$

$$\Rightarrow \theta = \cos^{-1}\left(\frac{1}{\sqrt{30}}\right)$$

CamScanner

Math: Find the angle between the surface $x^2-y^2-z^2=11$ and $xy+yz-zx=18$ at the point $(6,4,3)$

$\phi_1 = x^2-y^2-z^2-11$

$\nabla\phi_1 = 2x\hat{i}-2y\hat{j}-2z\hat{k}$

$\nabla\phi_1(6,4,3) = 12\hat{i}-8\hat{j}-6\hat{k}$

$|\nabla\phi_1| = 2\sqrt{61}$

$n_1 = \frac{1}{\sqrt{61}}(6\hat{i}-4\hat{j}-3\hat{k})$

$\phi_2 = xy+yz-zx-18$

$\nabla\phi_2 = (y-z)\hat{i}+(x+z)\hat{j}+(y-x)\hat{k}$

$\nabla\phi_2(6,4,3) = \hat{i}+9\hat{j}-2\hat{k}$

$|\nabla\phi_2| = \sqrt{86}$

$n_2 = \frac{1}{\sqrt{86}}(\hat{i}+9\hat{j}-2\hat{k})$

$n_1 \cdot n_2 = \frac{1}{\sqrt{61}\sqrt{86}}(6-36+6)$

$= \frac{-24}{\sqrt{61}\sqrt{86}}$

$\cos\theta = \frac{24}{\sqrt{61}\sqrt{86}}$

$\Rightarrow \theta = \cos^{-1}\frac{24}{\sqrt{61}\sqrt{86}}$

Math: Find the constants a ~~ab~~ and b so that the surfaces $5x^2-2yz-9x=0$ and $ax^2y+bz^3=4$ may cut orthogonally at $(1,-1,2)$

10.09.24

Math (Example-49 Page-2425): A fluid motion is given by $\vec{V} = (y+z)\hat{i} + (z+x)\hat{j} + (x+y)\hat{k}$. Show that the motion is irrotational and hence find the velocity potential

$$\text{Curl}\,\vec{V} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \times \{(y+z)\hat{i} + (z+x)\hat{j} + (x+y)\hat{k}\}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y+z & z+x & x+y \end{vmatrix}$$

$$= (1-1)\hat{i} - (1-1)\hat{j} + (1-1)\hat{k} = 0$$

$\therefore$ Irrotational.

Ps.eo.01

To find the corresponding velocity potential ϕ, we can consider the following relation

$$\vec{V} = \nabla\phi$$

$$d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz$$

$$= \left(\hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \frac{\partial\phi}{\partial z}\hat{k}\right) \cdot (\hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz)$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\phi \cdot d\vec{r}$$

$$= \nabla\phi \cdot d\vec{r} = \vec{V} \cdot d\vec{r}$$

$$= \left[(y+z)\hat{i} + (z+x)\hat{j} + (x+y)\hat{k}\right] \cdot (\hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz)$$

$$= (y+z)dx + (z+x)dy + (x+y)dz$$

$$= y\,dx + z\,dx + z\,dy + x\,dy + x\,dz + y\,dz$$

$$\Rightarrow \phi = \int(y\,dx + x\,dy) + \int(z\,dy + y\,dz) + \int(z\,dx + x\,dz)$$

$$\Rightarrow \phi = xy + yz + zx + C$$

[Another way to find scalar potential]

Example 50 P-425

Math: A fluid motion is given by $\vec{V} = (y\sin z - \sin x)\hat{i} + (x\sin z + 2yz)\hat{j} + (xy\cos z + y^2)\hat{k}$ is the motion irrotational? If so, find the velocity potential.

$$\text{Curl}\,\vec{V} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \times \{(y\sin z - \sin x)\hat{i} + (x\sin z + 2yz)\hat{j} + (xy\cos z + y^2)\hat{k}\}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y\sin z - \sin x & x\sin z + 2yz & xy\cos z + yz \end{vmatrix}$$

$$= 0$$

Irrotational.

$\vec{V} = \vec{\nabla}\phi$ - where ϕ is called velocity potential.

$$d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz$$

$$= \left(\hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial y}\right)(\hat{i}dx + \hat{j}dy + \hat{k}dz)$$

$$= \vec{\nabla}\phi \cdot d\vec{r} = \vec{V} \cdot d\vec{r}$$

$$= [(y\sin z - \sin x)\hat{i} + (x\sin z + 2yz)\hat{j} + (xy\cos z + y^2)\hat{k}] \cdot [\hat{i}dx + \hat{j}dy + \hat{k}dz]$$

$$= (y\sin z - \sin x)dx + (x\sin z + 2yz)dy + (xy\cos z + y^2)dz$$

$$= y\sin z\,dx + x\,dy\sin z + xy\cos z\,dz - \sin x\,dx + (2yz\,dy + y^2dz$$

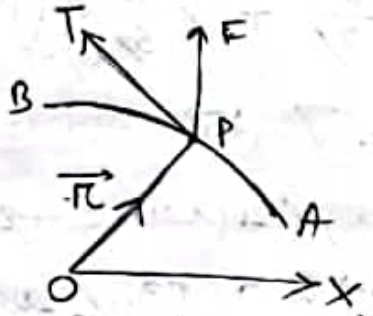
$$= d(xy\sin z) + d(\cos x) + d(y^2z)$$

$$\phi = \int d(xy\sin z) + \int d(\cos x) + \int d(y^2z)$$

$\Rightarrow \phi = xy\sin z + \cos x + y^2 z + c$

P-938

Line integral: Let $\vec{F}(x,y,z)$ be a vector func^n and a curve AB



Line integral of a vector func^n $\vec{F}$ along the curve AB ~~is~~ is defined as integral of the compont of $\vec{F}$ along the tangent ~~of~~ to the curve AB.

Component of $\vec{F}$ along a tangent PT at P
= Dot product of $\vec{F}$ and unit vector along PT

$= \vec{F} \cdot \frac{d\vec{r}}{ds}$ ($\frac{d\vec{r}}{ds}$ is a unit vector along tangent PT)

Line integral $= \Sigma \vec{F} \cdot \frac{d\vec{r}}{ds}$ from A to B along the curve.

$\therefore$ Line integral $= \int_C \left(\vec{F} \cdot \frac{d\vec{r}}{ds}\right) ds = \int_C \vec{F} \cdot d\vec{r}$

Note:

1. **Work:** If $\vec{F}$ represents the variable force acting on a particle along arc AB, the total work done $= \int_A^B \vec{F} \cdot d\vec{r}$

2. **Circulation:** If $\vec{V}$ represents the velocity of a liquid the $\oint_C \vec{V} \cdot d\vec{r}$ is called the circulation of V round the closed curve C.

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If the circulation of V round every closed curve is zero then V is said to be irrotational there.

3. When the path of integration is a closed curve then notation of integration is $\oint$ in place of $\int$.

P-439 Example-65

Math: If a force $\vec{F} = 2x^2y\hat{i} + 3xy\hat{j}$ displaces a particle in the xy-plane from (0,0) to (1,4) along a curve $y = 4x^2$. Find the work done

Work done $= \int_C \vec{F} \cdot d\vec{r}$

$= \int_C (2x^2y\hat{i} + 3xy\hat{j}) \cdot (dx\hat{i} + dy\hat{j})$

$= \int_C (2x^2y\,dx + 3xy\,dy)$

$\vec{r} = x\hat{i} + y\hat{j}$

$\Rightarrow d\vec{r} = dx\hat{i} + dy\hat{j}$

Putting the values of y & and dy, we get,

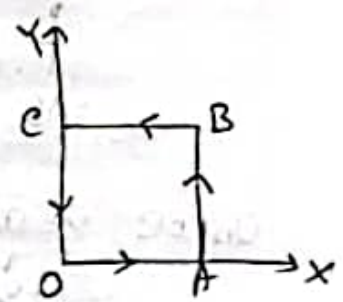
$= \int_0^1 [2x^2(4x^2)dx + 3x(4x^2 8x)dx]$

$= 104 \int_0^1 x^4 dx = 104 \left[\frac{x^5}{5}\right]_0^1 = \frac{104}{5}$

Math (Example-66 P-439): Evaluate $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F} = x^2\hat{i} + xy\hat{j}$ and C is the boundary of the square in the plane $z=0$ and bounded by the lines $x=0$, $y=0$, $x=a$ and $y=a$

Soln:

$$\int_C \vec{F} \cdot d\vec{r} = \int_{OA} \vec{F} \cdot d\vec{r} + \int_{AB} \vec{F} \cdot d\vec{r} + \int_{BC} \vec{F} \cdot d\vec{r} + \int_{CO} \vec{F} \cdot d\vec{r}$$



Here

$\vec{r} = x\hat{i} + y\hat{j}$

$d\vec{r} = dx\hat{i} + dy\hat{j}$

$\vec{F} = x^2\hat{i} + xy\hat{j}$

$\vec{F} \cdot d\vec{r} = x^2dx + xy\,dy$ ——— (1)

On OA, $y=0$

$\therefore \vec{F}\cdot d\vec{r} = x^2 dx$

$$\int_{OA} \vec{F}\cdot d\vec{r} = \int_0^a x^2 dx = \left[\frac{x^3}{3}\right]_0^a = \frac{a^3}{3} \quad \text{---} ②$$

On AB, $x=a$

$\therefore dx = 0$

① becomes,

$\vec{F}\cdot d\vec{r} = ay\,dy$

$$\int_{AB} \vec{F}\cdot d\vec{r} = \int_0^a ay\,dy = a\left[\frac{y^2}{2}\right]_0^a = \frac{a^3}{2}$$

--- ③

On BC, $y=a$

$\therefore dy = 0$

$\vec{F}\cdot d\vec{r} = x^2 dx$

$$\int_{BC} \vec{F}\cdot d\vec{r} = \int_a^0 x^2 dx = -\int_0^a x^2 dx = -\left[\frac{x^3}{3}\right]_0^a = -\frac{a^3}{3}$$

--- ④

On CO, $x=0$

$\therefore \vec{F}\cdot d\vec{r} = 0$

$$\int_{CO} \vec{F}\cdot d\vec{r} = 0 \quad \text{---} ⑤$$

On adding ②, ③, ④ & ⑤,

$$\int_C \vec{F}\cdot d\vec{r} = \frac{a^3}{3} + \frac{a^3}{2} + -\frac{a^3}{3} + 0 = \frac{a^3}{2} \text{ unit}^2$$

Math (Example-Q P-439): A vector field is given by $\vec{F} = (2y+3)\hat{i} + xz\hat{j} + (yz-x)\hat{k}$. Evaluate $\int_C \vec{F}\cdot d\vec{r}$ along the path C is $x=2t$, $y=t$, $z=t^3$ from $t=0$ to $t=1$.

$$\int_C \vec{F}\cdot d\vec{r} = \int_C (2y+3)\,dx + (xz)\,dy + (yz-x)\,dz$$

Since $x=2t$ $y=t$ $z=t^3$

$\frac{dx}{dt}=2$ $\frac{dy}{dt}=1$ $\frac{dz}{dt}=3t^2$

$$= \int_0^1 (2t+3)(2dt) + (2t)t^3 dt + (t^4-2t)(3t^2 dt)$$

$$= \int_0^1 (4t + 6t + 21t^4 + 3t^6 - 6t^3)\,dt$$

$$= \left[\frac{4t^2}{2} + 6t + \frac{21t^5}{5} + \frac{3t^7}{7} - \frac{6t^4}{4}\right]_0^1$$

$$= 7.32857$$

Math (Example 6 P-490): If ~~$\vec{A} = \ldots$~~ $\vec{A} = (3x^2 + 6y)\hat{i} - 14yz\hat{j} + 20xz^2\hat{k}$, evaluate the line integral $\oint \vec{A} \cdot d\vec{r}$ from (0,0,0) to (1,1,1) along the curve C.

$x = t,\ y = t^2,\ z = t^3$

$$\int_C \vec{A} \cdot d\vec{r} = \int_C \left[(3x^2 + 6y)\hat{i} - 14yz\hat{j} + 20xz^2\hat{k}\right] \cdot \left[\hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz\right]$$

$$= \int_C \left[(3x^2 + 6y)dx - 14yz\,dy + 20xz^2\,dz\right]$$

If $x = t,\ y = t^2,\ z = t^3$ then points (0,0,0) and (1,1,1) corresponds to $t = 0$ and $t = 1$ respectively

Now, $\int_C \vec{A} \cdot d\vec{r} = \int_{t=0}^{t=1} \left[(3t^2 + 6t^4)dt - 14t^2 t^3 d(t^2) + 20t(t^3)^2 d(t^3)\right]$

$$= \int_0^1 \left[3t^2 dt - 14t^5 2t\,dt + 20t^7 3t^2 dt\right]$$

$$= \int_0^1 (3t^2 - 28t^6 + 60t^9)\,dt$$

$$= 5$$

17.9.24

Math (Example-68 P-490): The acceleration of a particle at time t is given by

$\vec{a} = 18\cos t\,\hat{i} - 8\sin 2t\,\hat{j} + 6t\,\hat{k}$

If the velocity $\vec{v}$ and displacement $\vec{r}$ be zero at $t=0$, find $\vec{v}$ and $\vec{r}$ at any point t.

$$\vec{a} = \frac{d\vec{v}}{dt} = 18\cos t\,\hat{i} - 8\sin 2t\,\hat{j} + 6t\,\hat{k}$$

On integrating, we have

$$\vec{v} = \frac{d\vec{r}}{dt} = \hat{i}\int 18\cos 3t\,dt + \hat{j}\int -8\sin 2t\,dt + \hat{k}\int 6t\,dt$$

$$\Rightarrow \vec{v} = 6\sin 3t\,\hat{i} + 4\cos 2t\,\hat{j} + 3t^2\hat{k} + \vec{c} \quad \text{——(1)}$$

At $t=0$, $\vec{v} = \vec{0}$

$$\vec{0} = 4\hat{j} + \vec{c} \Rightarrow \vec{c} = -4\hat{j}$$

$$\vec{v} = \frac{d\vec{r}}{dt} = 6\sin 3t\,\hat{i} + 4\cos 2t\,\hat{j} + 3t^2\hat{k} + \vec{c}$$

$$= 6\sin 3t\,\hat{i} + 4(\cos 2t - 1)\hat{j} + 3t^2\hat{k}$$

Again integration;

$$\vec{r} = \hat{i}\int 6\sin 3t\,dt + \hat{j}\int 4(\cos 2t - 1)\,dt + \hat{k}\int 3t^2\,dt$$

$$= -2\cos 3t\,\hat{i} + (2\sin 2t - 4t)\hat{j} + t^3\hat{k} + \vec{c_1} \quad \text{——(2)}$$

At $t=0$, $\vec{r} = 0$,

$$\vec{0} = -2\hat{i} + \vec{c_1} \Rightarrow \vec{c_1} = 2\hat{i}$$

$$\vec{r} = 2(1-\cos 3t)\hat{i} + 2(\sin 2t - 2t)\hat{j} + t^3\hat{k}$$

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Math (Example 20, P-491): Evaluate $\iint_S \vec{A}\cdot\hat{n}\,ds$ where $\vec{A} = (x+y^2)\hat{i} - 2x\hat{j} + 2yz\hat{k}$ and S is the surface of the plane $2x+y+2z=6$ in the 1st octant.



Soln:

$\nabla(2x+y+2z-6) =$

$\left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(2x+y+2z-6)$

$= 2\hat{i} + \hat{j} + 2\hat{k}$

$\hat{n}$ = a unit vector normal to surface S.

$= \frac{2\hat{i}+\hat{j}+2\hat{k}}{\sqrt{4+1+4}} = \frac{2}{3}\hat{i} + \frac{1}{3}\hat{j} + \frac{2}{3}\hat{k}$

$\Rightarrow \hat{k}\cdot\hat{n} = \hat{k}\cdot\left(\frac{2}{3}\hat{i} + \frac{1}{3}\hat{j} + \frac{2}{3}\hat{k}\right) = \frac{2}{3}$

$\iint_S \vec{A}\cdot\hat{n}\,ds = \iint_R \vec{A}\cdot\hat{n}\,\frac{dx\,dy}{\hat{k}\cdot\hat{n}}$

where R is the projection of S.

Now,

$\vec{A}\cdot\hat{n} = \left[(x+y^2)\hat{i} - 2x\hat{j} + 2yz\hat{k}\right]\cdot\left(\frac{2}{3}\hat{i} + \frac{1}{3}\hat{j} + \frac{2}{3}\hat{k}\right)$

$= \frac{2}{3}(x+y^2) - \frac{2}{3}x + \frac{4}{3}yz$

$= \frac{2}{3}y^2 + \frac{4}{3}yz$ ——— (1)

Putting the value of z in (1), we get,

$\vec{A}\cdot\hat{n} = \frac{2}{3}y^2 + \frac{4}{3}y\left(\frac{6-2x-y}{2}\right)$ | $2x+y+2z=6 \Rightarrow z = \frac{6-2x-y}{2}$

$= \frac{2}{3}y(y+6-2x-y)$

$= \frac{4}{3}y(3-x)$ ——— (2)

$\iint_S \vec{A}\cdot\hat{n}\,ds = \iint_R \vec{A}\cdot\vec{n}\,\frac{dx\,dy}{|\hat{k}\cdot\hat{n}|}$ ——— (3)

Surface integral of F over S

$= \oint \Sigma \vec{F} \cdot \hat{n}$

$= \iint_S (\vec{F} \cdot \hat{n}) ds$

Note: 1) Flux $= \iint_S (\vec{F} \cdot \hat{n}) ds$ where $\vec{F}$ represents the velocity of a liquid.

If $\iint_S (\vec{F} \cdot \hat{n}) ds = 0$, then $\vec{F}$ is said to be a solenodial vector point func^n

Math (Example 21 P-494): Complute $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F} = \frac{\hat{i}y - \hat{j}x}{x^2+y^2}$ and C is the circle $x^2+y^2=1$ traversed counter clockwise

$\vec{r} = \hat{i}x + \hat{j}y + \hat{k}z \quad d\vec{r} = \hat{i}dx + \hat{j}dy + \hat{k}dz$

$$\int_C \vec{F} \cdot d\vec{r} = \int_C \frac{\hat{j}y - \hat{j}x}{x^2+y^2} \cdot (i\,dx + j\,dy + k\,dz)$$

$$= \int_C \frac{y\,dx - x\,dy}{x^2+y^2} = \int_C y\,dx - x\,dy \Big|_{x^2+y^2=1} \quad \text{---- (1)}$$

Parametric eqn of the circle are

$x = \cos\theta \quad y = \sin\theta$

Putting $x = \cos\theta$, $y = \sin\theta$, $dx = -\sin\theta$, $dy = \cos\theta\, d\theta$ in (1),

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \sin\theta(-\sin\theta\, d\theta) - \cos\theta(\cos\theta\, d\theta)$$

$$= -\int_0^{2\pi} (\sin^2\theta + \cos^2\theta\, d\theta) = -\int_0^{2\pi} d\theta = -(\theta)_0^{2\pi}$$

$$= -2\pi$$

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Putting the value of $\vec{A}.\hat{n}$ from (2) in (3), we get,

$$\iint_S \vec{A}.\hat{n}\,ds = \iint_R \frac{4}{3}y(3-x)\cdot\frac{3}{2}\,dx\,dy$$

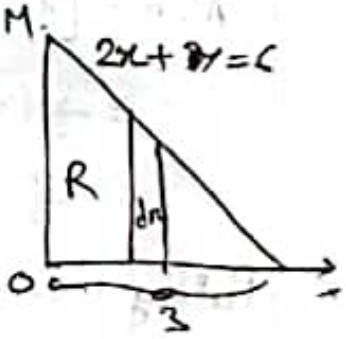
$$= \int_0^3 \int_0^{6-2x} 2y(3-x)\,dy\,dx$$

$$= \int_0^3 2(3-x)\left[\frac{y^2}{2}\right]_0^{6-2x} dx$$

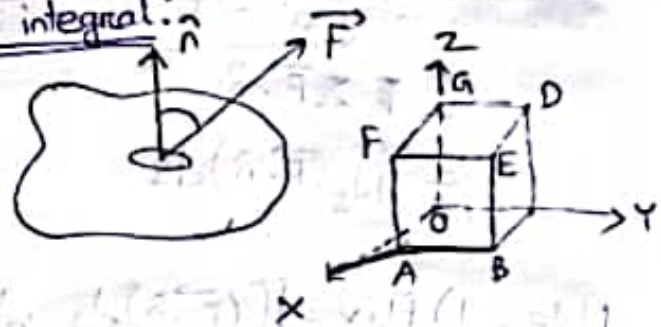
$$= \int_0^3 2(3-x)(6-2x)^2\,dx$$

$$= 4\int_0^3 (3-x)^3\,dx$$

$$= 4\left[\frac{(3-x)^4}{4(-1)}\right]_0^3 \quad \text{L} - (0-81) = 81$$



Surface integral:



A surface $r = f(u,v)$ is called smooth if $f(u,v)$ posses continuous & 1st order partial derivative.

Let $\vec{F}$ be a vector func^n and S be the given surface.

Surface integral of a vector func^n $\vec{F}$ over the surface S is defined as the integral of the components of $\vec{F}$ along the normal to the surface

Components of $\vec{F}$ along to the normal =

$\vec{F}.\hat{n}$ | where $\hat{n}$ is the unit normal vector to an element ds and $\hat{n} = \frac{\text{grad}f}{|\text{grad}f|}$ $ds = \frac{dx.dy}{(\hat{n}.\hat{ak})}$

29.9.29

Directional derivative: The directional derivative of ϕ at a given point P in the direction of $\bar{a}$ is

$$= \text{grad}\,\phi_P \cdot \frac{\bar{a}}{|\bar{a}|}$$

Math: Find the directional derivative of $\phi = xy^2 + yz^3$ at $(2,1,1)$ along $\hat{i}+2\hat{j}+2\hat{k}$

Soln:

$$\phi = xy^2 + yz^3$$

$$\text{grad}\,\phi = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)(xy^2 + yz^3)$$

$$= y^2\hat{i} + (2xy + z^3)\hat{j} + 3z^2y\hat{k}$$

$$\nabla\phi_{(2,1,1)} = \hat{i} + 5\hat{j} + 3\hat{k}$$

$$\bar{a} = \hat{i} + 2\hat{j} + 2\hat{k}$$

$$|\bar{a}| = 3$$

Directional derivative $=$

$$(\hat{i} + 5\hat{j} + 3\hat{k}) \cdot \left(\frac{\hat{i}+2\hat{j}+2\hat{k}}{3}\right)$$

$$= \frac{1+10+6}{3} = \frac{17}{3}$$

Math: Evaluate $\int_C \vec{F}\cdot d\vec{r}$ where $\vec{F} = x\hat{i} + y\hat{j} + z\hat{k}$ and C is the curve where $r = t\hat{i} + t^2\hat{j} + t^3\hat{k}$ from $t=0$ to $t=1$

Soln:

$$\vec{r} = t\hat{i} + t^2\hat{j} + t^3\hat{k}$$

$$\vec{F} = t\hat{i} + t^2\hat{j} + t^3\hat{k}$$

$$d\vec{r} = (\hat{i} + 2t\hat{j} + 3t^2\hat{k})\,dt$$

$$\int_C \vec{F}\cdot d\vec{r} = \int_0^1 (t\hat{i} + t^2\hat{j} + t^3\hat{k}) \cdot (\hat{i} + 2t\hat{j} + 3t^2\hat{k})\,dt$$

$$= \int_0^1 (t + 2t^3 + 3t^5)\,dt$$

$$= \left[\frac{t^2}{2} + \frac{2t^4}{4} + \frac{3t^6}{6}\right]_0^1 = \frac{5}{4}$$

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$$\iint_{DEFG} (4xz\hat{i} - y^2\hat{j} + yz\hat{k}) \cdot \hat{k}\, dx\, dy$$

$$= \iint_{DEFG} yz\, dx = \int_0^1 \int_0^1 y(1)\, dx\, dy$$

$$= \int_0^1 dx \left[\frac{y^2}{2}\right]_0^1 = [x]_0^1 \frac{1}{2} = \frac{1}{2}$$

$$\iint_{OAGF} (4xz\hat{i} - y^2\hat{j} + yz\hat{k}) \cdot (-\hat{j})\, dx\, dz$$

$$= \iint_{OAGF} y^2\, dx\, dz = 0 \quad (\because y = 0)$$

$$\iint_{BCED} (4xz\hat{i} - y^2\hat{j} + yz\hat{k}) \cdot \hat{j}\, dx\, dz$$

$$= \iint_{BCED} (-y^2)\, dx\, dz \qquad [As\ y = 1]$$

$$= -\int_0^1 dx \int_0^1 dz = -(x)_0^1 (z)_0^1 = -1$$

$$\iint_{ABDG} (4xz\hat{i} - y^2\hat{j} + yz\hat{k}) \cdot \hat{j}\, dy\, dz$$

$$= \iint 4xz\, dy\, dz$$

$$= \int_0^1 \int_0^1 4(1)z\, dy\, dz$$

$$= 4(y)_0^1 \left[\frac{z^2}{2}\right]_0^1 = 4(1)\left(\frac{1}{2}\right) = 2$$

$$\iint_{OCEF} (4xz\hat{i} - y^2\hat{j} + yz\hat{k}) \cdot (-\hat{i})\, dy\, dz$$

$$= \int_0^1 \int_0^1 -4xz\, dy\, dz = 0 \qquad (as\ x = 0)$$

Now from ①,

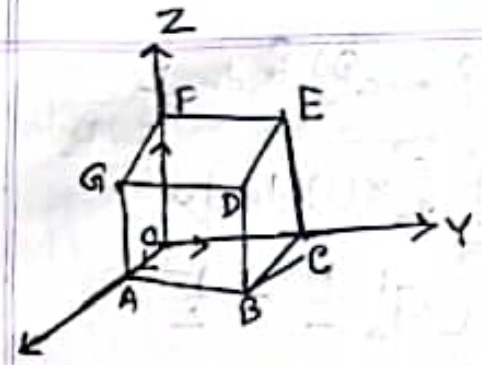
$$\iint_S \vec{F} \cdot \hat{n}\, ds = 0 + \frac{1}{2} + 0 - 1 + 2 + 0 = \frac{3}{2}$$

Important

Math (Ex-XX P-494): Show that $\iint_S \vec{F}\cdot\hat{n}\,ds = \frac{3}{2}$, where $\vec{F} = 4xz\hat{i} - y^2\hat{j} + yz\hat{k}$ and S is the surface of the cube bounded by planes.

$x=0,\ x=1,\ y=0\ \ y=1,\ z=0\ \ z=1$

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iint_{OABC} \vec{F}\cdot\hat{n}\,ds + \iint_{DEFG} \vec{F}\cdot\hat{n}\,ds + \iint_{OAGF} \vec{F}\cdot\hat{n}\,ds + \iint_{BCED} \vec{F}\cdot\hat{n}\,ds + \iint_{ABDG} \vec{F}\cdot\hat{n}\,ds + \iint_{OCEF} \vec{F}\cdot\hat{n}\,ds \quad \text{---(1)}$$



S.no	Surface	Outward normal	ds	
1	OABC	$-k$	$dxdy$	~~$z=0$~~
2	DEFG	k	$dxdy$	$z=1$
3	OAGF	$-j$	$dxdz$	$y=0$
4	BCED	j	$dxdz$	$y=1$
5	ABDG	i	$dydz$	$x=1$
6	OCEF	$-i$	$dydz$	$x=0$

Now $\iint_{OABC} \vec{F}\cdot n\,ds = \iint_{OABC} (4xz\hat{i} - y^2\hat{j} + yz\hat{k})(-k)\,dxdy$

$= \int_0^1 \int_0^1 -yz\,dxdy = 0 \quad (\text{as } z=0)$

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$$\int_C \vec{F}\cdot d\vec{r} =$$

$$\int_0^{2\pi} (6\cos\theta - 3\sin\theta)(-3\sin\theta)\,d\theta + (3\cos\theta + 3\sin\theta)(3\cos\theta)\,d\theta$$

$$= \int_0^{2\pi} (-18\sin\theta\cos\theta + 9\sin^2\theta + 9\cos^2\theta + 9\sin\theta\cos\theta)\,d\theta$$

$$= \int_0^{2\pi} (9\sin\theta\cos\theta + 9)\,d\theta$$

$$= \int_0^{2\pi} \left(\frac{9}{2}\sin 2\theta + 9\right) d\theta$$

$$= \left[-\frac{9}{2}\,\frac{-\cos 2\theta}{2} + 9\theta\right]_0^{2\pi}$$

$$= \frac{9}{4}\cos 2\pi + 18\pi - \frac{9}{4}\cos 0$$

$$= \frac{9}{4} + 18\pi - \frac{9}{4} = 18\pi$$

1·10·24

Green's Theorem (P-448):

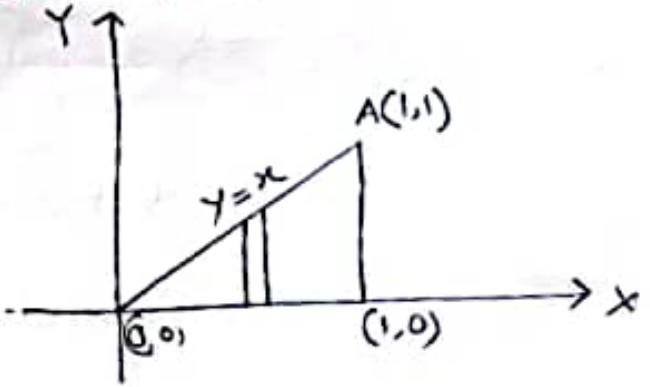
If $\phi(x,y)$, $\psi(x,y)$, $\frac{\partial\phi}{\partial y}$ and $\frac{\partial\psi}{\partial x}$ be continuous func^n over a region R bounded by simple closed curved C in x-y plane, then

$$\oint_C (\phi\,dx + \psi\,dy) = \iint_R \left(\frac{\partial\psi}{\partial x} - \frac{\partial\phi}{\partial y}\right) dx\,dy$$

Proof (Page- 448 to 449)

Math (Exp-80 P-449): Using Green's Theorem, evaluate $\int_C (x^2y\,dx + x^2\,dy)$ where C is the boundary described counter clockwise of the triangle with vertices (0,0), (1,0), (1,1)

Soln:



Math: Evaluate $\int_C \vec{F}.d\vec{r}$ where $\vec{F} = (5xy - 6x^4)\hat{i} + (2y - 4x)\hat{j}$ along the curve $y = x^3$ from $(1,1)$ to $(2,8)$

$d\vec{r} = \hat{i}dx + \hat{j}dy + \hat{k}dz$

$\vec{F}.d\vec{r} = (5xy - 6x^4)dx + (2y - 4x)dy$

$y = x^3$

$= (5x \times x^3 - 6x^4)dx + (2x^3 - 4x)3x^2 dx$

$= (5x^4 - 6x^4)dx + (6x^5 - 12x^3)dx$ | $y = x^3$, $dy = 3x^2 dy$

$\int \vec{F}.d\vec{r} =$

$\int_1^2 (5x^4 - 6x^2 + 6x^5 - 12x^3)\,dx$

$= \left[\frac{5x^5}{5} - \frac{6x^3}{3} + \frac{6x^6}{6} - \frac{12x^4}{4}\right]_1^2$

$= 32 - 16 + 64 - 48 - 1 + 2 - 1 + 3$

$= 35$

Important

Math: Evaluate $\int_C \vec{F}.d\vec{r}$ where $\vec{F} = (2x - y + z)\hat{i} + (x + y - z)\hat{j} + (3x - 2y + 4z)\hat{k}$ and C is the circle $x^2 + y^2 = 9$ in xy-plane

Soln:

$z = 0$

$\vec{F} = (2x - y)\hat{i} + (x + y)\hat{j} + (3x - 2y)\hat{k}$

$d\vec{r} = dx\hat{i} + dy\hat{j} + dz\hat{k}$

$\vec{F}.d\vec{r} = (2x - y)dx + (x + y)dy + (3x - 2y)dz$

$\int_C \vec{F}.d\vec{r} = \int (2x - y)dx + (x + y)dy$

Let,

$x = 3\cos\theta$ $\quad 0 \le \theta < 2\pi$

$y = 3\sin\theta$

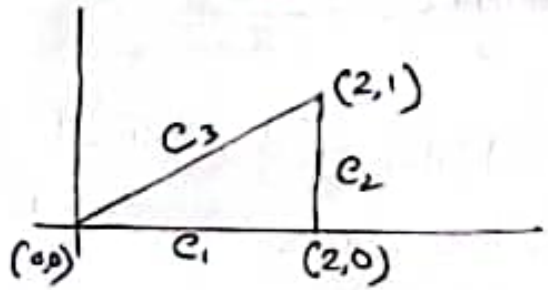
$dx = -3\sin\theta\, d\theta$

$dy = 3\cos\theta\, d\theta$

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Math: Evaluate $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F} = y\hat{i} + z\hat{j} + x\hat{k}$ and C is the triangle with vertices (0,0), (2,0), (2,1)

Soln:



$\vec{F} = y\hat{i} + z\hat{j} + x\hat{k}$

$d\vec{r} = \hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz$

$\vec{F} \cdot d\vec{r} = y\,dx + z\,dy + x\,dz$

For C_1,

$y = 0$

$dy = 0$

$\int_{C_1} (y\,dx + z\,dy + x\,dz) = \int_0^2 y\,dx = 0$

For C_2,

$x_2 = 2$

$dx = 0$

$z = 0$

$\int_{C_2} y\,dx + z\,dy + x\,dz = 0$

For C_3,

$y = mx$

$\Rightarrow 1 = 2m \Rightarrow m = \frac{1}{2}$

$y = \frac{1}{2}x \Rightarrow x - 2y = 0 \Rightarrow z = 0$

$\Rightarrow y = \frac{x}{2}$

$dy = \frac{1}{2}dx$

$\int_{C_3} \vec{F} \cdot d\vec{r} = \int_2^0 \frac{x}{2}\,dx = -\frac{1}{2}\int_0^2 x\,dx = \frac{1}{2}\left[\frac{x^2}{2}\right]_0^2$

$= -1$

$\int_C = \int_{C_1} + \int_{C_2} + \int_{C_3} = -1$

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By Green's Theorem, we have,

$$\int_C (\phi dx + \psi dy) = \iint_R \left(\frac{\partial \psi}{\partial x} - \frac{\partial \phi}{\partial y}\right) dx\,dy$$

$$\int_C (x^2 y\,dx + x^2 dy) = \iint_R (2x - x^2)\,dx\,dy$$

$$= \int_0^1 (2x - x^2)\,dx \int_0^x dy = \int_0^1 (2x - x^2)\,dx\,[y]_0^x$$

$$= \int_0^1 (2x - x^2)\,x\,dx = \int_0^1 (2x^2 - x^3)\,dx$$

$$= \left[\frac{2x^3}{3} - \frac{x^4}{4}\right]_0^1 = \frac{2}{3} - \frac{1}{4} = \frac{5}{12}$$

Math(Ex-8.1 P-449): State and verify Green's Theorem in the plane for $\oint (3x^2 - 8y^2)dx + (4y - 6xy)dy$ where C is the boundary of the region bounded by $x \geq 0, y \leq 0$ and $2x - 3y = 6$

Soln: Here the closed curve C consists of straight lines OB, BA and AO, where co-ordinates of A and B are (3, 0) and (0, -2) respectively. Let R be the region bounded by C.

By Green's Theorem in plane, we have

$$\oint [(3x^2 - 8y^2)dx + (4y - 6xy)dy]$$

$$= \iint_R \left[\frac{\partial}{\partial x}(4y - 6xy) - \frac{\partial}{\partial y}(3x^2 - 8y^2)\right] dx\,dy \quad \text{---(1)}$$

$$= \iint_R (-6y + 16y)\,dx\,dy = \iint_R 10y\,dx\,dy$$

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$$= 10 \int_0^3 dx \int_{\frac{1}{3}(2x-6)}^{0} y \, dy$$

$$= 10 \int_0^3 dx \left[\frac{y^2}{2}\right]_{\frac{1}{3}(2x-6)}^{0}$$

$$= -\frac{5}{9} \int_0^3 dx \, (2x-6)^2$$

$$= -\frac{5}{9} \left[\frac{(2x-6)^3}{3 \times 2}\right]_0^3$$

$$= -\frac{5}{54} (0+6)^3 = -20 \quad \text{——} (2)$$

Now we evaluate L.H.S of (1) along OB, BA and AO.

Along OB, $x = 0$.
$dx = 0$
y varies from 0 to -2

Along AO BA, $x = \frac{1}{3}(6+3y)$.
$dx = \frac{3}{2}$
y varies from -2 to 0

Along AO, $y = 0$
$dy = 0$
x varies from 3 to 0

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L.H.S. of ①,

$$\oint [(3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy]$$

$$= \int_{OB} [(3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy] +$$

$$\int_{BA} [(3x^2 - 8y^2)\,dx + (4x - 6xy)\,dy] +$$

$$\int_{AO} [(3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy]$$

$$= \int_0^{-2} 4y\,dy + \int_{-2}^{0} \left(\frac{3}{4}(6+3y)^2 - 8y^2\right)\left(\frac{3}{4}\right) dy$$

$$+ \left[4y - 3(6+3y)y\right] dy + \int_3^0 3x^2\,dx$$

$$= \left[2y^2\right]_0^{-2} + \int_{-2}^{0} \left[\frac{9}{8}(6+3y)^2 - 12y^2 + 9y - 18y - 9y^2\right] dy$$

$$+ \left[x^3\right]_3^0$$

$$= 2 \times 4 + \int_{-2}^{0} \left[\frac{9}{8}(6+3y)^2 - 21y^2 - 18y\right] dy$$

$$- 27$$

$$= 8 + \left[\frac{9}{8}\,\frac{(6+3y)^3}{3\times 3} - 7y^3 - 7y^2\right]_{-2}^{0} - 27$$

$$= -19 + \left[\frac{216}{8} + 7(-2)^3 + 7(-2)^2\right]$$

$$= -20 \quad \text{—③}$$

With the help of ② & ③, we find that ① is true and so Green's Theorem is verified.

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$x = r\sin\theta\cos\phi$

$y = r\sin\theta\sin\phi$

$z = r\cos\theta$

$$= 3\iiint r^2 (r^2 \sin\theta \, dr \, d\theta \, d\phi)$$

$$= 3 \times 8 \int_0^{\pi/2} d\phi \int_0^{\frac{3}{2}} \sin\theta \, d\theta \int_0^{a} r^4 \, dr$$

$$= 24 (\phi)_0^{\frac{3}{2}} (-\cos\theta)_0^{\frac{3}{2}} \left(\frac{r^5}{5}\right)_0^{a}$$

$$= 0 \cdot \frac{12\pi a^5}{5}$$

$$= \iint (21 - 12y)\, dx\, dy$$

Let us put $x = r\cos\theta$
$y = r\sin\theta$

$$= \int\int (21 - 12r\sin\theta)\, r\, d\theta\, dr$$

$$= \int_0^{2\pi} d\theta \int_0^2 (21r - 12r^2\sin\theta)\, dr$$

$$= \int_0^{2\pi} d\theta \left[\frac{21r^2}{2} - 4r^3\sin\theta\right]_0^2$$

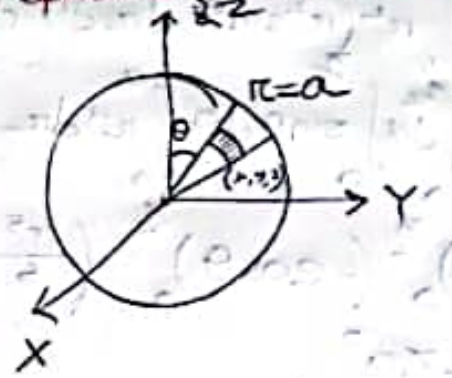
$$= \int_0^{2\pi} (42 - 32\sin\theta)\, d\theta$$

$$= \left[42\theta + 32\cos\theta\right]_0^{2\pi}$$

$$= 84\pi + 32 - 32$$

$$= 84\pi$$

Math (Exp-5 P- 472): Use divergence theorem to evaluate $\iint_S \vec{A} \cdot d\vec{s}$ where $\vec{A} = x^3\hat{i} + y^3\hat{j} + z^3\hat{k}$ and S is the surface of the sphere $x^2 + y^2 + z^2 = a^2$



$$\iint_S \vec{A} \cdot d\vec{s} = \iiint_V \text{div}\, \vec{A}\, dV$$

$$= \iiint_V \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot (x^3\hat{i} + y^3\hat{j} + z^3\hat{k})\, dV$$

$$= \iiint_V (3x^2 + 3y^2 + 3z^2)\, dV$$

$$= 3\iiint_V (x^2 + y^2 + z^2)\, dV$$

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Gauss's Theorem of Divergence (P-470):

(Relation between surface and vol integral)

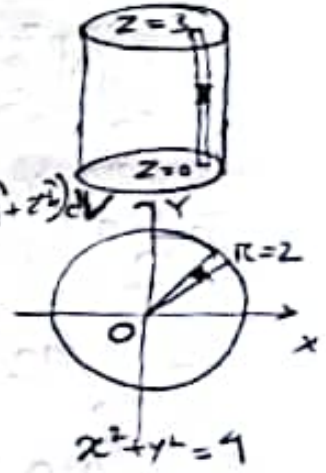
The surface integral of the normal component of a vector function F taken around a closed surface S is equal to the integral of the divergence of F taken over the vol V enclosed by surface S.

~~Math~~

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \text{div}\,\vec{F}\,dV$$

Proof: (P-470)

Math (Exp-108 P-423): Use divergence theorem to evalute $\iint_S \vec{F}\cdot d\vec{s}$ where $\vec{F} = 4x\hat{i} - 2y^2\hat{j} + z^2\hat{k}$ and S is the surface bounding the region $x^2+y^2=4$, $z=0$, $z=3$



Soln:

$$\iint_S \vec{F}\cdot ds = \iiint_V \text{div}\,\vec{F}\,dV$$

$$= \iiint_V \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot(4x\hat{i} - 2y^2\hat{j} + z^2\hat{k})\,dV$$

$$= \iiint_V (4 - 4y + 2z)\,dx\,dy\,dz$$

$$= \iint dx\,dy \int_0^3 (4 - 4y + 2z)\,dz$$

$$= \iint dx\,dy \left[4z - 4yz + \frac{2z^2}{2}\right]_0^3$$

$$= \iint dx\,dy\,(12 - 12y + 9)\,dx\,dy$$

Math(Example-2c P-446): Evaluate $\iint_S (yz\hat{i} + zx\hat{j} + xy\hat{k}) \cdot d\vec{S}$ where S is the surface of the sphere $x^2+y^2+z^2=a^2$ in the 1st octant.

Here $\phi = x^2+y^2+z^2-a^2$

Vector normal to the surface $= \nabla\phi$

$$= \left(\hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}\right)$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot(x^2+y^2+z^2-a^2)$$

$$= 2x\hat{i} + 2y\hat{j} + 2z\hat{k}.$$

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2x\hat{i}+2y\hat{j}+2z\hat{k}}{\sqrt{4x^2+4y^2+4z^2}}$$

$$= \frac{x\hat{i}+y\hat{j}+z\hat{k}}{\sqrt{x^2+y^2+z^2}} = \frac{x\hat{i}+y\hat{j}+z\hat{k}}{a}$$

$$\vec{F} = yz\hat{i} + zx\hat{j} + xy\hat{k}$$

$$\vec{F}\cdot\hat{n} = (yz\hat{i}+zx\hat{j}+xy\hat{k})\cdot\frac{x\hat{i}+y\hat{j}+z\hat{k}}{a}$$

$$= \frac{3xyz}{a}$$

$$\iint_S F\cdot\hat{n}\,ds = \iint_S (\vec{F}\cdot n)\frac{dx\,dy}{\hat{n}\cdot\hat{k}}$$

$$= \int_0^a \int_0^{\sqrt{a^2-x^2}} \frac{3xyz\,dx\,dy}{a\left(\frac{z}{a}\right)}$$

$$= 3\int_0^a \int_0^{\sqrt{a^2-x^2}} xy\,dy\,dx = 3\int_0^a x\left[\frac{y^2}{2}\right]_0^{\sqrt{a^2-x^2}} dx$$

$$= \frac{3}{2}\int_0^a x(a^2-x^2)\,dx = \frac{3}{2}\left[\frac{a^2x^2}{2} - \frac{x^4}{4}\right]_0^a$$

$$= \frac{3}{2}\left(\frac{a^4}{2} - \frac{a^4}{4}\right) = \frac{3a^4}{8}$$

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Math: Find div $\vec{F}$ for $\vec{F} = (x^2+yz)\hat{i} + (y^2+zx)\hat{j} + z^2xy\hat{k}$

$$\text{div}\,\vec{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot \left\{(x^2+yz)\hat{i} + (y^2+zx)\hat{j} + z^2xy\hat{k}\right\}$$

$$= \frac{\partial}{\partial x}(x^2+yz) + \frac{\partial}{\partial y}(y^2+zx) + \frac{\partial}{\partial z}xyz^2$$

$$= 2x + 2y + 2z$$

Now find the curl of

Q.4 Find the const a and b so that the surfaces $5x^2 - 2yz - 9x = 0$ and $ax^2y + bz^3 = 4$ may cut orthogonally at $(1,-1,2)$

Soln: let, $\phi_1 = 5x^2 - 2yz - 9x$ ⟸ (1), (2)

$$\nabla\phi_1 = i\frac{\partial\phi_1}{\partial x} + j\frac{\partial\phi_1}{\partial y} + k\frac{\partial\phi_1}{\partial z}$$

$$= i(10x - 9) + j(-2z) + (-2y)$$

$$\nabla\phi_1\Big|_{(1,-1,2)} = i - 4j + 2k$$

$$|\nabla\phi_1|_{(1,-1,2)} = \sqrt{1+16+4} = \sqrt{21}$$

$$\hat{n}_1 = \frac{i - 4j + 2k}{\sqrt{21}}$$

let, $\phi_2 = ax^2y + bz^3 - 4$

$$\nabla\phi_2 = i(2axy) + j(ax^2) + k(3bz^2)$$

$$\nabla\phi_2(1,-1,2) = -2ai + aj + 12bk$$

$$\Rightarrow \left|\nabla \phi_{2(1,-1,2)}\right| = \sqrt{4a^2+a^2+144b^2}$$

$$\hat{n}_2 = \frac{-2a\hat{i} + a\hat{j} + 12b\hat{k}}{\sqrt{5a^2+144b^2}}$$

$$\hat{n}_1 \cdot \hat{n}_2 = -2a - 4a + 24b$$

$$\Rightarrow 6a - 24b = 0 \quad \text{---(1)}$$

So its lies on $ax^2y + bz^3 = 4$

$$\Rightarrow a(1)^2(-1) + b(2)^3 = 4$$

$$\Rightarrow -a + 8b = 4 \quad \text{---(2)}$$

(1), (2) $\Rightarrow$

$$a = 4$$

$$b = 1$$

Exercise 5.10

(2) Find the work done when a force $\vec{F} = (x^2 - y^2 + x)\hat{i} - (2xy + y)\hat{j}$ moves a particle from origin to (1,1) along a parcabola

$y^2 = x$

$\Rightarrow$ $r = x\hat{i} + y\hat{j}$

$dr = dx\hat{i} + dy\hat{j}$

$$W = \int_C \vec{F}\, d\vec{r}$$

$$= \int_C \left[(x^2 - y^2 + x)\hat{i} - (2xy + y)\hat{j}\right](dx\hat{i} + dy\hat{j})$$

$$= \int_C (x^2 - y^2 + x)dx - (2xy + y)dy$$

(3) Sh
Con
suc
$\vec{V}$
$\Rightarrow$

given, $y^2 = x$

$$= \int_{(0,0)}^{(1,1)} (x^2 - y^2 + y^2)\,dx - (2y^2 y + y)\,dy$$

$$= \int_{(0,0)}^{(1,1)} x^2\,dx - (2y^3 + y)\,dy$$

$$= \int_0^1 x^2\,dx - \int_0^1 (2y^3 + y)\,dy$$

$$= \frac{1}{3} - \left[\frac{2}{4} + \frac{1}{2}\right]$$

$$= -2/3$$

$\therefore W = 2/3$ [as work can't be neg]

(3) Show that $\vec{V} = (2xy + z^3)\hat{i} + x^2\hat{j} + 3xz^2\hat{k}$ is a conservative field. Find its scalar potential ϕ such that $\vec{V} = \text{grad}\,\phi$, work done by the force $\vec{V}$ in moving a particle from $(1,-2,1)$ to $(3,1,4)$

$$\Rightarrow \text{Curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy + z^3 & x^2 & 3xz^2 \end{vmatrix}$$

$$= \hat{i}(0-0) - \hat{j}(3z^2 - 3z^2) + \hat{k}(2x - 2x)$$

$$= 0$$

$\therefore$ conservative field

if the scalare potential is φ

$$d\varphi = \vec{F} \cdot \vec{r}$$

$$\Rightarrow d\varphi = (2xy + z^3)dx + x^2 dy + 3xz^2 dz \quad \text{---(1)}$$

Integrating (1),

$$\varphi = x^2y + xz^3 + c$$

for work,

$$\varphi_{1(1,-2,1)} = (1)^2(-2) + (1)(1)^3$$

$$= -2 + 1 = -1$$

$$\varphi_{2(3,1,4)} = 3^2 \times 1 + 3 \times (4)^3$$

$$= 9 + 192$$

$$= 201$$

$$W = \varphi_2 - \varphi_1$$

$$= 201 - (-1)$$

$$= 202$$

(5) Find the work done in moving a particle once round the ellipse $\frac{x^2}{25}+\frac{y^2}{16}=1$, $z=0$ under the field of force given by $\vec{F}=(2x-y+z)\hat{i}+(x+y-z^2)\hat{j}+(3x-2y+4z)\hat{k}$. Is the field of force conservative?

⇒ We write eqn of ellipse as

$x=5\cos\theta$, $y=4\sin\theta$, $z=0$; $0\le\theta\le 2\pi$

$$\vec{F}\cdot d\vec{r} = \{(10\cos\theta-4\sin\theta)\hat{i}+(5\cos\theta+4\sin\theta)\hat{j}+(15\cos\theta-8\sin\theta)\hat{k}\}\{-5\sin\theta\, d\theta\hat{i}+4\cos\theta\, d\theta\hat{j}\}$$

$$= -50\cos\theta\sin\theta\, d\theta+20\sin^2\theta d\theta+20\cos^2\theta d\theta+16\sin\theta\cos\theta\, d\theta$$

$$\therefore \text{Work done} = \int_0^{2\pi}\vec{F}\cdot d\vec{r}$$

$$= \int_0^{2\pi} 34\cos\theta\sin\theta\, d\theta$$

$$= 2\pi - 17\left[\frac{-\cos 2\theta}{2}\right]_0^{2\pi}$$

$$= 40\pi+\frac{17}{2}(\cos 4\pi-\cos 0)$$

$$= 40\pi$$

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