Laplace Transform

The Laplace transform is of prime importance to scientists and engineers since it is a simple technique which can be used to solve differential and integral equations.

Definition: If $f(t)$ is a function defined for $t \geq 0$, then the Laplace transform of $f(t)$, denoted by $L[f(t)]$ is given by

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt = F(s)$$

provided the integral exists, s being a real parameter.

Sufficient condition for the existence of the Laplace transform:

The Laplace transform of a function $f(t)$, defined for $t \geq 0$ exists if $f(t)$ is piecewise continuous in every finite interval in the range $t \geq 0$ and is of exponential order.

Note: A function $f(t)$ is said to be piecewise continuous in the interval (a, b), if $f(t)$ is continuous in each of the subinterval of (a, b).

A function $f(t)$ is said to be of exponential order if $\lim_{t \to \infty} e^{-st} f(t) = 0$

Useful Laplace Transforms –

i) Consider $f(t) = c$

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt = \int_0^\infty e^{-st} \cdot c\,dt = c \cdot \frac{e^{-st}}{-s}\Big|_0^\infty$$

$$= \frac{c}{s} = F(s)$$

$$\therefore \mathcal{L}\{f(t)\} = \mathcal{L}\{c\} = \frac{c}{s}$$

ii) Consider $f(t) = t^n$

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt = \int_0^\infty e^{-st}\, t^n\,dt$$

$$\Rightarrow F(s) = \int_0^\infty e^{-p} \left(\frac{p}{s}\right)^n \frac{dp}{s} \qquad \text{putting } st = p$$

$$= \frac{1}{s^{n+1}} \int_0^\infty e^{-p}\, p^n\,dp$$

$$= \frac{\overline{|(n+1)}}{s^{n+1}}, \text{ where } n > -1 \text{ and } s > 0$$

$$= \frac{\lfloor n}{s^{n+1}}, \text{ where } n \in \mathbb{N}$$

iii) Consider $f(t) = e^{at}$

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} e^{at}\,dt = \int_0^\infty e^{-(s-a)t}\,dt = \frac{1}{s-a}$$

$$\Rightarrow F(s) = \frac{1}{s-a}, \text{ provided } s > a$$

$$\Rightarrow \mathcal{L}\{e^{at}\} = \frac{1}{s-a} = F(s)$$

Put $a = i\omega$

$$\mathcal{L}\{e^{i\omega t}\} = \frac{1}{s - i\omega} = \frac{s + i\omega}{s^2 + \omega^2}$$

$$\Rightarrow \mathcal{L}\{\cos\omega t\} = \frac{s}{s^2 + \omega^2}, \quad \mathcal{L}\{\sin\omega t\} = \frac{\omega}{s^2 + \omega^2}$$

Properties of Laplace Transform –

1) Linearity property –

$$\mathcal{L}\{af(t) \pm bg(t)\} = a\mathcal{L}\{f(t)\} \pm b\mathcal{L}\{g(t)\}$$

2) Change of scale property –

If $\mathcal{L}\{f(t)\} = F(s)$, then $\mathcal{L}\{f(at)\} = \frac{1}{a}F\left(\frac{s}{a}\right)$

3) First shifting property –

If $\mathcal{L}\{f(t)\} = F(s)$, then $\mathcal{L}\{e^{at}f(t)\} = F(s-a)$

4) If $\mathcal{L}\{f(t)\} = F(s)$, then $\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s)$

5) If $\mathcal{L}\{f(t)\} = F(s)$, then $\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(u)\,du$

provided $\lim_{t\to 0} \frac{f(t)}{t}$ exists.

6) Laplace transform of derivatives –

If $\mathcal{L}\{f(t)\} = F(s)$ exists, where $f(t)$ is continuous and tends to $f(0)$ as $t \to 0$, then the L.T of its derivative $f'(t)$ is given by

$$\int_0^\infty e^{-st} f'(t)\,dt = e^{-st} f(t)\Big|_0^\infty + s\int_0^\infty e^{-st} f(t)\,dt$$

$$\Rightarrow \mathcal{L}\{f'(t)\} = -f(0) + sF(s) = sF(s) - f(0)$$

Again, if $\lim_{t\to\infty} e^{-st} f'(t) = 0$ and $f'(t) \to f'(0)$ as $t \to 0$

then $\mathcal{L}\{f''(t)\} = -f'(0) + s\,\mathcal{L}\{f'(t)\}$

$= -f'(0) + s\,[sF(s) - f(0)]$

$\Rightarrow \mathcal{L}\{f''(t)\} = s^2 F(s) - sf(0) - f'(0)$

By similar procedure

$$\mathcal{L}\{f^n(t)\} = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - sf^{n-2}(0) - f^{n-1}(0).$$

7) Laplace Transform of integral-

If $\mathcal{L}\{f(t)\} = F(s)$, then $\mathcal{L}\left\{\int_0^t f(u)\,du\right\} = \frac{1}{s} F(s)$.

Example1: Obtain the Laplace transform of

$$f(t) = \begin{cases} \cos t, & 0 < t < 2\pi \\ 0, & t > 2\pi \end{cases}$$ by using definition.

Solution: By definition

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$

$$= \int_0^{2\pi} e^{-st} \cos t\,dt + \int_{2\pi}^\infty e^{-st} \cdot 0\,dt$$

$$= \left[\frac{e^{-st}}{(-s)^2+1^2}\{-s\cos t + 1 \cdot \sin t\}\right]_0^{2\pi}$$

$$= \left\{\frac{e^{-2\pi s}}{s^2+1}(-s+0)\right\} - \left\{\frac{1}{s^2+1}[-s+0]\right\}$$

$$= \frac{-s}{s^2+1} e^{-2\pi s} + \frac{s}{s^2+1}$$

$$= \frac{s}{s^2+1}\left(1 - e^{-2\pi s}\right)$$

Example2: obtain the laplace transform of

$$f(t) = \begin{cases} t, & 0 < t < 1 \\ 0, & t > 1 \end{cases}$$

Solution: By definition

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$

$$= \int_0^1 e^{-st} \cdot t\,dt + \int_1^\infty e^{-st} \cdot 0\,dt$$

$$= \left[\left\{t\left(\frac{e^{-st}}{-s}\right) + \int \frac{1}{s} e^{-st}\,dt\right]\right]_0^1$$

$$= \left[\frac{te^{-st}}{-s} - \frac{1}{s^2} e^{-st}\right]_0^1 = \left\{\left(\frac{e^{-s}}{-s} - \frac{1}{s^2}e^{-s}\right) - \left(0 - \frac{1}{s^2}\right)\right\}$$

$$= (-se^{-s} - e^{-s} + 1)/s^2$$

Example3: Obtain the Laplace transform of

$$f(t) = \begin{cases} 0 ; & 0 < t < 3 \\ 5 ; & t > 3 \end{cases}$$

Solution: $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$

$$= \int_0^3 e^{-st} \cdot 0\,dt + \int_3^\infty e^{-st} \cdot 5\,dt$$

$$= 5 \int_3^\infty e^{-st}\,dt$$

$$= 5 \cdot \left[\frac{e^{-st}}{-s}\right]_3^\infty$$

$$= 5\left\{0 + \frac{1}{s} e^{-3s}\right\}$$

$$= \frac{5}{s} e^{-3s}$$

Example1: Find the value of

(a) $\mathcal{L}\{e^t \cos^2 t\}$, (b) $\mathcal{L}\{(1+t^2e^{-3t})^2\}$, (c) $\mathcal{L}\{2\cos^2 4t\}$

Solution: (a) Here $f(t) = \cos^2 t = \frac{1}{2}(1+\cos 2t)$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\left\{\frac{1}{2}(1+\cos 2t)\right\} = \frac{1}{2}\mathcal{L}\{1\} + \frac{1}{2}\mathcal{L}\{\cos 2t\}$$

$$= \frac{1}{2}\frac{1}{s} + \frac{1}{2}\frac{s}{s^2+2^2}$$

$$= \frac{1}{2}\frac{1}{s} + \frac{1}{2}\frac{s}{s^2+4} = F(s)$$

$$\mathcal{L}\{e^t \cos^2 t\} = F(s-1)$$

$$= \frac{1}{2}\cdot\frac{1}{s-1} + \frac{1}{2}\frac{s-1}{(s-1)^2+4}$$

(b) $\mathcal{L}\{(1+t^2e^{-3t})^2\}$

$$= \mathcal{L}\{1 + 2t^2e^{-3t} + t^4e^{-6t}\}$$

$$= \mathcal{L}\{1\} + 2\mathcal{L}\{t^2e^{-3t}\} + \mathcal{L}\{t^4e^{-6t}\}$$

$$= \frac{1}{s} + 2\cdot\frac{2!}{(s+3)^3} + \frac{4!}{(s+6)^5}$$

$$= \frac{1}{s} + \frac{4}{(s+3)^3} + \frac{24}{(s+6)^5}$$

(c) $\mathcal{L}\{2\cos^2 4t\} = \mathcal{L}\{1+\cos 8t\}$

$$= \frac{1}{s} + \frac{s}{s^2+64}$$

Example-2: Find the value of

(a) $\mathcal{L}\{\sin 3t \sin 5t\}$, (b) $\mathcal{L}\{\sin 3t \cos 2t\}$, (c) $\mathcal{L}\{\cos^3 t\}$

Solution: (a) $\mathcal{L}\{\sin 5t \sin 3t\} = \frac{1}{2}\mathcal{L}\{2\sin 5t \sin 3t\}$

$$= \frac{1}{2}\mathcal{L}\{\cos 2t - \cos 8t\} = \frac{1}{2}\left(\frac{s}{s^2+4} - \frac{s}{s^2+64}\right) = \frac{30s}{(s^2+4)(s^2+64)}$$

(b) $\mathcal{L}\{\sin 3t \cos 2t\} = \frac{1}{2}\mathcal{L}\{2\sin 3t \cos 2t\}$

$= \frac{1}{2}\mathcal{L}\{\sin 5t + \sin t\}$

$= \frac{1}{2}\left\{\frac{5}{s^2+25} + \frac{1}{s^2+1}\right\}$

$= \frac{3s^2+15}{(s^2+25)(s^2+1)}$

(c) $\mathcal{L}\{\cos^3 t\} = \frac{1}{4}\mathcal{L}\{3\cos t + \cos 3t\}$

$= \frac{1}{4}\left\{\frac{3s}{s^2+1} + \frac{s}{s^2+9}\right\}$

$= \frac{s^3+7s}{(s^2+1)(s^2+9)}$

Example-3: Use $\mathcal{L}\{f'(t)\} = sF(s) - f(0)$, find $\mathcal{L}\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\}$

Solution: Let $f(t) = \sin\sqrt{t}$, $f'(t) = \frac{\cos\sqrt{t}}{2\sqrt{t}}$, $f(0) = 0$

$\mathcal{L}\{f'(t)\} = sF(s) - f(0)$

$\Rightarrow \mathcal{L}\left\{\frac{\cos\sqrt{t}}{2\sqrt{t}}\right\} = s\mathcal{L}\{\sin\sqrt{t}\}$

$\Rightarrow \mathcal{L}\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\} = 2s\mathcal{L}\{\sin\sqrt{t}\}$ ——— (1)

Now $\mathcal{L}\{\sin\sqrt{t}\} = \mathcal{L}\left\{\sqrt{t} - \frac{(\sqrt{t})^3}{\lfloor 3} + \frac{(\sqrt{t})^5}{\lfloor 5} - \cdots\right\}$

$= \mathcal{L}\{t^{1/2}\} - \frac{1}{\lfloor 3}\mathcal{L}\{t^{3/2}\} + \frac{1}{\lfloor 5}\mathcal{L}\{t^{5/2}\} - \cdots$

$= \frac{\Gamma 3/2}{s^{3/2}} - \frac{1}{\lfloor 3}\cdot\frac{\Gamma 5/2}{s^{5/2}} + \frac{1}{\lfloor 5}\frac{\Gamma 7/2}{s^{7/2}} - \cdots$

$= \frac{\frac{1}{2}\Gamma 1/2}{s^{3/2}} - \frac{3/2\cdot 1/2\cdot\Gamma 1/2}{3\cdot 2\cdot s^{5/2}} + \frac{5/2\cdot 3/2\cdot 1/2\,\Gamma 1/2}{5\cdot 4\cdot 3\cdot 2\, s^{7/2}} - \cdots$

$= \frac{\sqrt{\pi}}{2s^{3/2}}\left\{1 - \frac{1}{2^2 s} + \frac{(1/2^2 s)^2}{\lfloor 2} - \cdots\right\}$

$$\mathcal{L}\{\sin\sqrt{t}\} = \frac{\sqrt{\pi}}{2s^{3/2}} \cdot e^{-\frac{1}{2\sqrt{s}}} = \frac{\sqrt{\pi}}{2s^{3/2}} e^{-\frac{1}{4s}}$$

From (1) we get

$$\mathcal{L}\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\} = 2s \cdot \frac{\sqrt{\pi}}{2s^{3/2}} e^{-\frac{1}{4s}}$$

$$= \frac{\sqrt{\pi}\, e^{-1/4s}}{s^{1/2}}$$

Example-4: If $\mathcal{L}\{f(t)\} = F(s)$, then $\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty f(u)\,du$

Use the property to show that

$$\int_0^\infty \frac{e^{-3t} - e^{-6t}}{t}\,dt = \log 2.$$

Solution: Let $f(t) = e^{-3t} - e^{-6t}$

$$\mathcal{L}\{f(t)\} = \frac{1}{s+3} - \frac{1}{s+6}$$

$$\therefore \mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty \left(\frac{1}{u+3} - \frac{1}{u+6}\right)du$$

$$= \lim_{M\to\infty} \Big[\log(u+3) - \log(u+6)\Big]_s^M$$

$$= \lim_{M\to\infty} \left\{\log\frac{M+3}{M+6} - \log\frac{s+3}{s+6}\right\}$$

$$= \lim_{M\to\infty} \log\frac{1+3/M}{1+6/M} - \log\frac{s+3}{s+6}$$

$$\mathcal{L}\left\{\frac{e^{-3t} - e^{-6t}}{t}\right\} = \log\frac{s+6}{s+3}$$

$$\Rightarrow \int_0^\infty e^{-st}\,\frac{e^{-3t} - e^{-6t}}{t}\,dt = \log\frac{s+6}{s+3}$$

$$\Rightarrow \int_0^\infty \frac{e^{-3t} - e^{-6t}}{t}\,dt = \log 2, \quad \text{take } s \to 0$$

Example-5 Show that $\mathcal{L}\left\{\int_0^t \frac{\sin u}{u}\,du\right\} = \frac{1}{s}\tan^{-1}1/s$

Solution: Let $f(t) = \sin t$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{\sin t\} = \frac{1}{s^2+1} = F(s)$$

$$\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty \frac{1}{u^2+1}\,du$$

$$= \lim_{M\to\infty} \int_s^M \frac{1}{u^2+1}\,du$$

$$= \lim_{M\to\infty} \tan^{-1}u\Big|_s^M$$

$$= \lim_{M\to\infty} (\tan^{-1}M - \tan^{-1}s)$$

$$= \lim_{M\to\infty} \tan^{-1}\frac{M-s}{1+Ms} = \lim_{M\to\infty} \tan^{-1}\frac{1-s/M}{\frac{1}{M}+s}$$

$$= \tan^{-1}\frac{1}{s}$$

$$\therefore \mathcal{L}\left\{\int_0^t \frac{\sin u}{u}\,du\right\} = \frac{1}{s}\tan^{-1}\frac{1}{s}$$

Example-6 Show that $\mathcal{L}\left\{\int_0^t \frac{1-e^{-u}}{u}\,du\right\} = \frac{1}{s}\log\left(1+\frac{1}{s}\right)$

Solution: Let $F(t) = 1-e^{-t}$

$$\mathcal{L}\{F(t)\} = \mathcal{L}\{1-e^{-t}\} = \frac{1}{s} - \frac{1}{s+1} = f(s)$$

$$\mathcal{L}\left\{\frac{F(t)}{t}\right\} = \mathcal{L}\left\{\frac{1-e^{-t}}{t}\right\} = \int_s^\infty f(u)\,du$$

$$= \int_s^\infty \left(\frac{1}{u} - \frac{1}{u+1}\right)du$$

$$= \lim_{M\to\infty} \int_s^M \left(\frac{1}{u} - \frac{1}{u+1}\right)du$$

$$= \lim_{M\to\infty} \left[\log u - \log(u+1)\right]_s^M$$

$$= \lim_{M\to\infty} \left[\log \frac{u}{u+1}\right]_s^M$$

$$= \lim_{M\to\infty} \left\{\log \frac{M}{M+1} - \log \frac{s}{s+1}\right\}$$

$$= \log\left(\frac{s+1}{s}\right)$$

$$= \log\left(1+\frac{1}{s}\right)$$

$$\therefore \mathcal{L}\left\{\int_0^t \frac{1-e^{-u}}{u}\,du\right\} = \frac{1}{s}\log\left(1+\frac{1}{s}\right)$$

Example-7 Find the value of $\int_0^\infty \frac{\cos 6t - \cos 4t}{t}\,dt$

Solution: Let $f(t) = \cos 6t - \cos 4t$

$$\mathcal{L}\{f(t)\} = \frac{s}{s^2+6^2} - \frac{s}{s^2+4^2}$$

$$\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty \left(\frac{u}{u^2+6^2} - \frac{u}{u^2+4^2}\right) du$$

$$= \lim_{M\to\infty} \int_s^M \left(\frac{u}{u^2+6^2} - \frac{u}{u^2+4^2}\right) du$$

$$= \lim_{M\to\infty} \left[\frac{1}{2}\log(u^2+36) - \frac{1}{2}\log(u^2+16)\right]_s^M$$

$$= \frac{1}{2}\lim_{M\to\infty} \log\frac{u^2+36}{u^2+16}\Big|_s^M$$

$$= \frac{1}{2}\lim_{M\to\infty} \left\{\log\frac{M^2+36}{M^2+16} - \log\frac{s^2+36}{s^2+16}\right\}$$

$$= -\frac{1}{2}\log\frac{s^2+36}{s^2+16}$$

$$= \log\left(\frac{s^2+16}{s^2+36}\right)^{1/2}$$

$$\Rightarrow \mathcal{L}\left\{\frac{\cos 6t - \cos 4t}{t}\right\} = \log\left(\frac{s^2+16}{s^2+36}\right)^{1/2}$$

$$\Rightarrow \int_0^\infty e^{-st}\,\frac{\cos 6t - \cos 4t}{t}\,dt = \log\left(\frac{s^2+16}{s^2+36}\right)^{1/2}$$

If we take $s \to 0$, then

$$\int_0^\infty \frac{\cos 6t - \cos 4t}{t}\,dt = \log\left(\frac{2}{3}\right)$$

Example 8: Using the theorem (If $L\{f(t)\} = F(s)$, then $L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s)$) Find the value of

$$\int_0^\infty t e^{-2t} \sin t \, dt \text{ and } \int_0^\infty t e^{-2t} \cos t \, dt$$

Solution: We know that $L\{\sin t\} = \frac{1}{s^2+1}$.

Therefore, by the theorem, we have

$$L\{t \sin t\} = -\frac{d}{ds}\left(\frac{1}{s^2+1}\right) = \frac{2s}{(s^2+1)^2}$$

$$\Rightarrow \int_0^\infty e^{-st} t \sin t \, dt = \frac{2s}{(s^2+1)^2}$$

$$\Rightarrow \int_0^\infty t e^{-2t} \sin t \, dt = \frac{4}{25}$$

Similarly, $L\{\cos t\} = \frac{s}{s^2+1}$

$$L\{t \cos t\} = -\frac{d}{ds}\left(\frac{s}{s^2+1}\right)$$

$$= -\frac{(s^2+1)\cdot 1 - s \cdot 2s}{(s^2+1)^2}$$

$$= \frac{s^2-1}{(s^2+1)^2}$$

$$\int_0^\infty e^{-st} t \cos t \, dt = \frac{s^2-1}{(s^2+1)^2}$$

$$\int_0^\infty t e^{-2t} \cos t \, dt = \frac{3}{25}$$

Example-9: Find the value of $\mathcal{L}\{t(\sin 2t - 2\cos 3t)\}$

Solution:

$$\mathcal{L}\{t(\sin 2t - 2\cos 3t)\} = \mathcal{L}\{t\sin 2t\} - 2\mathcal{L}\{t\cos 3t\}$$

$$= -\frac{d}{ds}\left(\frac{2}{s^2+4}\right) + 2\frac{d}{ds}\left(\frac{s}{s^2+9}\right)$$

$$= \frac{4s}{(s^2+4)^2} + 2\cdot\frac{9-s^2}{(s^2+9)^2}$$

Example-10: Find the value of $\mathcal{L}\{t^2\sin 2t + t^3\cos t\}$

Soln: $\mathcal{L}\{t^2(\sin 2t) + t^3\cos t\}$

$$= \mathcal{L}\{t^2\sin 2t\} + \mathcal{L}\{t^3\cos t\}$$

$$= (-1)^2\frac{d^2}{ds^2}\left(\frac{2}{s^2+4}\right) + (-1)^3\frac{d^3}{ds^3}\left(\frac{s}{s^2+1}\right)$$

Example-11: Find $\mathcal{L}\left\{\frac{e^{-3t}}{\sqrt{t}}\right\}$

Soln: We know $\mathcal{L}\{t^n\} = \frac{\Gamma(n+1)}{s^{n+1}}$, $n > -1$, $s > 0$

$$\mathcal{L}\{t^{-\frac{1}{2}}\} = \frac{\Gamma(-\frac{1}{2}+1)}{s^{-\frac{1}{2}+1}} = \frac{\Gamma\frac{1}{2}}{s^{\frac{1}{2}}} = \frac{\sqrt{\pi}}{\sqrt{s}}$$

$$\therefore \mathcal{L}\left\{\frac{1}{\sqrt{t}}\right\} = \frac{\sqrt{\pi}}{\sqrt{s}}$$

$$\therefore \mathcal{L}\left\{\frac{e^{-3t}}{\sqrt{t}}\right\} = \frac{\sqrt{\pi}}{\sqrt{s+3}}$$

Example12: Find $\mathcal{L}\left\{\int_t^\infty \frac{\cos u}{u}\,du\right\}$

Solution:

$$\mathcal{L}\left\{\int_t^\infty \frac{\cos u}{u}\,du\right\} = \mathcal{L}\left\{\int_1^\infty \frac{\cos tv}{tv}\cdot t\,dv\right\}$$

Let $u = tv$, $du = t\,dv$, $u = t \Rightarrow v = 1$, $u = \infty \Rightarrow v = \infty$

$$= \mathcal{L}\left\{\int_1^\infty \frac{\cos tv}{v}\,dv\right\}$$

$$= \int_0^\infty e^{-st}\left\{\int_1^\infty \frac{\cos tv}{v}\,dv\right\}dt$$

$$= \int_1^\infty \frac{1}{v}\left\{\int_0^\infty e^{-st}\cos tv\,dt\right\}dv$$

$$= \int_1^\infty \frac{1}{v}\,\mathcal{L}\{\cos tv\}\,dv$$

$$= \int_1^\infty \frac{1}{v}\cdot\frac{s}{s^2+v^2}\,dv$$

$$= \frac{1}{s}\int_1^\infty \left(\frac{1}{v} - \frac{v}{s^2+v^2}\right)dv$$

$$= \frac{1}{s}\left[\log v - \frac{1}{2}\log\left(s^2+v^2\right)\right]_1^\infty$$

$$= \frac{1}{s}\left[\log\frac{v}{\sqrt{s^2+v^2}}\right]_1^\infty$$

$$= \frac{1}{s}\left[\log\frac{1}{\sqrt{s^2/v^2+1}}\right]_1^\infty$$

$$= \frac{1}{s}\left\{\log\frac{1}{\sqrt{0+1}} - \log\frac{1}{\sqrt{s^2+1}}\right\}$$

$$= -\frac{1}{s}\log\frac{1}{\sqrt{s^2+1}} = \frac{1}{s}\log\sqrt{s^2+1}$$

$$= \frac{\log(s^2+1)}{2s}.$$

Example 13: Find $\mathcal{L}\left\{\int_t^\infty \frac{e^{-u}}{u}\,du\right\}$

Solution:

$$\mathcal{L}\left\{\int_t^\infty \frac{e^{-u}}{u}\,du\right\} = \mathcal{L}\left\{\int_1^\infty \frac{e^{-vt}}{vt}\,t\,dv\right\} \quad \Big| \begin{matrix} u = vt \\ du = t\,dv \end{matrix}$$

$$= \int_0^\infty e^{-st}\left\{\int_1^\infty \frac{e^{-vt}}{v}\,dv\right\}dt$$

$$= \int_1^\infty \frac{1}{v}\left\{\int_0^\infty e^{-st}e^{-vt}\,dt\right\}dv$$

$$= \int_1^\infty \frac{1}{v}\,\mathcal{L}\{e^{-vt}\}\,dv$$

$$= \int_1^\infty \frac{1}{v}\cdot\frac{1}{(s+v)}\,dv$$

$$= \frac{1}{s}\int_1^\infty \left(\frac{1}{v} - \frac{1}{s+v}\right)dv$$

$$= \frac{1}{s}\log\frac{v}{s+v}\Big|_1^\infty$$

$$= \frac{1}{s}\log\frac{1}{s/v+1}\Big|_1^\infty$$

$$= \frac{1}{s}\left\{0 - \log\frac{1}{s+1}\right\}$$

$$= \frac{1}{s}\log(s+1)$$